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Abstract

A technique for obtaining asymptotic expansions for complex hypergeometric functions is demonstrated. The method is used to find an asymptotic expansion for the function ${}_2\widetilde{F}_1$.

1. Introduction.

Several papers give asymptotic expansions for real hypergeometric functions, for example, Muirhead (1978) and Srivastava (1980). However, only a few papers on asymptotic expansions for complex hypergeometric functions have appeared, which seems strange since the two kinds of functions formally look very much the same. One would expect that the methods used for the real functions also would work for the complex functions. This is not so, however, because certain functions appearing in the integral representations for the complex hypergeometric functions obtain their maximum on a whole subspace of $\mathbb{R}^m$, and not just at isolated points. In this paper it is shown how it is possible to solve this problem by reducing the number of variables using the lemma found in Section 3. Li et al. (1970) had to solve the same problem but their way of doing this was not explained in detail. We have chosen to consider the complex hypergeometric function ${}_2\widetilde{F}_1$ for an example; this function appears in the density for the distribution of complex canonical correlation coefficients (James (1964), (112)). Furthermore, we have chosen to apply the method used for the real function ${}_2F_1$ by Glynn (1980).

2. Notations.

In this paper we consider matrices with complex elements. $\mathcal{H}_+(p,\mathbb{C})$ denotes the set of positive definite hermitian $p \times p$ matrices, $\mathcal{U}(p)$ the group of unitary $p \times p$ matrices and $V(p,q)$ the set of $q \times p$ matrices Q for which $\overline{Q}'Q = I$, where $\overline{Q}'$ is the conjugate transpose of Q and $p \leq q$. The normed invariant measure on $V(p,q)$ is called dQ. We let

$$v(p,q) = \pi^{p(q-1)} \cdot \widetilde{\Gamma}_p(q)^{-1}, \quad \text{where}$$
$$\widetilde{\Gamma}_p(q) = \pi^{\frac{1}{2}p(p-1)} \prod_{i=1}^{p} \Gamma(q-i+1)$$

For $A = A_1 + iA_2 \in \mathcal{H}_+(p,\mathbb{C})$ the notation dA will be used for $dA_1 dA_2$, where

$$dA_1 = \prod_{i \geq j} da_{ij,1} \quad \text{and} \quad dA_2 = \prod_{i>j} da_{ij,2}\,.$$

If $A \in \mathcal{H}_+(p, \mathbb{C})$ we can find a non-singular hermitian matrix B so that $A = B^2$, and in this case B is called $A^{\frac{1}{2}}$. The set of diagonal matrices with diagonal elements $u_1, -, u_p$ is denoted $D(U)$.

3. Integral representation of ${}_2\widetilde{F}_1$.

Let P^2 and R^2 be diagonal matrices. $P^2 = \text{diag}(\rho_i^2)$ and $R^2 = \text{diag}(\gamma_i^2)$, where $1 > \rho_1 > \ldots > \rho_p > 0$ and $1 > \gamma_1 > \ldots > \gamma_p > 0$. The complex hypergeometric function ${}_2\widetilde{F}_1(N, N, q; P^2, R^2)$ (which in the following will be called ${}_2\widetilde{F}_1$) can be expressed as

$$(1)\quad \widetilde{\Gamma}_p(N)^{-2} \int_{\Omega(1)} |AB|^{N-p} \exp\left(tr(-A - B + 2\,Re([A^{\frac{1}{2}}B^{\frac{1}{2}}PQ'R:0]F))\right) dAdBdQdF$$

where $\Omega(1) = \mathcal{H}_+(p, \mathbb{C}) \times \mathcal{H}_+(p, \mathbb{C}) \times \mathcal{U}(p) \times V(p, q)$, and $q \geq p$ (see James (1964), (86) and 91–92)).

We shall derive the asymptotic behavior of ${}_2\widetilde{F}_1$ for large N using the following: if the function $f(x) = f(x_1, \ldots, x_m)$ has an absolute maximum at an interior point ξ of a domain S in a real m-dimensional space, then under suitable conditions, as $N \to \infty$

$$(2)\qquad \int_S f(x)^N h(x)dx \sim (2\pi/N)^{\frac{1}{2}m} f(\xi)^N h(\xi)\Delta(\xi)^{-\frac{1}{2}}$$

where $a \sim b$ means that $\lim_{N\to\infty} a/b = 1$ and

$$\Delta(x) = \det(-\delta^2 \log f/\delta x_i \delta x_j)$$

This result is due to Hsu (1948).

To be able to apply (2) we first make the substitutions $A \to NA$ and $B \to NB$ in (1); then ${}_2\widetilde{F}_1$ can be expressed as

$$(3)\qquad C_1 \int_{\Omega(1)} g(A, B, P, Q, R, F)^N \cdot |AB|^{-p} dAdBdQdF$$

where $C_1 = \widetilde{\Gamma}_p(N)^{-2} \cdot N^{2pN}$, and

$$g(A,B,P,Q,R,F) = |AB| \exp\left(tr(-A-B+2\,Re([A^{\frac{1}{2}}B^{\frac{1}{2}}PQ'R:0]F))\right)$$

We can write a matrix $A \in \mathcal{H}_+(p,\mathbb{C})$ on the form $A = GU^2\overline{G}'$, where $G \in \mathcal{U}(p)$ and $U \in D(U)$. Using (93) of James (1964) we can express (3) as

$$(4) \qquad C_2 \int_{\Omega(2)} f(U,G,V,H,P,Q,R,F)^N h(U,V) dU dG dV dH dQ dF$$

where $\Omega(2) = D(U) \times \mathcal{U}(p) \times D(V) \times \mathcal{U}(p) \times \mathcal{U}(p) \times V(p,q)$,

$$f(U,G,V,H,P,Q,R,F) = |UV|^2 \exp\left(tr(-U^2 - V^2 + 2\,Re([UGVHPQ'R:0]F))\right)$$

$$h(U,V) = |UV|^{-2p+1} \prod_{i<j} (u_i^2 - u_j^2)^2 (v_i^2 - v_j^2)^2$$

and $$C_2 = C_1 \cdot v(p,p) \cdot 2^{2p}$$

The asymptotic behavior for the real hypergeometric function ${}_2F_1$ was obtained by applying (2) to an integral of the form (4) after making a proper parametrization of the compact groups. That does not work in the complex case since f does not have an isolated maximum. In order to solve this problem we will now show how it is possible to reduce the number of parameters.

Let $\mathcal{U}_1(p)$ be the manifold consisting of matrices in $\mathcal{U}(p)$ with real positive diagonal elements; let $\mathcal{U}_2(p)$ be the subgroup of $\mathcal{U}(p)$ consisting of diagonal matrices.

$\mathcal{U}_1(p)$ can be identified with the coset space $\mathcal{U}(p)/\mathcal{U}(1) \times \ldots \times \mathcal{U}(1)$. As a group of transformations on this coset space $\mathcal{U}(p)$ is transitive and we let dG_1 denote the normed invariant measure on the coset space.

For $G \in \mathcal{U}(p)$ we have a unique decomposition $G = G_1 G_2$ with $G_1 \in \mathcal{U}_1(p)$ and $G_2 \in \mathcal{U}_2(p)$. Let dG_i be the normed invariant measure on $\mathcal{U}_i(p) (i = 1,2)$.

Lemma. *Let R be a diagonal matrix and let φ be a function of the elements of a $p \times p$ matrix, then*

$$(5) \qquad \int_{\mathcal{U}(p) \times V(p,q)} \varphi([QR:0]F) dQ dF = \int_{\mathcal{U}_1(p) \times V(p,q)} \varphi([Q_1 R:0]F) dQ_1 dF$$

Proof. Write the left side of (5) as

$$\int_{\mathcal{U}_1(p)\times\mathcal{U}_2(p)\times V(p,q)} \varphi([Q_1Q_2R:0]F)dQ_1dQ_2dF \quad (6)$$

Now for fixed Q_1 and Q_2 use the invariance of dF to write (6) as

$$\int_{\mathcal{U}_1(p)\times\mathcal{U}_2(p)\times V(p,q)} \varphi([Q_1Q_2R:0]\widetilde{Q}_2F)dQ_1dQ_2dF$$

where $\widetilde{Q}_2 = \begin{pmatrix} \overline{Q}_2 & 0 \\ 0 & 0 \end{pmatrix}$ is a $q\times q$ matrix.

But $Q_2R\overline{Q}_2 = R$ from which the lemma follows. □

Using the lemma three times we can replace G, H, and Q in (4) by G_1, H_1, and Q_1, all three belonging to $\mathcal{U}_1(p)$, to get an integral representation of ${}_2\widetilde{F}_1$ which will be denoted ${}_2\widetilde{I}_1$, and to which we can apply (2).

4. Asymptotic representation of ${}_2\widetilde{F}_1$.

We are now ready to prove

Theorem 1. *Let $P^2 = diag(\rho_i^2)$ and $R^2 = diag(\gamma_i^2)$, where $1 > \rho_1 > \ldots > \rho_p > 0$ and $1 > \gamma_1 > \ldots > \gamma_p > 0$, then for large N*

$$\begin{aligned} &{}_2\widetilde{F}_1(N,N,q;P^2,R^2) \sim \\ &C(N)\prod_{i=1}^{p}(1-\gamma_i\rho_i)^{-2N+q+p-1}(\gamma_i\rho_i)^{-\frac{1}{2}-q+p}\cdot\prod_{i<j}^{p}\left((\rho_i^2-\rho_j^2)(\gamma_i^2-\gamma_j^2)\right)^{-1} \end{aligned} \quad (7)$$

where

$$C(N) = N^{2pN-\frac{1}{2}p(2p+2q-1)}e^{-2pN}\pi^{\frac{1}{2}p}\widetilde{\Gamma}_p(N)^{-2}\widetilde{\Gamma}_p(p)\widetilde{\Gamma}_p(q) \quad (8)$$

Proof. We follow Glynn (1980) and concentrate upon results which are not found in his paper. For fixed U and V the function f occurring in ${}_2\widetilde{I}_1$ has a unique maximum given by

$G_1 = H_1 = Q_1 = I$ and $F = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$; this follows from Corollary 4.1 in Glynn (1980). To see this, write any complex matrix $A_1 + iA_2$ on the real form

$$\begin{pmatrix} A_1 & -A_2 \\ A_2 & A_1 \end{pmatrix} \tag{9}$$

and take into account that when a diagonal matrix is written on real form all the diagonal elements appear twice. Then maximizing over U and V it follows that the maximum value of f is

$$e^{-2p} \prod_{i=1}^{p} (1 - \gamma_i \rho_i)^{-2} \tag{10}$$

and the maximum is obtained at

$$U = V = \text{diag}\left((1 - \gamma_i \rho_i)^{-\frac{1}{2}} \right).$$

We have to use a parametrization of the compact groups to be able to apply (2), since $\Omega(2)$ is not an open subset.

A matrix $G_1 \in \mathcal{U}_1(p)$ can be expressed as $G_1 = \exp(iS)$, where $\overline{S}' = S$. We are, however, only going to use the part $1 + iS - \frac{1}{2}S^2$ of $\exp(iS)$, so we can assume that the diagonal elements of S are 0. The jacobian of this transformation has the form

$$v(p,p)^{-1}(1 + 0(s_{ij}^2)).$$

(This follows from Li et al. (1970)). Similarly $H_1 = \exp(iT)$ and $Q_1 = \exp(iW)$.

Finally $F \in V(p,q)$ is parametrized by writing

$$[F : -] = \exp(iZ) = \exp\left(i \begin{pmatrix} Z_{11} & -\overline{Z}'_{21} \\ Z_{21} & 0 \end{pmatrix} \right)$$

where $[F : -]$ is a $q \times q$ unitary matrix whose first p columns are F, Z_{11} is $p \times p$ hermitian and Z_{21} is a $(q-p) \times p$ matrix. The jacobian is

$$(2\pi)^{-p} v(p,q)^{-1} \left(1 + 0(z_{ij}^2)\right)$$

Using this we can apply (2). The maximum value of f is obtained at the point ξ defined by

$$U = V = \operatorname{diag}\left((1-\gamma_i\rho_i)^{-\frac{1}{2}}\right), \quad S = T = W = 0 \quad \text{and} \quad Z = 0.$$

The number of variables to be integrated is $m = p(2p+2q-1)$.

We have to find the Hessian of $-\log(f)$. Fortunately we can use the calculations done by Glynn (1980) for the corresponding Hessian related to ${}_2F_1$; to see this write (once more) any complex matrix $A_1 + iA_2$ on real form (9). We find

$$\Delta = 2^m 2^{2p} \prod_{i=1}^{p} \left((1-\gamma_i\rho_i)(\gamma_i\rho_i/(1-\gamma_i\rho_i))^{q-p+\frac{1}{2}}\right)^2 \times \prod_{i<j}^{p} \left((\gamma_i\rho_i - \gamma_j\rho_j)^4 (1-\gamma_i\rho_i)^{-4}(1-\gamma_j\rho_j)^{-4}(\rho_i^2-\rho_j^2)(\gamma_i^2-\gamma_j^2)\right)^2 \tag{11}$$

Inserting in (2) and reducing the expression proves Theorem 1. □

Without proof we state the following theorem. The steps of the proof are exactly the same as for Theorem 1.

Theorem 2. *Let $P^2 = diag(\rho_i^2)$ and $R^2 = diag(\gamma_i^2)$, where $1 > \rho_i > \ldots \rho_k > \rho_{k+1} = \ldots = \rho_p = 0$ and $1 > \gamma_1 > \ldots > \gamma_p > 0$, then for large N*

$$ {}_2\widetilde{F}_1 \cdot (N, N, q; P^2, R^2) \sim d(N) \prod_{i=1}^{k} (1-\gamma_i\rho_i)^{-2N+q+p-1} (\gamma_i\rho_i)^{-\frac{1}{2}+p-q} \cdot \prod_{i<j}^{k} \left((\rho_i^2-\rho_j^2)(\gamma_i^2-\gamma_j^2)\right)^{-1} \times \prod_{i=1}^{k} \prod_{j=k+1}^{p} \left((\gamma_i^2 - \gamma_j^2)\rho_i^2\right)^{-1} \tag{12}$$

where

$$d(N) = N^{2Nk-\frac{1}{2}k(2p+2q-1)} e^{-2kN} \pi^{\frac{1}{2}k} \widetilde{\Gamma}_k(N)^{-2} \widetilde{\Gamma}_k(p) \widetilde{\Gamma}_k(q). \tag{13}$$

5. Numerical comparisons.

For $p = 1$ and $p = 2$ it is possible to calculate the exact values of ${}_2\widetilde{F}_1$ from the series expansion in complex zonal polynomials (James (1964), (88)). Below is the result obtained on a computer for some values of N, q, p, P and R. The asymptotic values of ${}_2\widetilde{F}_1$ obtained from (7)-(8) are also given.

$$p = 1,\ q = 3,\ \gamma = 0.45,\ \rho = 0.35$$

N	25	50	100	200
exact	51.5	$5.11 \cdot 10^4$	$2.57 \cdot 10^4$	$3.54 \cdot 10^{25}$
asymptotic	57.4	$5.36 \cdot 10^4$	$2.63 \cdot 10^4$	$3.58 \cdot 10^{25}$

$$p = 2,\ q = 3,\ \gamma_1 = 0.3,\ \gamma_2 = 0.1,\ \rho_1 = 0.4,\ \rho_2 = 0.2$$

N	25	50	100	200
exact	6.9	565	$5.36 \cdot 10^7$	$1.25 \cdot 10^{19}$
asymptotic	11.9	640	$5.50 \cdot 10^7$	$1.25 \cdot 10^{19}$

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