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Some structural hypotheses in a multivariate cointegration analysis of the purchasing power parity and the uncovered interest parity for UK. *

by

Søren Johansen and Katarina Juselius

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Abstract.

The paper developes some new tests for structural hypotheses in the framework of a multivariate error correction model with Gaussian errors.

The tests are constructed by an analysis of the likelihood function, and motivated by an empirical investigation of the PPP relation and the UIP relation for the United Kingdom.

There are three types of tests discussed. First we consider the same linear restrictions on all the cointegrating relations, then we consider the hypothesis that certain relations are assumed to be cointegrating, and finally we formulate a general hypothesis that contains the previous ones. This hypothesis can be expressed by the condition that some of the cointegrating relations are subject to given linear restrictions, while others are unconstrained.

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1. Introduction.

This paper developes some tests for structural hypotheses on the cointegrating relations in a multivariate error correction model. We consider a VAR model in levels, under the assumption of cointegration. This model is used to describe the statistical variation of the data without imposing the economically interesting relations. Instead the parameteric formulation allows us to formulate the structural economic hypotheses as statistical hypotheses concerning the cointegrating relations. This again allows for a likelihood analysis if we assume Gaussian errors, and the tests proposed are the likelihood ratio tests, see Johansen (1988) and Johansen and Juselius (1990).

The empirical purpose of this paper is to investigate the international transmission effects between countries through the exchange rate, the interest rate and the price determination as assumed by the purchasing power parity and the uncovered interest rate parity relation. This has been the focus of interest in a vast number of empirically oriented papers, see for instance Adler and Lehman (1983), Baillie and Selover (1987), Corbae and Ouliaris (1988), Edison and Klovland (1987), Hakkio (1984) and Schotman (1989). Generally the empirical evidence of these fundamental relations have been weak. Here we will try to suggest possible reasons why so many studies have failed in this respect. This will point to the importance of considering the interaction between exchange rates, interest rates and prices in the goods and the asset markets in a simultaneous model as well as the importance of distinguishing between short–run and long–run effects. The link between the goods and the asset markets can be found in the determination of the exchange rate, which seems to play a crucial role in this context.

We will analyze some time series data from the UK economy using an econometric modelling approach which differs from the standard ones in two importants ways. Firstly, we will analyze the data in a full system of equations model, thus allowing for possible interactions in the determination of prices, interest rates and exchange rates. This would eliminate the single equation bias likely to have affected many of the previous studies. Secondly, we will adopt a model specification that explicitely allows for different short–run and long–run dynamics using recent results on nonstationary time series. The distinction between short–run and long–run is crucial in this empirical problem, since the short–run effects from a highly volatile asset market seem to be fundamentally different from the long–run movements in the goods and possibly also from medium long–run movements in the asset markets.

The econometric analysis is based on a full system cointegration model which is well designed for this type of empirical work by the explicit classification into nonstationary and stationary components providing an interpretation in terms of the dynamics of long–run and short–run effects.

2. The economic and statistical framework.

The most popular models that have been applied for exchange rate determination include the flexible price monetary model of Frenkel (1976) and the overshooting monetary model of Dornbush (1976). For the interest rate determination various versions of the uncovered real or nominal interest rate parity model have been the standard reference. The international price determination is as a rule assumed to be determined by various versions of the purchasing power parity relation. The link between them can be found in the real interest rate differential model of Frankel (1979) relating an interest rate differential between two countries to i) a possible covered interest rate differential, ii) an expected change in exchange rates and iii) an expected change in the purchasing power parity between the countries. Here we will concentrate on the empirical investigation of the two fundamental equilibrium relations, the PPP, relating the price levels in the two countries

$$p_1 - p_2 = e_{12}$$

and the UIP, relating the interest rates to the exchange rates

$$i_1 - i_2 = \Delta e^e_{12}$$

where p_i indicates the price level in country i, i_i is the interest rate in country i, e_{12} is the exchange rate denominated in the currency of country 1 and a superscript e indicates expectation. All variables are assumed to be in logarithms. The two equilibrium relations are fundamentally different. The first one can be assumed to be a backward looking long–run relation, the adjustment towards which can be expected to be very slow, possibly might not be there at all. The second one is a forward looking market clearing relation and thus can be expected to be more short–run. However, most empirical works on UIP and in particular on PPP have been disappointing by not being able to verify this relation. Therefore it seems reasonable to consider a model formulation that does not directly impose the UIP and/or the PPP relation but, more indirectly, assumes a tendency in the market to react according to these relations. These considerations motivate a model specification of the error–correction type:

Prices:

$$\Delta p_{1t} = f_{p1}(\Delta X_{t-j}, j=1,..) + \underset{(-)}{\gamma_1}(p_1-p_2-e_{12})_{t-1} + \underset{(+)}{\gamma_2}(i_1-i_2)_{t-1} + \epsilon_{p1_t}$$

$$\Delta p_{2t} = f_{p2}(\Delta X_{t-j}, j=1,..) + \underset{(+)}{\gamma_3}(p_1-p_2-e_{12})_{t-1} + \underset{(-)}{\gamma_4}(i_1-i_2)_{t-1} + \epsilon_{p2_t}$$

Exchange rates:

$$\Delta e_{12t} = f_e(\Delta X_{t-j}, j=1,..) + \underset{(+)}{\gamma_7}(p_1-p_2-e_{12})_{t-1} + \underset{(\pm)}{\gamma_8}(i_1-i_2)_{t-1} + \epsilon_{e_t}$$

Interest rates:

$$\Delta i_{1t} = f_{i1}(\Delta X_{t-j}, j=1,..) + \underset{(+)}{\gamma_9}(p_1-p_2-e_{12})_{t-1} + \underset{(-)}{\gamma_{10}}(i_1-i_2)_{t-1} + \epsilon_{i1_t}$$

$$\Delta i_{2t} = f_{i2}(\Delta X_{t-j}, j=1,..) + \underset{(-)}{\gamma_{11}}(p_1-p_2-e_{12})_{t-1} + \underset{(+)}{\gamma_{12}}(i_1-i_2)_{t-1} + \epsilon_{i2_t}$$

where $f_i(\Delta X)$ indicates a linear function of the process ΔX_t, where $X_t = [p_1, p_2, e_{12}, i_1, i_2]$. No assumption is made at this stage on the specific form of the short run dynamics, nor on the simulteaneous structure of the model. In the empirical application the lag length will be specified generally enough for the residuals to be uncorrelated. The hypothetical parameter restrictions implied by the long–run PPP and UIP relations will not be imposed but instead tested for data admissibility with the unrestricted cointegration space based on the multivariate cointegration model (Johansen and Juselius, 1990) to be considered below.

A priori we expect the deviations from the purchasing power parity and the interest rate differential to affect both prices, interest rates and exchange rates with the coefficients γ_i, the expected signs of which are given in the parenthesis. The coefficient to the interest rate differential γ_8 can either be negative due to the Keynesian assumption of a rise in domestic interest rates leading to a currency appreciation, or positive due to the monetarist assumption that inflationary expectations will lead to currency depreciation.

Before defining the basic econometric model, the data will will be briefly described. The basic variables of interest are:

p_1 = a UK wholesale price index.

p_2 = a trade weighted foreign price index.

e_{12} = the UK effective exchage rate.

i_1 = the three months treasury bill rate in UK.

i_2 = the three months Eurodollar interest rate.

All variables are in logarithms and the sample period is 1972.1 to 1987.3 thus covering the the post Bretton Woods floating exchange rate system. The graphs of the differenced data are given in fig. 3.1 and illustrate the large fluctuations in the data. These are partly the result of the two oil crises, and the great turbulence in the exchange market during this period. We indicate below the major events of importance:

1973.3—1973.4 : The first oil crisis.

1979 :The tight monetary policy introduced by Margareth Thatcher.

1979 : The abandonment of the exchange control in the UK.

1979 : The second oil crisis.

1980 : The depository institutions deregulation and monetary control act in USA.

1982 : The depository institutions act of 1982.

The last two events have exerted an influence on the interest rate determination outside the borders of USA. Altogether the above events are likely to have changed some of the parameters of the model, motivating some care when interpreting the empirical results. However, it seems more likely that these are in the short–run parameters of the model rather than the long–run and therefore might be of less importance for our study.

With these precautions in mind we will turn to the basic model, the five dimensional vector autoregressive model with Gaussian errors

$$X_t = A_1X_{t-1} + \dots + A_kX_{t-k} + \mu + \Psi D_t, \; t = 1,\dots,T \tag{2.1}$$

where $X_t = [p_1, p_2, e_{12}, i_1, i_2]$ as defined above, $X_{-k+1},\dots,X_0$ are fixed, $\epsilon_1,\dots,\epsilon_T$ are i.i.d $N_p(o,\Sigma)$ and D_t are centered seasonal dummies. We write the model in the error correction form

$$\Delta X_t = \Gamma_1 \Delta X_{t-1} + \dots + \Gamma_{k-1}\Delta X_{t-k+1} + \Pi X_{t-k} + \mu + \Psi D_t + \epsilon_t, \; t = 1,..,T \tag{2.2}$$

and assume in the following the hypothesis

$$\mathscr{H}_1(r) : \;\; \Pi = \alpha\beta', \tag{2.3}$$

where α and β are p×r matrices. The hypothesis $\mathscr{H}_1(r)$ is the hypothesis of reduced rank of Π implying that under certain conditions (see Johansen, 1989b) the process ΔX_t is stationary, X_t is nonstationary, but also that $\beta' X_t$ is stationary. Thus we can interprete the relations $\beta' X_t$ as the stationary relations among nonstationary variables, i.e. as cointegrating relations. The importance of the model formulation (2.2) with the hypothesis of cointegration (2.3) is that it allows the precise formulation of a number of interesting economic hypotheses in such a way that they can be tested. In section 4 we will consider the testing of three different structural hypotheses which are briefly described below in terms of hypotheses about the PPP relation.

First we ask whether the cointegration space contains the purchasing power parity restriction for all cointegration vectors. Then we ask whether the PPP–relation

$(p_1-p_2-e_{12})$ is stationary by itself without involving the other variables of the system. This type of hypothesis has been widely tested using the Dickey–Fuller type of univariate testing procedure (Dickey and Fuller, 1979). Our procedure differs in the sense that it uses all information in the data in an optimal way, thus in most cases increasing the efficiency considerably. Finally we consider the hypothesis whether some linear combination of p_1, p_2 and e_{12} is stationary. This could be of interest if the previous hypotheses were rejected.

It was shown in Johansen (1988), Johansen and Juselius (1990) how one calculates the maximum likelihood estimator in the general error correction model. In section 4 it will be shown that simple modifications of this procedure yield the estimates and test statistics under the three hypotheses discussed above. Below we give a brief description of the estimation procedure to introduce the necessary notation and the most important concepts.

The likelihood function is first concentrated with respect to the parameters $\Gamma_1,...,\Gamma_{k-1}$ μ and Ψ by regressing ΔX_t and X_{t-k} on $\Delta X_{t-1},...,\Delta X_{t-k+1}$, 1, and D_t. This defines residulas R_{0t} and R_{kt} and residual product moment matrices

(2.4) $$S_{ij} = T^{-1} \sum_{t=1}^{T} R_{it}R'_{jt}, \quad i,j = 0,k.$$

The concentrated likelihood function has the form of a reduced rank regression

(2.5) $$R_{0t} = \alpha\beta' R_{kt} + \text{error}.$$

For fixed β (2.5) can be solved for α by regression

(2.6) $$\hat{\alpha}(\beta) = S_{0k}\beta(\beta' S_{kk}\beta)^{-1},$$

and β is determined by solving the eigenvalue problem:

(2.7) $$|\lambda S_{kk} - S_{k0}S_{00}^{-1}S_{0k}| = 0.$$

This has solutions $\hat{\lambda}_1 > ... > \hat{\lambda}_p > 0$ with the corresponding eigenvectors $\hat{V} = (\hat{v}_1,...,\hat{v}_p)$ normalized by $\hat{V}' S_{kk}\hat{V} = I$. The maximum likelihood estimator for β is then found as

(2.8) $$\hat{\beta} = (\hat{v}_1,...,\hat{v}_r),$$

which with the above normalization gives

(2.9) $\hat{\alpha} = S_{0k}\hat{\beta}$.

The maximized likelihood function is found to be

(2.10) $$L_{max}^{-2/T} = |S_{00}| \prod_{i=1}^{r} (1-\hat{\lambda}_i),$$

and the likelihood ratio test for the hypothesis $\mathcal{H}_1(r)$ the full VAR model (2.2), $\mathcal{H}_0$, is given by

(2.11) $$-2\ln Q(\mathcal{H}_1(r) | \mathcal{H}_0) = -T \sum_{i=r+1}^{p} \ln(1-\hat{\lambda}_i)$$

which is called the trace statistics. An alternative test called the λ_{max} is based on the comparison of $\mathcal{H}_1(r-1)$ against $\mathcal{H}_1(r)$:

(2.12) $$-2\ln Q(\mathcal{H}_1(r-1) | \mathcal{H}_1(r)) = -T\ln(1-\hat{\lambda}_r)$$

3. <u>The empirical results of the cointegration analysis.</u>

Model (2.2) was first estimated for k=2, but the residuals did not pass the normality test due to excess kurtosis. Large residuals were found to coincide with the occurence of the late 1973 and 1979 oil price shocks. This motivated the inclusion of the world oil prices in the models information set. Since this variable obviously is exogenously determined in this system we have reformulated (2.2) in the following way:

(3.1) $$\Delta X_t = \Gamma_1 \Delta X_{t-1} + C_0 \Delta x_{6t} + C_1 \Delta x_{6t-1} + \Pi X_{t-2} + \mu + \Psi D_t + \epsilon_t,\ t=1,..,T$$

where x_{6t} is the logarithm of the world oil price at time t and X_t is defined as before.

The misspecification tests for this model are reported in table 3.1. below. The results are now more satisfactory. However, in eq.2 and eq.5 there are still indication of excess kurtosis, causing the Jarque–Bera test statistic for normality to become significant. This is not surprising since the combined price index, x_2, and the eurodollar rate, x_5, were chosen to explain the variation in UK price, exchange rate and interest rates but not vice versa, meaning that the selected variable set is not sufficient to account for the variation in x_2 and x_5. The variable x_2 and x_5 might well be weakly exogenous for the

Table 3.1. Residual misspecification tests in model (3.1)

	stand.dev.	skewness	excess kurtosis	normality test $\chi^2(2)$	autocorr. test $\chi^2(20)$
eq.1.	.007	.29	1.27	4.84	6.09
eq.2	.007	.28	2.16	12.44	9.59
eq.3	.030	.30	.17	.95	13.54
eq.4	.011	.58	.25	3.55	9.11
eq.5	.013	−.51	3.76	37.95	16.41

long–run parameters of interest, which would make the deviation less important. The weak exogeneity hypothesis is testable as will be shown below. Here we conclude that the residuals from UK prices, exchange rates and interest rates can be assumed to follow a Gaussan process and the residuals from the combined prices and the eurodollar interest rates follow an innovation process. The graphs of the residuals $\hat{\epsilon}_{it}$ and the first differences Δx_{it} (i=1,...,5) are presented in fig 3.1. Note how well the large fluctuations in the original data have been accounted for by the chosen information set.

The test of the rank of Π is performed using the two likelihood ratio tests (2.11) and (2.12), the results of which are presented in table 3.2. below:

Table 3.2. Tests of the cointegration rank.

λ_i	$-T\ln(1-\lambda_i)$	$\lambda_{max}(.95)$	$-T\Sigma\ln(1-\lambda_i)$	$\lambda_{trace}(.95)$
.083	5.19	8.08	5.19	8.08
.102	6.47	14.60	11.66	17.84
.254	17.52	21.27	29.26	31.26
.285	20.16	27.34	49.42	48.41
.401	31.32	33.26	80.75	69.98

Based on the λ_{max} test statistic the hypotesis of no cointegration cannot be rejected at the standard 5 % level, whereas the trace statistic would lead us to accept two cointegration vectors. To make the decision still more difficult, note that $\hat{\lambda}_3 \approx \hat{\lambda}_2$ suggesting that possibly the third eigenvector should be considered among the stationary vectors. This illustrates the fact that the two test procedures not neccesarily give the same result and also the ambiguity when choosing the number of cointegrating vectors. Basically this ambiguity is due to the low power in cases when the cointegration relation is quite close to the non stationary boundary (Johansen, 1990b). This is a real nuisance considering that the null hypothesis of a unit root is not always reasonable from an economic point of view. The latter has been pointed out by a.o. Schotman and van Dijk (1989). This problem is often present in empirical work when the speed of adjustment to the hypothetical equlibrium state is very slow for instance due to regulations, high adjustment costs and other short–run effects which tend to push the process off the equlibrium path. This seems to indicate that the final determination of the number of cointegration vectors has to be based both on the result of the formal testing and the interpretability of the obtained coefficients as well as the graphs. In fig. 3.2 the graphs of $\hat{v}_i'X_t$ are given at the l.h.s. of the figure and $\hat{v}_i'R_{kt}$ at the r.h.s. The estimated values $\hat{v}_i$ are given in table 3.3 below and R_{kt} is defined at p. If r=2 we would expect the first two processes to look stationary, albeit not like white noise processes. The ordering of the relations based on the ordering of $\hat{\lambda}_i$ means that the first relation is the most correlated with the stationary part of the process and the second is the next most correlated, etc. The graphs of the cointegrating relations corrected for the short run dynamics, $\hat{v}_i'R_{kt}$, look much more satisfactory in this respect compared to the graphs of $\hat{v}_i'X_t$. This gives an illustration of an important property of this model, namely its ability to describe an inherent tendency to move towards the equilibrium states, without necessarily ever reaching it because of frequent and often large shocks pushing it away from the equilibrium path. In that sense the graphs $\hat{v}_i'X_t$ describe the actual deviation from the equilibrium path as a function of short–run effects, whereas $\hat{v}_i'R_{kt}$ describes the adjustment path corrected for the short–run dynamics of the model. If the short–run dynamics can be expected to be substantially different from the long–run the two graphs will usually look quite different. This illustrates the importance of specifying the short–run as well as the long–run, even if the main interest is in the long–run. For instance if the interest is only in the PPP relation it would be tempting to restrict the basic variable set to $\{p_1, p_2, e_{12}\}$. But this would result in a model where the important short–run effects from the asset markets measured by the interest rates would not be accounted for, thus possibly invalidating the estimation of the long–run PPP.

Since the graphical examination supported the choice of r=2, the subsequent analysis will be based on the assumption of two stationary relations and three common trends in this data set. We now turn to the analysis of the individual estimates which are given in table 3.3. below.

Before commenting on the individual estimates it should be pointed out that any linear combination of the stationary vectors is also a stationary vector and therefore a direct interpretation is not always interesting. This points to the need to use testing as a device to find out whether any specified structural relation can be contained in the space spanned by β. However, in this case the first eigenvector seems to contain the assumed PPP relation among the first three variables and the second seems to contain the interest rate differential among the last two variables. The α_{i1} coefficients seem to indicate that the first eigenvector is most important for the UK price and the UK effective exchange rate equations, whereas the coefficient $\hat{\alpha}_{21}$ in the combined price equation is essentially zero. the α_{i2} coefficients indicate that the interest rate differential is important only in the two interest rate equations. However, formal testing would be needed to make these statements more precise. The joint hypothesis $\{\alpha_{i.} = 0\}$ can be tested by the likelihood ratio test procedure described in Johansen (1990a), Johansen & Juselius (1990). Testing each α_{ij} individually is also possible but reasonable only under the assumption that the actually estimated β_i vectors (based on the normalization $\beta' S_{kk} \beta' = I$) are the cointegrating vectors of interest instead of linear combinations of them. Since this is not neccesarily the case we will in the first stage test the hypothesis $\{\alpha_{2j} = 0, j=1,2\}$ and $\{\alpha_{5j} = 0, j=1,2\}$.

Table 3.3. The estimated eigenvectors and the corresponding weights.

The eigenvectors

$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{v}_3$	$\hat{v}_4$	$\hat{v}_5$
1.00	.03	.36	1.00	1.00
–.91	–.03	–.46	–2.40	–1.45
–.93	–.10	.41	1.12	–.48
–3.38	1.00	1.00	–.41	2.28
–1.89	–.94	–1.03	2.98	.76

The weights:

α_1	α_2	α_3	α_4	α_5
–.07	.04	–.01	.00	–.01
–.02	.00	–.04	.01	.01
.10	–.01	–.15	–.04	–.05
.03	–.15	.03	.01	–.02
.06	.29	.01	.03	–.01

Before the testing of α we will show the connection between the $\hat{\alpha}_{ij}$ values and the eigenvalues $\hat{\lambda}_i$. This interpretation of the eigenvalues is also useful for the understanding of the behaviour of the test statistic (2.11).

Let $\hat{\gamma} = S_{0k}(\hat{v}_{r+1},...,\hat{v}_p)$ be the coefficients we would have obtained to $\hat{v}_i'X_{t-k}$ if we had left them in the model. It follows from the definition of $\hat{v}_i$ as eigenvectors that

$$\hat{\gamma}' S_{00}^{-1}\hat{\gamma} = (\hat{v}_{r+1},...,\hat{v}_p)' S_{k0}S_{00}^{-1}S_{0k}(\hat{v}_{r+1},...,\hat{v}_p) = \mathrm{diag}(\hat{\lambda}_{r+1},...,\hat{\lambda}_p).$$

Thus the test statistic (2.11) which is approximately $T\sum_{i=r+1}^{p}\hat{\lambda}_i = T\mathrm{tr}(\hat{\gamma}' S_{00}^{-1}\hat{\gamma})$ is really measuring the size of the coefficients of the supposedly nonstationary components in the full regression model. Consider the second cointegration vector β_2 and the corresponding values of α_{i2} which, for i=1,2,3, are approximately zero. The value of $\hat{\lambda}_2$ is thus affected by the number of zero and nonzero coefficients in each column, indicating that a low $\hat{\lambda}_i$ value might be the result of many $\alpha_{ij} = 0$ for that particular β_i.

The hypothesis $\alpha_{i.} = 0$ is the equivalent of testing whether Δx_{it} is weakly exogenous in the model when the parameters of interest are the long–run coefficients β_{ij}. See

Johansen (1989a) for a full discussion of this topic. We will first test the hypothesis

$\mathscr{H}_2 : \alpha_{2j}=0$ for j=1,2

which is of particular interest in this case since the Gaussian assumption about the residuals $\hat{\epsilon}_{2t}$ was not completely satisfied due to excess kurtosis. If $\mathscr{H}_2$ is accepted the system could be reduced to a four–dimensional system by conditioning on Δx_{2t} without affecting the estimates of β. Solving the model under the restriction $\mathscr{H}_2$ gives rise to the p–1 new eigenvalues to be compared with the eigenvalues of the unrestricted model H_1:

	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{\lambda}_3$	$\hat{\lambda}_4$	$\hat{\lambda}_5$	$-T\ln(1-\hat{\lambda}_i)$				
$\mathscr{H}_1$:	.407	.285	.254	.102	.083	31.3	20.2	17.6	6.5	5.2
$\mathscr{H}_2$:	.400	.277	.158	.088		30.6	19.5	10.3	5.5	

The hypothesis is tested by comparing the the restricted model within the unrestricted model $\mathscr{H}_1$ using the likelihood ratio test procedure derived in Johansen (1990a). This amounts to comparing the r first eigenvalues under $\mathscr{H}_2$ with the corresponding eigenvalues under $\mathscr{H}_1$ in the following way:

$$-2\ln Q(\mathscr{H}_2 | \mathscr{H}_1) = -60\ln\{\frac{(1-.400)(1-.277)}{(1-.407)(1-.285)} = .65 + .66 = 1.31$$

The test statistic is asymptotically distributed as $\chi^2(2)$ and therefore not significant. Thus we conclude that the combined price x_2 is weakly exogenous for β. Next we test the hypothesis that the eurodollar interest rate is weakly exogenous for β, i.e.:

$\mathscr{H}_3$: $\alpha_{5j}=0$ for j=1,2

giving the test statistic:

$$-2\ln Q(\mathscr{H}_3 | \mathscr{H}_1) = -60\ln\{\frac{(1-.387)(1-.231)}{(1-.406)(1-.285)}\} = 1.96 + 4.38 = 6.34$$

The $\chi^2(2)$ test statistic is now more significant indicating that the eurodollar rate cannot be considered weakly exogenous for β.

Finally we will investigate the estimates of the restricted $\hat{\Pi} = \hat{\alpha}\hat{\beta}'$ for r=2 as given in table 3.4 below. These estimates measure the combined effect of the two cointegrating relations in each of the five equations.

Table 3.4. The estimates of $\Pi = \alpha\beta'$ for r = 2.

	p_1	p_2	e_{12}	i_1	i_2
eq.1	–.067	.061	.060	.272	.090
eq.2	–.018	.016	.016	.064	.030
eq.3	.101	–.091	–.093	–.345	–.186
eq.4	.030	–.026	–.018	–.263	.072
eq.5	.066	–.062	–.082	.097	–.382

Note that the PPP relation seems to be present in all equations with the greatest weight in the exchange rate equation followed by the UK price equation and the eurodollar rate equation. It is amazing how closely the estimates follow the hypothetical PPP $(a_i,-a_i,-a_i)$ relation where a_i is the weight coefficient in equation i. Note also that the sign of a_i is consistent with the expected signs for the UK equations 1, 3 and 4. For the combined price level the coefficients were found to be zero, and for the eurodollar rate it is more difficult a priori to know what sign to expect. The coefficients to the interest rates are also quite close to the expected ones. However in the first three equations the weighted sum of the interest rates instead of the interest rate differential seems to be relevant, albeit the coefficient to the eurodollar rate is small enough to be statistically insignificant in the price equations. The same result was found in a similar study of prices, exchange rates and interest rates between Denmark and Germany (Juselius, 1989) which makes the observation more interesting.

4. A class of tests for linear structural hypotheses on the cointegration vectors.

The hypotheses are formulated in terms of the cointegrating relations β, since these describe the long–run relations in which most economic strutural hypotheses are formulated. Examples and motivation are given below, but for later reference the hypotheses are the following:

(4.1) $\mathscr{H}_4 : \beta = H_4\varphi,$ $H_4(p\times s),\ \varphi(s\times r),\ r \le s \le p,$

(4.2) $\mathscr{H}_5 : \beta = (H_5,\psi)$ $H_5(p\times r_1),\ \psi(p\times r_2),\ r = r_1+r_2,$

(4.3) $\mathscr{H}_6 : \beta = (H_6\varphi,\psi)$ $H_6(p\times s),\ \varphi(s\times r_1),\ \psi(p\times r_2),\ r_1 \le s \le p,\ r = r_1+ r_2.$

These hypotheses are linear hypotheses on the cointegrating relations, which are structural in the sense that the do not depend on any normalization of the parameter β.

It will be shown by an analysis of the likelihood function how the estimators and test statistics can be calculated. The calculations are reduced to an eigenvalue problem, such that the analysis of the hypotheses under the various restrictions is similar to the one that is outlined in the beginning of section 2.

Since these hypothesis are really hypotheses about the space spanned by β, the cointegrating space, we can also formulate the hypotheses as

(4.1a) $sp(\beta) \subset sp(H_4)$

(4.2a) $sp(H_5) \subset sp(\beta)$

(4.3a) $\dim(sp(\beta) \cap sp(H_6)) \ge r_1$

The first hypothesis formulates the same (p–s) linear restrictions on all the r cointegrating relations. Here $r \le s \le p$, where $r = s$ means that all cointegrating relations are assumed known, and $s = p$ indicates that no restrictions are imposed, such that the hypothesis reduces to the model considered in Section 2.

The next hypothesis (4.2) assumes r_1 of the r cointegrating relations known, as specified by the matrix H_5. The remaining r_2 relations (ψ) are chosen without restrictions. If $r_2 = 0$, i.e. $r = r_1$, the hypothesis $\mathscr{H}_5$ reduces to a special case of $\mathscr{H}_4$, with $s = r = r_1$.

Finally $\mathscr{H}_6$ puts the restrictions on r_1 of the cointegrating relations (φ) which are chosen in the space $sp(H_6)$ and the remaining (ψ) are chosen without any restrictions. It follows that if $r_2 = 0$, then all relations are chosen in $sp(H_6)$ and the hypothesis reduces to $\mathcal{H}_4$ and if $r_1 = s$, then $\mathcal{H}_6$ reduces to $\mathcal{H}_5$.

It was shown in Johansen (1990a) that the asymptotic distribution of the maximum likelihood estimator for β is a mixture of Gaussian distributions. This implies that hypotheses as the ones considered give rise to a likelihood ratio test that is asymptotically distributed as χ^2. We shall apply this general result in the following, and concentrate on the derivation and interpretation of the test statistics. In each case we calculate the degrees of freedom for the test. The models $\mathcal{H}_4, \mathcal{H}_5$ and $\mathcal{H}_6$ are tested against $\mathcal{H}_1$. Throughout the estimate under the model $\mathcal{H}_1$ is denoted by ˆ and under any of the submodels by ⌒.

4.1. <u>The hypothesis $\beta = H_4\varphi$.</u>

This hypothesis where H_4 (p×s) is known and φ(s×r) is unknown was treated in Johansen (1988), Johansen and Juselius(1990) and Johansen (1990a).

Since the hypothesis only involve the parameters in β, it is convenient to use equation (2.5). For $\beta = H_4\varphi$ (2.5) becomes:

(4.4) $$R_{0t} = \alpha\varphi' H_4' R_{kt} + \text{error},$$

which immediately shows that the solution is similar to that of (2.5) only with R_{kt} replaced by $H_4' R_{kt}$. Thus the estimation procedure is the same only the levels are transformed into the set of variables where cointegration is to be found.

This shows that the estimator under $\mathcal{H}_4$, $\hat{\varphi}$, is found as the eigenvectors of the equation

(4.5) $$|\lambda H_4' S_{kk} H_4 - H_4' S_{k0} S_{00}^{-1} S_{0k} H_4| = 0.$$

This equation has solutions $\hat{\lambda}_1 > ... > \hat{\lambda}_s > 0$ and eigenvectors $\hat{V} = (\hat{v}_1,...,\hat{v}_s)$. The estimate of $\hat{\varphi}$ is then $(\hat{v}_1,...,\hat{v}_r)$ such that

(4.6) $$\hat\beta = H_4(\hat v_1,\ldots,\hat v_r),$$

and

(4.7) $$L_{max}^{-2/T} = |S_{00}| \prod_{i=1}^{r} (1-\hat\lambda_i).$$

From this we find the first result

THEOREM 1. The hypothesis $\beta = H_4\varphi$, where H_4 is p×s can be tested by the likelihood ratio test given by

(4.8) $$-2\ln Q(\mathcal{H}_5 | \mathcal{H}_4) = T \sum_{i=1}^{r} \ln\{(1-\tilde\lambda_i)/(1-\hat\lambda_i)\}$$

which is asymptotically distributed as χ^2 with f = (p–s)r degrees of freedom. The estimate of β is found from (4.6) and α from (2.9).

Proof. The result follows from the above derivation. The degrees of freedom are calculated as follows: Normalize the β matrix such that $\beta' = (I,\tau')$, with τ (p–r)×r. This gives pr +(p–r)r free parameters under $\mathcal{H}_1$. Under the restriction (4.1), where $\beta = H_4\varphi$, φ is normalized in the same way leaving pr + (s–r)r free parameters under the hypothesis. The difference is the degrees of freedom for the test.

Two hypotheses of type (4.1) are relevant for our empirical problem. The first one is the hypothesis of the purchasing power parity formulated as: The variables p_1,p_2 and e_{12} enter into the cointegrating relations with coefficients proportional to (1,–1,–1), i.e. the cointegration relations are of the form $(a_i,-a_i,-a_i,*,*)$ for i = 1,...r. This can be formulated as a hypothesis of the type (4.1) with

$$H_{4.1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The solution to the eigenvalue problem (4.5) with $H_4 = H_{4.1}$ and S_{kk}, S_{k0} and S_{00} as defined by (2.4) is the s = 3 eigenvalues reported below. These are compared with the eigenvalues from the unrestricted model $\mathcal{H}_1$.

	λ_1	λ_2	λ_3	λ_4	λ_5	$-T\ln(1-\lambda_i)$			
$\mathcal{H}_1$:	.407	.285	.254	.102	.083	31.3	20.2	17.6	6.5
$\mathcal{H}_{4.1}$:	.386	.278	.090			29.2	19.5	5.6	

For r = 2 the likelihood ratio test (4.8) becomes:

$$-2\ln Q(\mathcal{H}_{5.1} \mid \mathcal{H}_4) = 60\ln\{\frac{(1-.386)(1-.278)}{(1-.407)(1-.285)}\} = 2.09 + .59 = 2.68$$

The test statistic is asymptoticlly distributed as $\chi^2(4)$ and thus this hypothesis cannot be rejected. This is of course consistent with the observation that the estimated coefficients in $\Pi = \alpha\beta'$ for r = 2 closely approximated the PPP restriction in all equations of the system. Note however that the PPP restriction is not valid for the third vector $\hat{v}_3$ as can be seen by the drop in the $-T\ln(1-\hat{\lambda}_3)$ from 17.6 to 5.6. This is intuitively reasonable since $\hat{v}_3$ has been found to lie in the nonstationarity space. If we erroneously would have included the third vector among the stationary relations, the beautiful PPP structure found in $\hat{\Pi}=\hat{\alpha}\hat{\beta}$ for r = 2 would have been diffused by the third vector. This serves as an illustration of how careful one has to be not to get misleading results because of the complicated interaction between stationary and nonstationary processes, as well as between short–run and long–run dynamics. It also illustrates a methodological point, namely that this test procedure has the property that once the calculation of eigenvectors has been performed one can conduct the inference for different values of r without recalculating estimates and test statistics.

The second hypothesis of interest states that only the nominal interest differential enters all cointegration relations. This can be formulated as a hypothesis of type (4.1) with $H_4 = H_{4.2}$ below:

$$H_{4.2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

The solution to the eigenvalue problem (4.5) with $H_4 = H_{4.2}$ gives the s = 4 eigenvalues reported below:

$\mathcal{H}_1$:	.407	.285	.254	.102	.083
$\mathcal{H}_{4.2}$:	.286	.254	.146	.093	

In this case the likelihood ratio test statistic becomes 13.17 to be compared to a $\chi^2(2)$ distribution. The hypothesis $\mathcal{H}_{4.2}$ is thus strongly rejected. Note that the main contribution to the test statistic comes from the first eigenvector. This is a consequence of the fact already commented on that in the unrestricted model the interest rates enter the first vector with equal signs.

Given the results of the tests so far it might be of interest to test whether the PPP relation and/or the nominal interest rate differential are stationary processes by themselves, i.e. to test whether $[1,-1,-1, 0, 0]'X_t$ and/or $[0, 0, 0, 1,-1]'X_t$ are stationary. This is the next structural test to be discussed below.

4.2. The hypothesis $\beta = (H_5, \psi)$

In this hypothesis H_5 is a known $(p\times r_1)$ matrix and $\psi(p\times r_2)$ is unknown ($r = r_1+r_2$). Thus r_1 relations are assumed known and the remaining $r_2 = r-r_1$ are to be estimated independently of the vectors in H_5. We split α accordingly into $\alpha = (\alpha_1,\alpha_2)$ so that (2.5) now becomes

(4.9) $$R_{0t} = \alpha_1 H_5' R_{kt} + \alpha_2 \psi' R_{kt} + \text{error}.$$

This model is easily estimated by first concentrating the model with respect to α_1 by regression. Thus if we assume that $H_5'X_t$ is stationary we start the analysis by regressing on the stationary components, just as we regress on the variables $\Delta X_{t-1},\dots,\Delta X_{t-k+1}$.

This gives new residuals $R_{0.ht}$ and $R_{k.ht}$, and the concentrated likelihood function gives rise to the reduced rank regression problem

(4.10) $\quad R_{0.ht} = \alpha_2\psi' R_{k.ht} + \text{error}.$

The equation (4.10) has the same form as (2.5) and can hence be solved by the same maximization procedure, i.e.

$$|\psi'(S_{kk.h} - S_{k0.h}S_{00.h}^{-1}S_{0k.h})\psi| / |\psi' S_{kk.h}\psi|$$

has to be maximzed over all $p\times r_1$ matrices ψ. Here

$$S_{ij.h} = S_{ij} - S_{ik}H_5(H_5'S_{kk}H_5)^{-1}H_5'S_{kj}, \quad i,j = 0,k.$$

Note that $S_{kk.h}$ is of rank $p-r_1$ and hence singular, but that $S_{k0.h}S_{00.h}^{-1}S_{0k.h}$ has the same singularity, hence Lemma 1 proved in the Appendix shows how to maximize, by first diagonalizing $S_{kk.h}$ and then reduce $S_{k0.h}S_{00.h}^{-1}S_{0k.h}$ by the eigenvectors of $S_{kk.h}$ to a $(p-r_1)\times(p-r_1)$ matrix whose eigenvectors determine the estimate of ψ. Thus we first solve the eigenvalue problem

$$|\tau I - S_{kk.h}| = 0$$

and pick out the eigenvectors corresponding to the $p-r_1$ positive eigenvalues and define C $= (e_1,\dots,e_{p-r_1})\text{diag}(\tau_1^{-1/2},\dots,\tau_{p-r_1}^{-1/2})$. Next we solve the eigenvalue problem

(4.11) $\quad |\lambda I - C'S_{k0.h}S_{00.h}^{-1}S_{0k.h}C| = 0.$

This equation has solutions $\hat\lambda_1 > \dots > \hat\lambda_{p-r_1} > 0$ and eigenvectors $\hat V = (\hat v_1,\dots,\hat v_{p-r_1})$. Thus $\hat\psi = (\hat v_1,\dots,\hat v_{r_2})$ such that

(4.12) $\quad \hat\beta = (H_5, \hat v_1,\dots,\hat v_{r_2}),$

and

(4.13) $\quad L_{\max}^{-2/T} = |S_{00.h}| \prod_{i=1}^{r_2}(1-\hat\lambda_i).$

It gives a convenient formulation to solve the eigenvalue problem

(4.14) $\quad |\rho H_5'S_{kk}H_5 - H_5'S_{k0}S_{00}^{-1}S_{0k}H_5| = 0$

for the eigenvalues $\hat\rho_1 > \dots > \hat\rho_{r_1}$, since we can then write

(4.15) $\quad |S_{00.h}| = |S_{00}|\,|H_5'(S_{kk} - S_{k0}S_{00}^{-1}S_{0k})H_5| / |H_5'S_{kk}H_5| = |S_{00}| \prod_{i=1}^{r_1}(1-\hat\rho_i).$

In the present context we have that $\hat{\alpha}_2' S_{00.h}^{-1} \hat{\alpha}_2 = \text{diag}(\hat{\lambda}_1,\ldots,\hat{\lambda}_{r_1})$, such that $\hat{\lambda}_i$ measures the size of the coefficients to the cointegrating relations with respect to the matrix $S_{00.h}$. An intuitive interpretation of $\hat{\rho}_i$ similar to that of $\hat{\lambda}_i$ is however not possible.

Combining the above results we then get

THEOREM 2. The hypothesis $\mathcal{H}_5$: $\beta = (H_5, \psi)$ can be tested by the likelihood ratio test

$$-2\ln Q(\mathcal{H}_5 \mid \mathcal{H}_1) = T \left\{ \sum_{i=1}^{r_1} \ln(1-\hat{\rho}_i) + \sum_{i=1}^{r_2} \ln(1-\tilde{\lambda}_i) - \sum_{i=1}^{r} \ln(1-\hat{\lambda}_i) \right\} \tag{4.16}$$

which is asymptotically distributed as χ^2 with $f = (p-r)r_1$ degrees of freedom. The estimates of β is found from (4.12) and α is given by (2.9).

Proof. We shall calculate the degrees of freedom. We again normalize β as before with τ $(p-r)\times r$. Now fixing the first r_1 columns of τ amounts to fixing $r_1(p-r)$ parameters, hence the degrees of freedom $r(p-r) - r_2(p-r) = r_1(p-r)$.

The first hypotesis, of the type $\mathcal{H}_5$, we consider in this context is whether the PPP relation is stationary on its own. Let $H_{5.1} = [1,-1,-1, 0, 0]'$ in (4.2). The solution to (4.11) gives us $r_2 = 1$ eigenvalue $\tilde{\lambda}_1$ corresponding to the solution of (4.11) and, since $r_1 = 1$, an estimate $\hat{\rho}_1$ defined as the solution of (4.14).

	The eigenvalues					$-T\ln(1-\lambda_i)$			
$\mathcal{H}_1$: $\hat{\lambda}_i$	.407	.285	.254	.102	.083	31.3	20.2	17.6	6.5
$\mathcal{H}_{5.1}$: $\tilde{\lambda}_i$	.396	.281	.254	.101		30.2	19.8	17.6	6.4
$\hat{\rho}_1$:	.106					6.7			

The likelihood ratio test (4.16) becomes:

$$-2\ln Q(\mathcal{H}_{5.1}|\mathcal{H}_1) = 60\ln\{\frac{(1-.396)(1-.106)}{(1-.407)(1-.285)}\} = 14.53$$

The test statistic is asymptotically distributed as $\chi^2(3)$ and the hypothesis that PPP on its own is stationary is rejected.

The second hypothesis of interest $\mathcal{H}_{5.2}$ is whether the interest rate differential is stationary. We take $H_{5.2} = [0, 0, 0, 1,-1]'$ in (4.11) and get:

	λ_1	λ_2	λ_3	λ_4	λ_5	$-T\ln(1-\lambda_i)$				
$\mathcal{H}_1 : \hat{\lambda}_i$	.407	.285	.254	.102	.083	31.3	20.2	17.6	6.5	5.2
$\mathcal{H}_{5.2}:\hat{\lambda}_i$	.406	.260	.105	.101		31.3	18.1	6.7	6.4	
$:\hat{\rho}_1$	.263									

The likelihood ratio test statistic becomes:

$$-2\ln Q(\mathcal{H}_{5.2}|\mathcal{H}_1) = 60\ln\{\frac{(1-.406)(1-.263)}{(1-.407)(1-.285)}\} = 1.93$$

The test statistic is asymptotically distributed as $\chi^2(3)$ and thus not significant. We conclude that the interest rate differential by itself is a stationary process.

Based on the test results of this section one might ask the question whether there exists another linear combination between p_1, p_2 and e_{12} that is stationary. This type of structural hypothesis will be considered in the next section.

4.3. The hypothesis $\beta = (H_6\varphi,\psi)$.

As before we start from (2.5) which in this case has the form

(4.17) $$R_{0t} = \alpha_1\varphi' H_3' R_{kt} + \alpha_2\psi' R_{kt} + \text{error}.$$

where $\alpha_i(p\times r_i)$, $\varphi(s\times r_1)$, and $\psi(p\times r_2)$ have to be estimated and $H_6(p\times s)$ is known. Here $r_1 \le s \le p$.

This problem does not as easily reduce to an eigenvalue problem, but instead we shall apply a simple switching algorithm to maximize the likelihood function.

The algorithm is described briefly as follows:

1) For fixed φ concentrate with respect to α_1 by regression and then solve the reduced rank problem for α_2 and ψ.

2) Now fix ψ, concentrate the likelihood function with respect to α_2 and solve the reduced rank problem for α_1 and φ.

3) Repeat step 1) and 2) until convergence.

Since the likelihood function is maximized at each step and since β can be restricted to a compact set by the normalization $\beta' S_{kk}\beta = I$, the algorithm does converge, but it could be to a local maximum. We do not have a proof that the algorithm converges to the global maximum, but in the cases we have used it there has been no problems and convergence was attained after a few steps.

The algorithm is based on a Lemma given in the Appendix. To get started choose $\psi = 0$ and φ to solve the eigenvalue problem

$$|\lambda H_6' S_{kk} H_6 - H_6' S_{k0} S_{00}^{-1} S_{0k} H_6| = 0$$

for $\hat{\lambda}_1 \geq \dots \hat{\lambda}_s \geq 0$ and $(\hat{v}_1,\dots,\hat{v}_s)$ such that $\hat{\beta}_1 = H_6\hat{\varphi} = H_6(\hat{v}_1\dots,\hat{v}_{r_1})$.

The first step of the algorithm consists of fixing this value of $\hat{\beta}_1$, concentrate with respect to α_1, i.e. condition on $\hat{\beta}_1' R_{kt}$, and find the ψ that maximizes

(4.18) $$|\psi'(S_{kk.\hat{\beta}_1} - S_{k0.\hat{\beta}_1} S_{00.\hat{\beta}_1}^{-1} S_{0k.\hat{\beta}_1})\psi| / |\psi' S_{kk.\hat{\beta}_1}\psi|.$$

The matrix $S_{kk.\hat{\beta}_1}$ is singular, since $S_{kk.\hat{\beta}_1}\hat{\beta}_1 = 0$, but $S_{k0.\hat{\beta}_1} S_{00.\hat{\beta}_1}^{-1} S_{0k.\hat{\beta}_1}$ is singular with the same null space. Hence the Lemma shows how one can find the eigenvalues $\hat{\lambda}_1,\dots,\hat{\lambda}_{p-r_1}$ and eigenvectors $\hat{u}_1,\dots,\hat{u}_{p-r_1}$ such that $\hat{\beta}_2 = (\hat{u}_1,\dots,\hat{u}_{r_2})$.

The second step of the algorithm is to fix $\hat{\beta}_2$, concentrate with respect to α_2, and determine a new estimate of β_1 by maximizing

(4.19) $$|\varphi' H_6'(S_{kk.\hat{\beta}_2} - S_{k0.\hat{\beta}_2} S_{00.\hat{\beta}_2}^{-1} S_{0k.\hat{\beta}_2})H_6\varphi| / |\varphi' H_6' S_{kk.\hat{\beta}_2} H_6\varphi|.$$

This problem can again be solve by the procedure in the Lemma, even though the matrix $H_6{}'S_{kk.\beta_2}H_6$ is nonsingular. This gives optimal eigenvalues $\hat{\omega}_1,...,\hat{\omega}_s$, and eigenvectors $\hat{v}_1,...,\hat{v}_s$, such that $\hat{\beta}_1 = H_6(\hat{v}_1,...,\hat{v}_{r_1})$.

The maximized likelihood function has two expressions corresponding to the two steps of the algorithm:

$$(4.20) \qquad L_{\max}^{-2/T} = |S_{00.\hat{\beta}_1}| \prod_{i=1}^{r_2}(1-\hat{\lambda}_i) = |S_{00.\hat{\beta}_2}| \prod_{i=1}^{r_1}(1-\hat{\omega}_i),$$

where $\hat{\beta}_1 = H_6\hat{\varphi}$ and $\hat{\beta}_2 = \hat{\psi}$. Again the maximized likelihood can be given the different expression by solving the eigenvalue problem

$$|\rho\hat{\beta}_1'S_{kk}\hat{\beta}_1 - \hat{\beta}_1'S_{k0}S_{00}^{-1}S_{0k}\hat{\beta}_1| = 0$$

for the eigenvalues $\hat{\rho}_1 > ... > \hat{\rho}_{r_1}$, since we can then write

$$|S_{00.\hat{\beta}_1}| = |S_{00}| \prod_{i=1}^{r_1}(1-\hat{\rho}_i).$$

The results above can be summarized in

THEOREM 3. The hypothesis $\mathcal{H}_6$: $\beta = (H_6\varphi, \psi)$ can be tested by a likelihood ratio test of the form

$$(4.21) \qquad -2\ln Q(\mathcal{H}_6 | \mathcal{H}_1) = T\,\{\sum_{i=1}^{r_1}\ln(1-\hat{\rho}_i) + \sum_{i=1}^{r_2}\ln(1-\hat{\lambda}_i) - \sum_{i=1}^{r}\ln(1-\hat{\lambda}_i)\}$$

which is asymptotically distributed as χ^2 with $f = (p-s-r_2)r_1$ degrees of freedom. The estimator is calculated by a swithcing algorithm by solving succesive maximization problems (4.18) and (4.19).

Proof. The proof follows from the above calculations, we shall here derive the degrees of freedom for the test. Let us write the parameters as $\alpha_1\varphi'$ and $\alpha_2\psi'$. The parameter φ can be normalized such that the first set of parameters contain $pr_1 + r_1(s-r_1)$ free parameters. In the second set of parameters we normalize ψ to contain $r_2(p-r_2)$ parameters. These are not free since at each step of the algorithm, the parameter ψ is chosen orthogonal to $\hat{\beta}_1$ which restricts the variation to a $(p-r_1)$–dimensional space.

Thus we have $pr_2 + (p-r_1-r_2)r_2$ parameters in the second set of parameters. This gives a total of $pr + (p-r)r_2 + (s-r_1)r_1$. Subtracting this from $pr + (p-r)r$ gives the degrees of freedom.

A hypothesis of this type can be formulated by asking if there is a vector of the form (a,b,c,0,0) in the cointegration space for some a,b, and c. In matrix formulation, we can define

$$H_6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and formulate the hypothesis as

(4.22) $\qquad \mathscr{H}_6\colon\ \beta = (H_3\varphi, \psi)$

In our empirical example $r = 2$, $r_1 = 1$ and $r_2 = 1$. The iterated solution to the eigenvalue problems (4.18) and (4.19) are given below and compared with the unrestricted results:

	The estimated eigenvalues					$-T\ln(1-\lambda_i)$				
$\mathscr{H}_1$: $\hat{\lambda}_i$	.407	.285	.254	.102	.083	31.3	20.2	17.6	6.5	5.2
$\mathscr{H}_6$: $\hat{\lambda}_i$	.407	.284	.102	.083		31.3	20.0	6.5	5.2	
: $\hat{\rho}_1$	.256					17.7				

The likelihood ratio test (4.21) becomes:

$$-2\ln Q(\mathscr{H}_6 \mid \mathscr{H}_1) = 60\ln\{\frac{(1-.256)(1-.407)}{(1-.407)(1-.285)}\} = 2.4$$

The test statistic is asymptotically distributed as $\chi^2(1)$ and not strongly significant suggesting that there does not exists a linear combination between p_1, p_2 and e_{12} that is stationary.

5. Concluding remarks.

In this paper some likelihood ratio tests are developed to test structural hypothesis on the cointegration space in a multivariate cointegration model. It is demonstrated how the multivariate analysis in combination with the hypothesis of cointegration allows a precise formulation of a number of interesting economic hypothesis in such a way that they can be tested. The importance of these tests is illustrated by an application to the purchasing power parity and the uncovered interest rate parity relation for UK versus a trade weighted foreign country. In a five–dimensional system of equations (two prices, exchage rate and two interest rates) we ask the question whether the PPP–relation is stationary by itself, i.e. without the interest rates and correspondingly whether the nominal interest rate differential is a stationary process. The answer is negative for the PPP–relation but positive for the interest rate differential. We also ask the question whether The PPP relation with some combination of the two interest rates is stationary and find that this hypothesis can indeed be accepted. The result seems to indicate that evidence on the PPP relation can be found by accounting for the interaction between the goods and the asset markets. Finally, we ask the question whether there exists any linear combination between the prices and the exchange rate that is stationary and get a negative answer.

Figure 3.1. Graphs of the differences of the data $\Delta Z_t = (\Delta p_1, \Delta p_2, \Delta e_{12}, \Delta i_1, \Delta i_2)$ and the corresponding residuals from eq.(3.1).

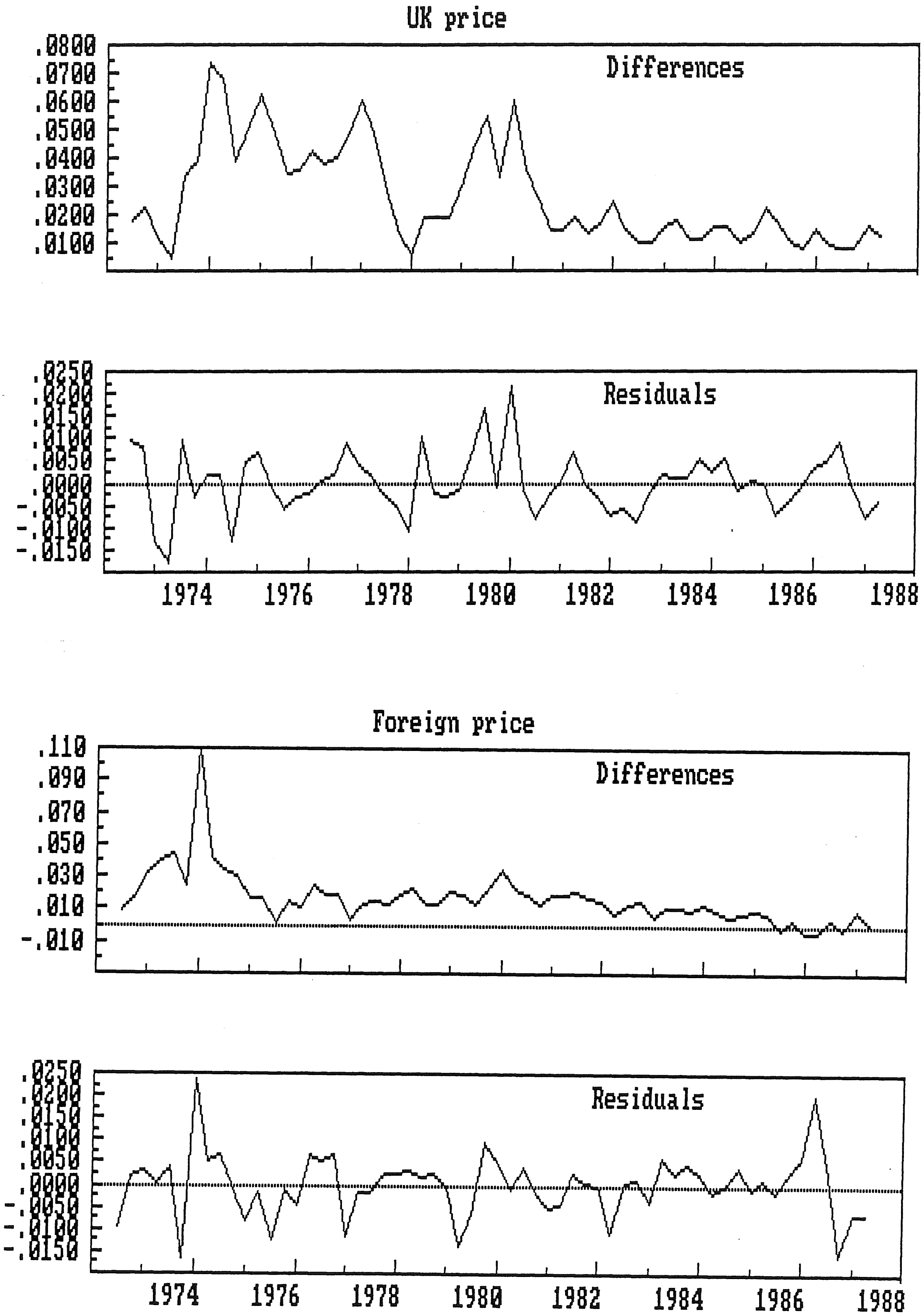
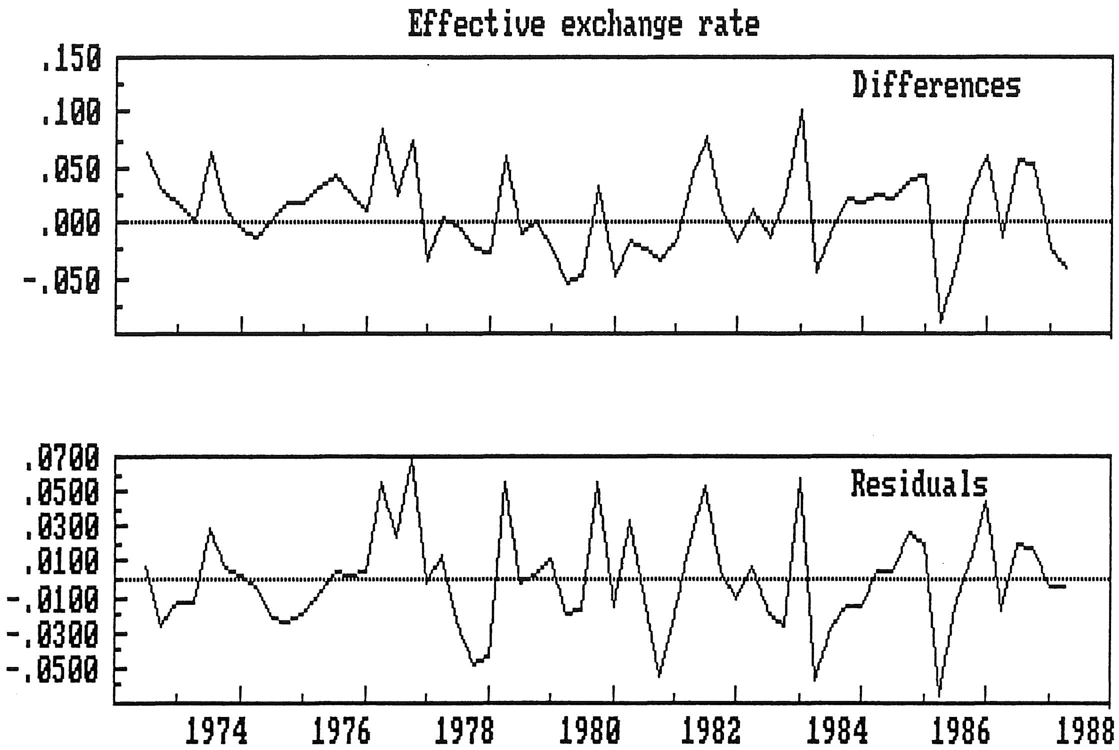


Figure 3.1. continues.



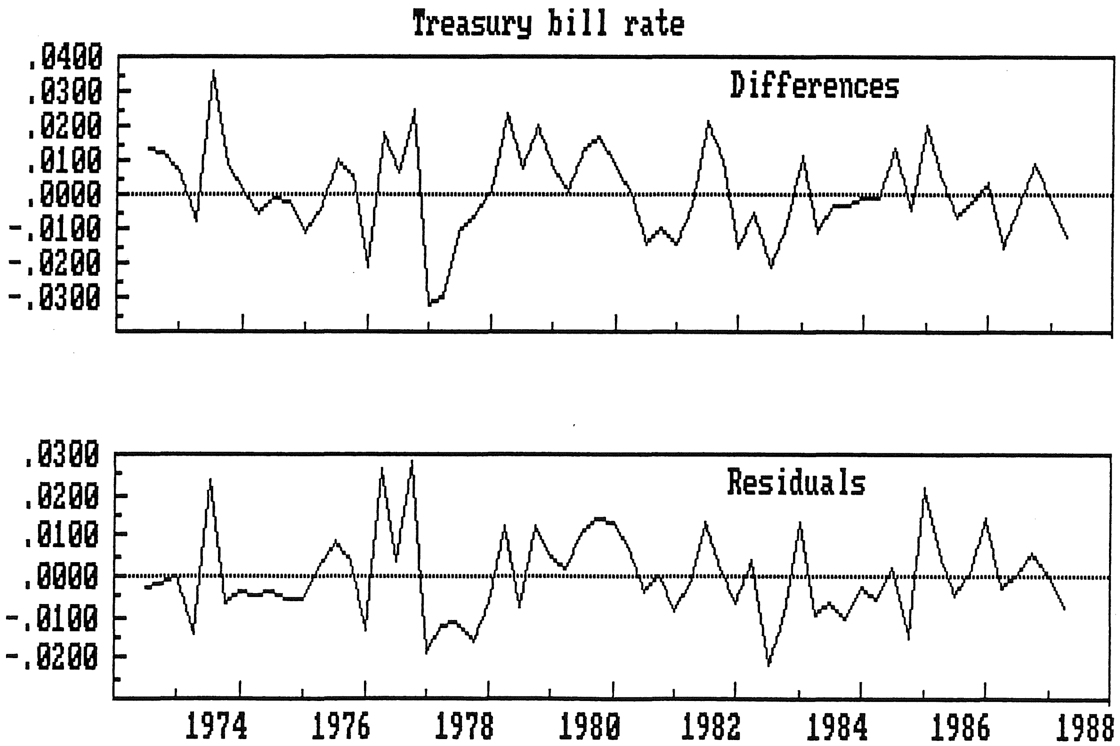


Figure 3.1. continues.

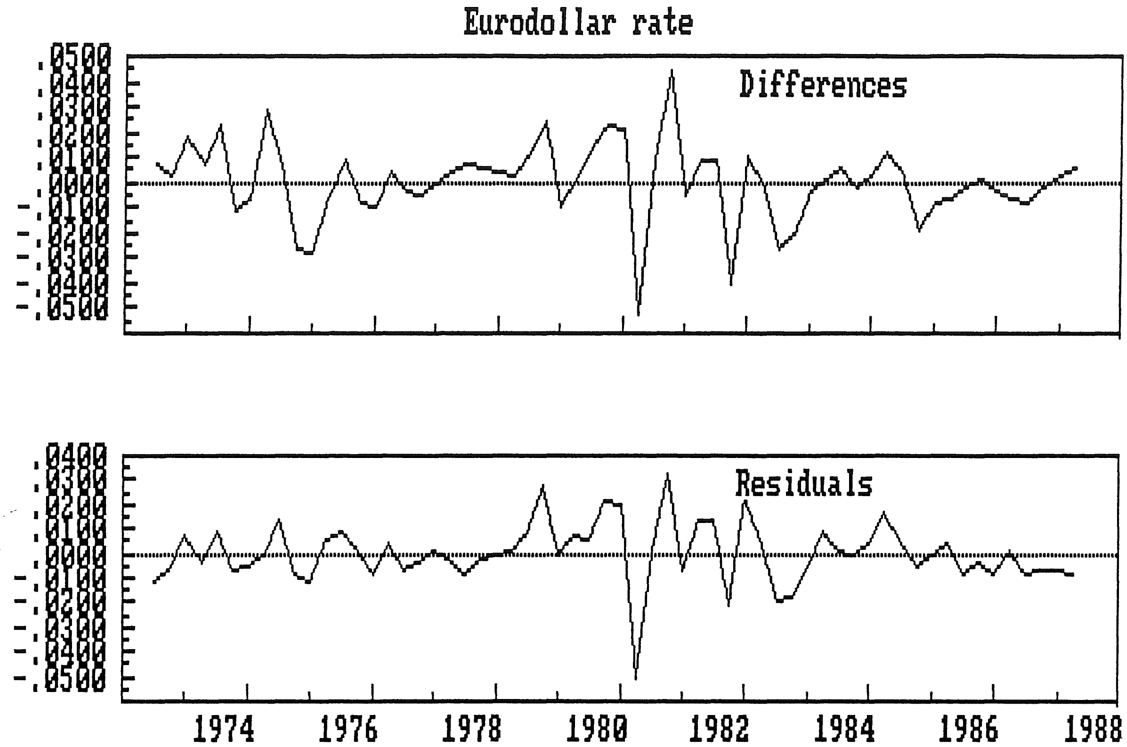


Figure 3.2. Graphs of the cointegration relations $\beta' Z_t$ (cf. eq. (3.1)) and $\beta' R_{kt}$ (cf. eq. (2.5)).

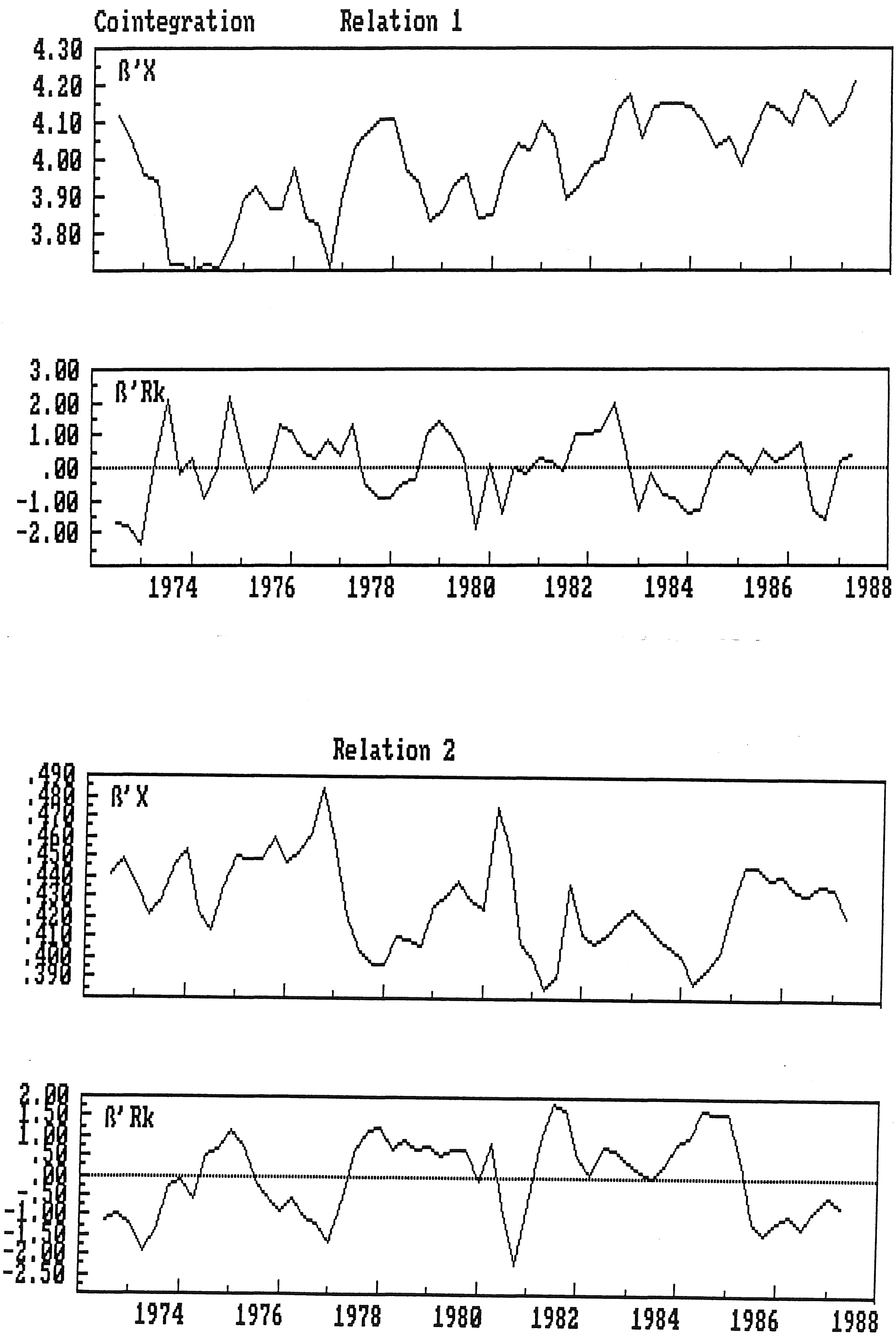
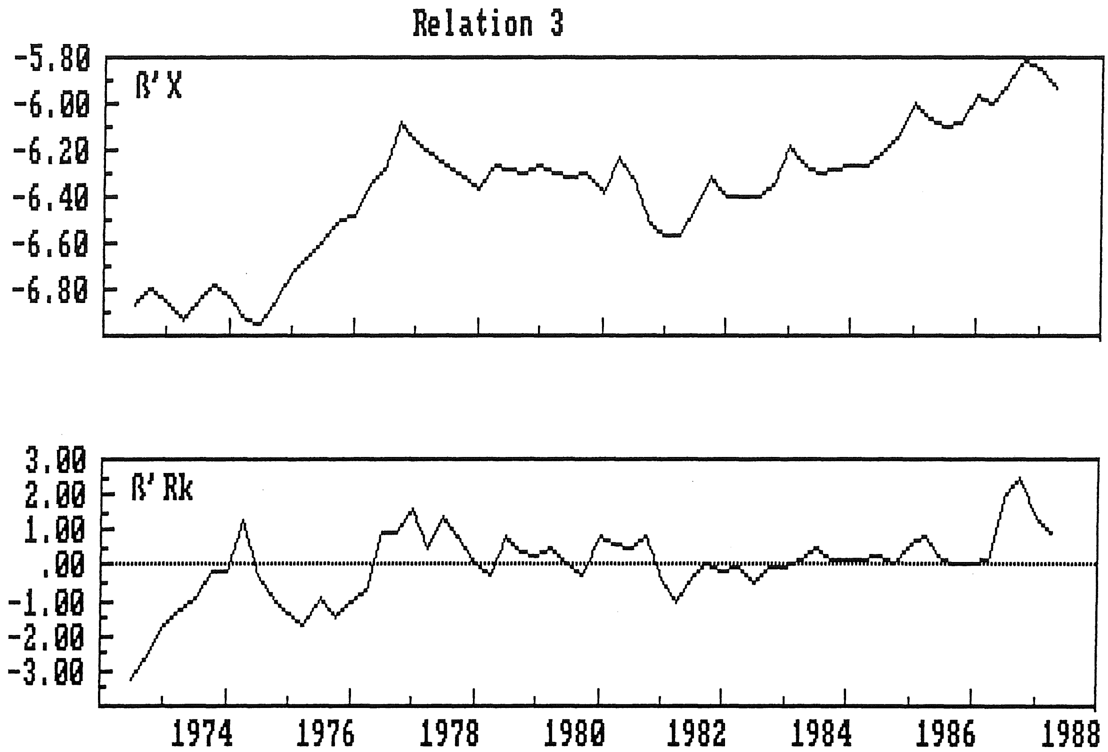


Figure 3.2. continues.



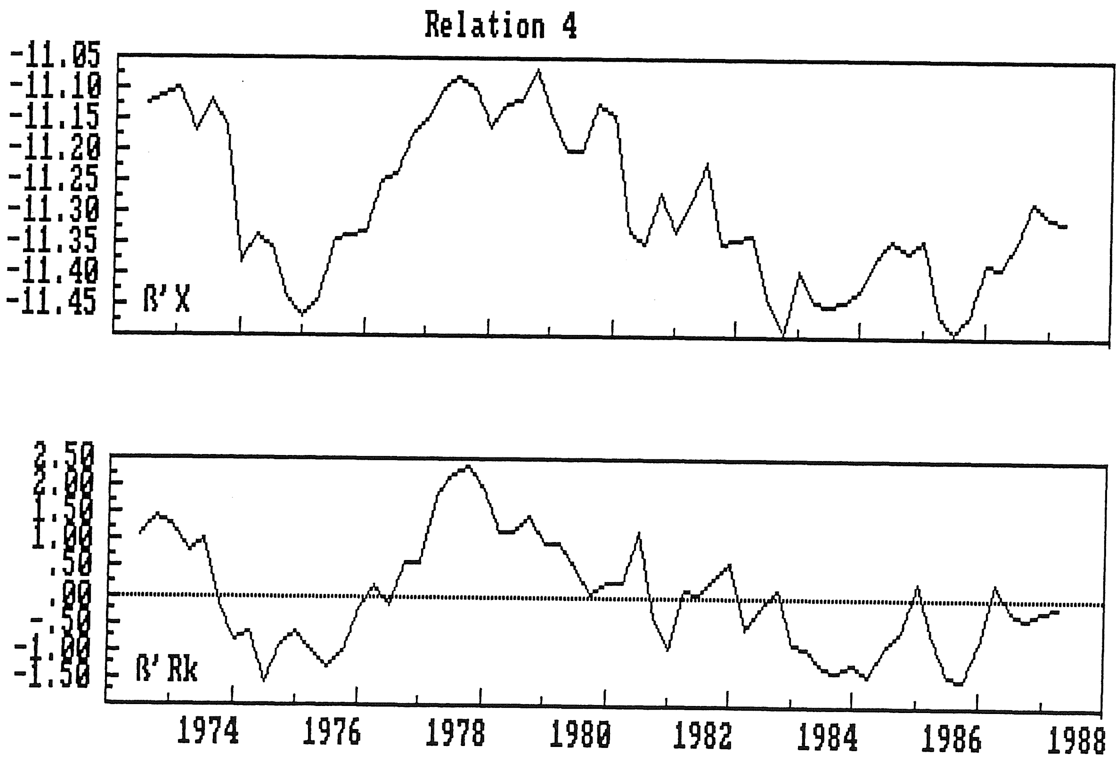
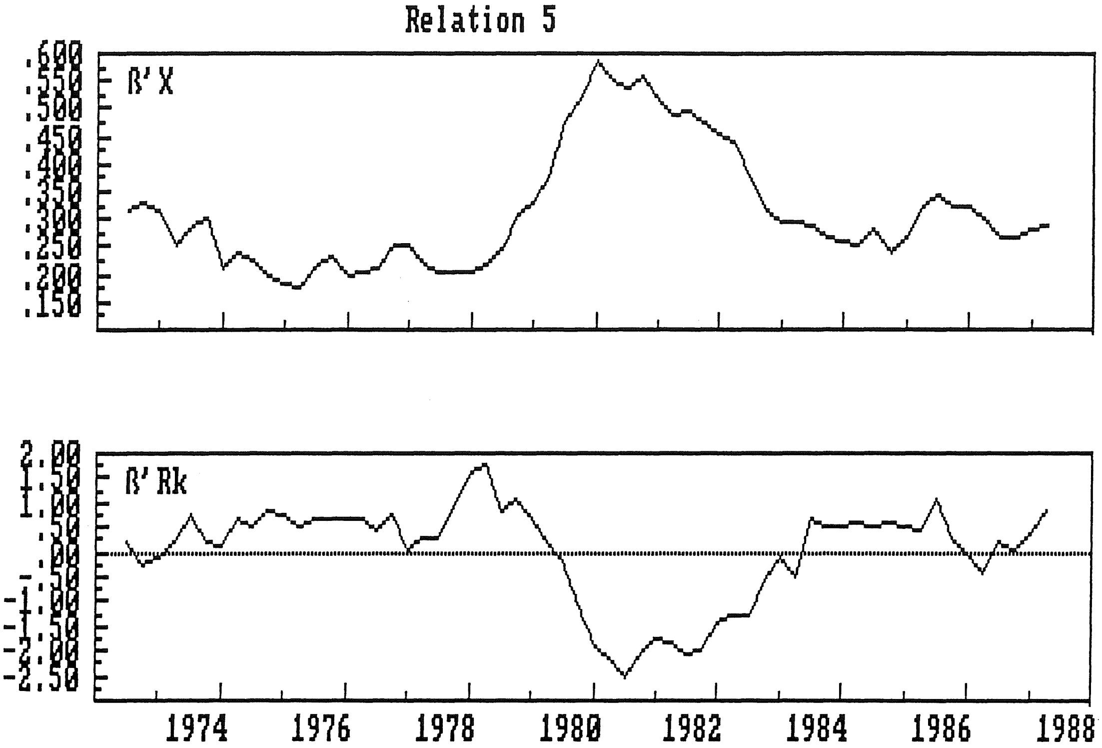


Figure 3.2. continues.



APPENDIX

The following result shows that the classical result about maximizing the ratio of determinants of quadratic forms also holds if the matrices are singular in a suitable sense.

LEMMA: Let A and B be p×p positive semidefinite matrices, such that A has rank m and such that $Ax = 0$ implies that $Bx = 0$ for $x \in R^p$.

The expression

$$f(\beta) = |\beta'(A-B)\beta| \,/\, |\beta' A\beta|, \quad \beta' A > 0,$$

is maximized among all p×r matrices β ($r \leq m$) by first solving the eigenvalue problem $|\rho I - A| = 0$, for $\rho_1 \geq \dots \geq \rho_m > \rho_{m+1} = \dots = \rho_p = 0$ and eigenvectors $(e_1,\dots,e_p)$, and define the p×m matrix

$$C = (e_1,\dots,e_m)\mathrm{diag}(\rho_1^{-1/2},\dots,\rho_m^{-1/2}),$$

such that $C'AC = I_{m\times m}$. Next solve the reduced eigenvalue problem $|\lambda I - C'BC| = 0$ for $\lambda_1 \geq \dots \geq \lambda_{p-m}$ and eigenvectors $u_1,\dots,u_{p-m}$. The solution of the maximization problem is then given by $\hat{\beta} = C(u_1,\dots,u_r)$ and the maximium value of f is $f(\hat{\beta}) = \prod_{i=1}^{r} (1-\lambda_i)$. The solution $\hat{\beta}$ is ortogonal to the null space of A.

Proof. From $A = E\mathrm{diag}(\rho_1,\dots,\rho_p)E'$ it follows, since $\rho_{m+1} = \dots = \rho_p = 0$, that

$$\beta' A\beta = \beta'(e_1,\dots,e_m)\mathrm{diag}(\rho_1,\dots,\rho_m)(e_1,\dots,e_m)'\beta.$$

Thus $\beta' A\beta$ only depends on β through its projection onto $(e_1,\dots,e_m)$. This also holds for $A - B$, since $Ae_i = 0$ implies that $Be_i = 0$, and hence it also holds for $f(\beta)$. Thus we reduce the problem to the m–dimensional space spanned by $(e_1,\dots,e_m)$ and solve it there using the classical tools from multivariate analysis, see Rao(1973).

The ortogonality of $\hat{\beta}$ to the null space of A follows from $\hat{\beta} \in sp\,(e_1,\dots,e_m)$ and that the null space is spanned by $(e_{m+1},\dots,e_p)$.

What is happening here is roughly the following: For non singular matrices the above maximization problem is solved by solving the eigenvalue problem

$$|\lambda A - B| = 0.$$

When A and also B are singular in the way described, the determinant is identically zero for all values of λ, and hence not very helpful for the solution. Hence one considers the equation reduced to the m–dimensional orthogonal complement of the null space of A, because A is nonsingular on this space. This is what is accompliced by multiplying by C' and C.

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