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## Some Simple Non-Parametric Tests for Misspecification of Regression Models

 Using Sign Changes of Residuals
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#### Abstract

We consider the residuals from the general linear model $Y=X \beta+\epsilon$ with Gaussian errors, and the misspecification tests obtained from the number of positive residuals and the number of sign changes. We then derive the asymptotic distribution of these test statistics and indicate asymptotic expressions for the power functions for local linear alternatives. We find that the statistics are asymptotically independent and Gaussian. The asymptotic mean and variance of the number of sign changes is $n / 4$ and $n / 16$, hence has the same value as it would have if the residuals were independent. The number of positive residuals has mean $\mathrm{n} / 2$ and asymptotic variance $\mathrm{n} / 4-1^{\prime} \mathrm{X}\left(\mathrm{X}^{\prime} \mathrm{X}\right)^{-1} \mathrm{X}^{\prime} 1 / 2 \pi$.

KEY WORDS: Hajek projektions, General linear model, Residuals.


1. INTRODUCTION.

Consider a simple regression model with Gaussian errors

$$
\begin{equation*}
Y_{i}=\alpha+\beta t_{i}+\sigma \epsilon_{i}, \quad i=1, \ldots, n \tag{1.1}
\end{equation*}
$$

where $\epsilon_{1}, \ldots, \epsilon_{n}$ are $\operatorname{NID}(0,1)$ and the parameters are estimated by ordinary least squares.

A simple diagnostic check on the validity of the model can be performed using the statistics

$$
\mathrm{N}_{+}=\sum_{\mathrm{i}=1}^{\mathrm{n}} 1\left\{\mathrm{Y}_{\mathrm{i}}-\hat{\alpha}-\hat{\alpha} \mathrm{K}_{\mathrm{i}}>0\right\}
$$

and

The mean of $N_{+}$under the model (1.1) is clearly $n / 2$, but the variance is more complicated to calculate, since the indicator functions in the sum are dependent.

The purpose of this paper is to find asymptotic expressions for the mean and variance of $\mathrm{N}_{+}$and $\mathrm{N}_{+-}$, to prove that $\mathrm{N}_{+}$and $\mathrm{N}_{+-}$are asymptotically normally distributed and to find asymptotic expressions for the power function for linear local alternatives.

The test statistic $N_{+-}$in a sequence of + and -'s, which is essentially half the number of runs, has always been used as a test
statistic, see Arbuthnott (1710). Note that $N_{+-}$and $N_{-+}$differ by at most 1 , and that the number of runs is $\mathrm{N}_{+-}+\mathrm{N}_{-+}+1$. Geary suggested using $N_{+-}+N_{-+}$as a test against autoregression assuming a binomial variation of $\mathrm{N}_{+-}$, and Draper and Smith (1966) discuss the distribution of $N_{+-}$given $N_{+}$under the assumption that the residuals are independent, referring to Swed and Eisenhart (1943) for the distribution. Both Draper and Smith and Geary warn that one should consider the dependence between the residuals, but they claim that the effect can be ignored.

Gastwirth and Selwyn (1980) find the asymptotic distribution of $N_{+}$ and $\mathrm{N}_{+-}$under the model $\mathrm{Y}_{\mathrm{i}}=\mu+\epsilon_{\mathbf{i}}$ where $\epsilon_{\mathrm{i}}$ are i.i.d. with density. Their proof exploits the exchangeability in the distribution of the residuals. Mikhail and Lester (1981) calculate moments of $\mathrm{N}_{+-}$for this model and show that the exact distribution is well approximated by a $\chi^{2}$ distribution.

We shall work in the general linear model with Gaussian errors and the method used for proving asymptotic normality is a slight variation of the projection method of Hajek, see Lehmann (1975) or Hajek (1969).
2.THE ASYMPTOTIC DISTRIBUTION OF $\mathrm{N}_{+}$AND $\mathrm{N}_{+-}$AND THE LOCAL POWER OF THE TESTS BASED ON THESE.

Consider the model

$$
\begin{equation*}
\mathrm{Y}=\mathrm{X} \beta+\sigma \epsilon \tag{2.1}
\end{equation*}
$$

where $\mathrm{X}(\mathrm{n} \times \mathrm{k})$ is a known design matrix, $\beta(\mathrm{k} \times 1)$ is the unknown parameter
and $\epsilon(\mathrm{n} \times 1)$ is a vector of i.i.d. $N(0,1)$ random variables.
We define the estimate

$$
\hat{\beta}=\left(\mathrm{X}^{\prime} \mathrm{X}\right)^{-1} \mathrm{X}^{\prime} \mathrm{Y}
$$

the "hat matrix" or projection

$$
H=X\left(X^{\prime} X\right)^{-1} X^{\prime}
$$

and finally the residuals

$$
E=(I-H) Y
$$

Under the model (2.1) the residuals E are distributed as $\mathrm{N}_{\mathrm{n}}\left(0, \sigma^{2}(\mathrm{I}-\mathrm{H})\right)$. Now define

$$
\begin{equation*}
N_{+}=\sum_{i=1}^{n} 1\left\{E_{i}>0\right\} \tag{2.2}
\end{equation*}
$$

and

$$
\begin{equation*}
N_{+-}=\sum_{i=1}^{n-1} 1\left\{E_{i}>0, E_{i+1}<0\right\} \tag{2.3}
\end{equation*}
$$

We can then prove

THEOREM 1. Assume $\mathrm{H}_{\mathrm{ij}} \in \mathrm{O}\left(\mathrm{n}^{-1}\right)$, then under the model (2.1), the asymptotic distribution of ( $N_{+}, N_{+-}$) calculated from the OLS residuals are the same as that of $\left(N_{+}^{* *}, N_{+-}^{* *}\right)$, where

$$
\begin{equation*}
N_{+}^{* *}=\sum_{i=1}^{n} 1\left\{\epsilon_{i}>0\right\}-1^{\prime} H \epsilon /(2 \pi)^{1 / 2} \tag{2.4}
\end{equation*}
$$

$$
\begin{equation*}
N_{+-}^{* *}=\sum_{i=1}^{n-1} 1\left\{\epsilon_{i}>0, \epsilon_{i+1}<0\right\} \tag{2.5}
\end{equation*}
$$

These are asymptotically Gaussian with means and variances given up to terms of order 1 by

$$
\begin{equation*}
E\left(N_{+}\right)=n / 2 \tag{2.6}
\end{equation*}
$$

(2.8) $\quad V\left(N_{+}\right)=n / 4-1$ 'H1/2 $\pi$
(2.9) $\quad V\left(N_{+-}\right)=n / 16$

$$
\begin{equation*}
V\left(N_{+}, N_{+-}\right)=0 \tag{2.10}
\end{equation*}
$$

If $N_{+}$and $N_{+-}$are calculated from the recursive residuals, which are independent, then the same results hold with $H$ replaced by 0 .

Remark. Notice that the variance of $N_{+}$is in general less than binomial since the indicators are correlated. If the linear model (2.1) contains the constant term, i.e. if $1 \in \operatorname{sp}(\mathrm{X})$, then $1^{\prime} H=1$, and $V\left(N_{+}\right)=$ $(1-2 / \pi) n / 4 \simeq n / 11$, as compared to the binomial variance $n / 4$.

Strangely enough the variance of $\mathrm{N}_{+-}$is the same as it would be if the indicator variables were independent. This supports the intuition of Geary(1967) who wrote about the dependence "the effect is believed to be negligible when T is not small, and is ignored here". Note also that the asymptotic variance of $\mathrm{N}_{+-}$is the same for any linear model (2.1).

The next result concerns the power functions $\beta_{+}$and $\beta_{+-}$of the two tests one can derive from $N_{+}$and $N_{+-}$. We consider a local alternative to the linear model of the form

$$
\begin{equation*}
\xi_{\mathrm{n}}=\xi_{0}+\mathrm{h}_{\mathrm{n}} \xi_{1}, \tag{2.11}
\end{equation*}
$$

where $\xi_{0} \in \operatorname{sp}(\mathrm{X})$ and $\xi_{1} \notin \mathrm{sp}(\mathrm{X})$, and let $\mathrm{h}_{\mathrm{n}}$ tend to zero with a suitably chosen power of $n$.

The power function of any of the above statistics ( N ) is given by

$$
\beta\left(\xi_{\mathrm{n}}\right)=\mathrm{P}_{\xi_{\mathrm{n}}}\left\{\left|\mathrm{~N}-\mathrm{E}_{\mathrm{O}}(\mathrm{~N})\right| \geq \mathrm{u}_{1-\alpha / 2} \mathrm{~V}_{0}(\mathrm{~N})^{1 / 2}\right\},
$$

where $u_{1-\alpha / 2}$ is the $1-\alpha / 2$ quantile of the Gaussian distribution and $E_{0}($. and $V_{0}($.$) are calculated for \xi_{\mathrm{n}}=\xi_{0}$. If N is asymptotically Gaussian we want to approximate this power function with an expression of the form

$$
\begin{equation*}
\beta\left(\xi_{\mathrm{n}}\right)=1-\Phi\left\{\mathrm{u}_{1-\alpha / 2}+\mathrm{c}\right\}+\Phi\left\{-\mathrm{u}_{1-\alpha / 2}+\mathrm{c}\right\} \tag{2.12}
\end{equation*}
$$

We can then state the results on the power function in terms of the value of $c$.

THEOREM 2. The power function $\beta_{+}$of the test based on $N_{+}$can be approximated as follows:

If $n^{-1} \sum_{i=1}^{n}\left((I-H) \xi_{1}\right)_{i} \rightarrow a_{1} \neq 0$, and if $n^{1 / 2} h_{n} \rightarrow b_{1}$, then $\beta_{+}$is given by (2.12) for $c=a_{1} b_{1} /\left(\pi / 2-1^{\prime} H 1 / n\right)^{1 / 2}$.

$$
\operatorname{If} \sum_{i=1}^{n}\left((I-H) \xi_{1}\right)_{i}=0 \text { and } n^{-1} \sum_{i=1}^{n}\left((I-H) \xi_{1}\right)_{i}^{3} \rightarrow a_{2} \neq 0 \text { and } n^{1 / 6} h_{n} \rightarrow b_{2} \text {, }
$$

then $\beta_{+}$is given by (2.12) for $c=a_{2} b_{2}^{3} / 6(\pi / 2-1 \text { 'H1/n })^{1 / 2}$.

$$
\text { If } n^{-1} \sum_{i=1}^{n-1}\left((I-H) \xi_{1}\right)_{i}\left((I-H) \xi_{1}\right)_{i+1} \rightarrow a_{3} \neq 0 \text { and if } n^{1 / 4} h_{n} \rightarrow b_{3}
$$

then the power function $\beta_{+-}$can be approximated by (2.12) with $c=$ $-2 a_{3} b_{3}^{2} / \pi$.

As an example of these results consider the very special case of (2.1) given by

$$
Y_{i}=\alpha+\beta t_{i}+\sigma \epsilon_{i}, \quad i=1, \ldots, n,
$$

and the local alternative given by

$$
Y_{i}=\gamma+\delta t_{i}+h_{n} t_{i}^{2}+\sigma \epsilon_{i} i=1, \ldots, n
$$

In this case $\left(\xi_{1}\right)_{i}=t_{i}^{2}$ and

$$
\left((I-H) \xi_{1}\right)_{i}=t_{i}^{2}-\overline{t^{2}}-\left(t_{i}-\bar{t}\right) \sum_{i=1}^{n} t_{i}^{3} /\left(\sum_{i=1}^{n} t_{i}^{2}\right)=R_{i}
$$

say. Since $1 \in \operatorname{sp}(X)$, we have $1^{\prime}(I-H)=0$ and hence that

$$
\sum_{i=1}^{n}\left((I-H) \xi_{1}\right)_{i}=\sum_{i=1}^{n} R_{i}=0
$$

Hence the power function $\beta_{+}$can be approximated by (2.12) with

$$
c=n^{1 / 2} h_{n}^{3} n^{-1} \sum_{i=1}^{n} R_{i}^{3} / 6\left(\pi / 2-1^{\prime} H 1 / n\right)^{1 / 2}
$$

The power function $\beta_{+-}$can be approximated by (2.12) with the choice

$$
c=-2 n^{1 / 2} h_{n}^{2} n^{-1} \sum_{i=1}^{n-1} R_{i} R_{i+1} / \pi
$$

The proofs of these results will be given in the next section.

## 3. MATHEMATICAL RESULTS.

We define the Hajek projections

$$
N_{+-}^{*}=\sum_{i=1}^{n-1}\left\{E\left(N_{+-} \mid Y_{i}, Y_{i+1}\right)-E\left(N_{+-}\right)\right\}-\sum_{i=2}^{n-1}\left\{E\left(N_{+-} \mid Y_{i}\right)-E\left(N_{+-}\right)\right\}
$$

and

$$
N_{+}^{*}=\sum_{i=1}^{n}\left\{E\left(N_{+} \mid Y_{i}\right)-E\left(N_{+}\right)\right\}
$$

It is easily seen that if $N_{n}$ is either $N_{+-}$or $N_{+}$then

$$
\mathrm{V}\left(\mathrm{~N}_{\mathrm{n}}-\mathrm{N}_{\mathrm{n}}^{*}, \mathrm{~N}_{\mathrm{n}}^{*}\right)=0
$$

and we shall apply the following result from Lehmann(1975) p. 349

LEMMA 1 If $N_{n}^{*} / V\left(N_{n}^{*}\right) 1 / 2$ converges in distribution to $F$ and if $V\left(N_{n}\right) / V\left(N_{n}^{*}\right) \rightarrow 1, n \rightarrow \infty$, then also $N_{n} / V\left(N_{n}\right)^{1 / 2}$ converges in distribution to F .

We shall derive the asymptotic distribution of the statistics under the model (2.1). Since the distribution of the residuals $E$ does not depend on $\beta$ and $\sigma$ we let $\beta=0$ and $\sigma=1$ in the calculations. The proof depends on an expansion of the Gaussian distribution which we give as

LEMMA 2 Consider an n-dimensional Gaussian variable Z with mean $\xi$ and variance matrix $I-H$, where we assume that as $n \rightarrow \infty$ we have $\xi \in O\left(n^{-\alpha}\right)$ for some $\alpha \leq 1$ and $H \in O\left(n^{-1}\right)$. Then

$$
P\left\{Z_{i}<0, \quad i=1, \ldots, p\right\}=
$$

$$
\begin{equation*}
\prod_{i=1}^{p} \Phi\left(-\xi_{i} /\left(1-H_{i i}\right)^{1 / 2}\right)\left(1-\sum_{i \neq j}^{p} H_{i j} / \pi+O\left(\xi n^{-1}\right)+O\left(n^{-2}\right)\right) \tag{3.1}
\end{equation*}
$$

Fur ther

$$
\begin{gathered}
\text { (3.2) } \Phi\left(-\xi_{\mathrm{i}} /\left(1-\mathrm{H}_{\mathrm{ii}}\right)^{1 / 2}\right)=1 / 2-\xi_{\mathrm{i}} /(2 \pi)^{1 / 2}+\xi_{\mathrm{i}}^{3} / 6(2 \pi)^{1 / 2}-\xi_{\mathrm{i}} \mathrm{H}_{\mathrm{ii}} / 2(2 \pi)^{1 / 2} \\
+O\left(\xi^{4}, \xi^{3} \mathrm{n}^{-1}, \xi^{-2}\right)
\end{gathered}
$$

Proof. Let $V_{i}=\left(Z_{i}-\xi_{i}\right) /\left(1-H_{i i}\right)^{1 / 2}$ denote the standardised variable and let $K_{i j}=H_{i j} /\left(\left(1-H_{i i}\right)\left(1-H_{j j}\right)\right)^{1 / 2}, i \neq j$ and $K_{i j}=0$ for $i=j$. Further let $\eta_{i}=-\xi_{i} /\left(1-H_{i i}\right)^{1 / 2}$, then

$$
P\left\{Z_{i}<0, i=1, \ldots, p\right\}=P\left\{V_{i}<\eta_{i} i=1, \ldots, p\right\}
$$

and V is distributed as $\mathrm{N}_{\mathrm{p}}(\mathrm{O}, \mathrm{I}-\mathrm{K})$ where $\mathrm{K}_{\mathrm{ii}}=0$.

$$
\begin{gathered}
\text { Now for } B=\left\{V_{i}<\eta_{i}, i=1, \ldots, p\right\} \text { we get } \\
P(B)=I(B) / I\left(R^{p}\right)
\end{gathered}
$$

where

$$
I(B)=\int_{B} \varphi_{p}(x) \exp \left\{-1 / 2 x^{\prime}\left[(I-K)^{-1}-I\right] x\right\} d x
$$

and

$$
\varphi_{\mathrm{p}}(\mathrm{x})=(2 \pi)^{-\mathrm{p} / 2} \exp \{-1 / 2 \mathrm{x} x\}
$$

Now use the expansion

$$
(\mathrm{I}-\mathrm{K})^{-1}=\mathrm{I}+\mathrm{K}+\mathrm{K}^{2}+\ldots
$$

to find

$$
\exp \left\{-1 / 2 \mathrm{x}^{\prime}\left[(\mathrm{I}-\mathrm{K})^{-1}-\mathrm{I}\right] \mathrm{x}\right\}=1-1 / 2 \mathrm{x}^{\prime} \mathrm{Kx}+\mathrm{O}\left(\mathrm{n}^{-2}\right)
$$

Integrating with respect to $\varphi_{p}(x)$ we get

$$
I\left(R^{p}\right)=1-1 / 2 \sum_{i \neq j}^{p} K_{i j} \int_{R^{2}} x_{i} x_{j} \varphi_{2}\left(x_{i}, x_{j}\right) d x_{i} d x_{j}+O\left(n^{-2}\right)=1+
$$

$O\left(n^{-2}\right)$,
and
$O\left(n^{-2}\right)$

$$
=\prod_{\mathrm{i}=1}^{\mathrm{p}} \Phi\left(\eta_{\mathrm{i}}\right)\left\{1-1 / 2 \sum_{\mathrm{i} \neq \mathrm{j}}^{\mathrm{p}} \mathrm{~K}_{\mathrm{ij}} \frac{\varphi\left(\eta_{\mathrm{i}}\right) \varphi\left(\eta_{\mathrm{j}}\right)}{\Phi\left(\eta_{\mathrm{i}}\right) \Phi\left(\eta_{\mathrm{j}}\right)}+\mathrm{O}\left(\mathrm{n}^{-2}\right)\right\}
$$

Now the result (3.1) follows from

$$
\varphi\left(\eta_{\mathrm{i}}\right) / \Phi\left(\eta_{\mathrm{i}}\right)=2 /(2 \pi)^{1 / 2}+O(\xi)
$$

and the result (3.2) follows from a Taylor's expansion of $\Phi$ around 0 .

We shall of ten need this result in the following simple form

COROLLARY 1. If $Z$ is distributed as $N_{n}(O, I-H)$ then

$$
\begin{equation*}
P\left(Z_{i}<0, i=1, \ldots, p\right)=2^{-p}\left(1-\sum_{i \neq j}^{p} H_{i j} / \pi+O\left(n^{-2}\right)\right) \tag{3.3}
\end{equation*}
$$

If Z is distributed as $\mathrm{N}_{\mathrm{n}}(\xi, \mathrm{I})$ and $\xi \in \mathrm{O}\left(\mathrm{n}^{-1}\right)$ then

$$
\begin{equation*}
P\left(Z_{i}<0, i=1, \ldots, p\right)=2^{-p}\left(1-2 \sum_{i=1}^{p} \xi_{i} /(2 \pi)^{1 / 2}+O\left(n^{-2}\right)\right) \tag{3.4}
\end{equation*}
$$

Note that probabilities like $\mathrm{P}\left(\mathrm{Z}_{1}>0, \mathrm{Z}_{2}<0\right)$ can be found by changing the sign in (3.3) and (3.4) of the relevant terms in $\xi$ and $H$.

Next we shall give a proof of the expressions for the mean and variance of the variables $\mathrm{N}_{+-}$and $\mathrm{N}_{+}$i.e. (2.6) to (2.10). The relation $E\left(N_{+}\right)=n / 2$ is obvious, and if we introduce the notation $U_{i}=1\left\{E_{i}>0\right\}$ $=1-\bar{U}_{i}$, we can calculate

$$
V\left(N_{+}\right)=V\left(\sum_{i=1}^{n} U_{i}\right)=\sum_{i=1}^{n} V\left(U_{i}\right)+\sum_{i=1}^{n} \sum_{j \neq i}^{n} V\left(U_{i}, U_{j}\right)
$$

We now have $\mathrm{V}\left(\mathrm{U}_{\mathrm{i}}\right)=1 / 4$ and by (3.3) $\mathrm{E}\left(\mathrm{U}_{\mathrm{i}} \mathrm{U}_{\mathrm{j}}\right)=1 / 4\left(1-2 \mathrm{H}_{\mathrm{ij}} / \pi\right)+\mathrm{O}\left(\mathrm{n}^{-2}\right)$ and hence $V\left(U_{i}, U_{j}\right)=-H_{i j} / 2 \pi+O\left(n^{-2}\right)$. Summing over $i$ and $j$ we get (2.8). The expectation of $\mathrm{N}_{+-}$is similar, but the variance is more complicated:

$$
\begin{aligned}
V\left(N_{+-}\right)= & V\left(\sum_{i=1}^{n-1} U_{i} \bar{U}_{i+1}\right)= \\
& \sum_{i=1}^{n-1} V\left(U_{i} \bar{U}_{i+1}\right)+2 \sum_{i=1}^{n-2} V\left(U_{i} \bar{U}_{i+1}, U_{i+1} \bar{U}_{i+2}\right) \\
& +2 \sum_{i=1}^{n-1} \sum_{j \neq i-1, i, i+1} V\left(U_{i} \bar{U}_{i+1}, U_{j} \bar{U}_{j+1}\right)
\end{aligned}
$$

For the first two terms we evaluate from (3.3)

$$
\begin{aligned}
& V\left(U_{i} \bar{U}_{i+1}\right)=3 / 16+O\left(n^{-1}\right) \\
& V\left(U_{i} \bar{U}_{i+1}, U_{i+1} \bar{U}_{i+2}\right)=-E\left(U_{i} \bar{U}_{i+1}\right) E\left(U_{i+1} \bar{U}_{i+2)}\right. \\
& =-1 / 16+O\left(n^{-1}\right)
\end{aligned}
$$

Thus these two terms contribute $\mathrm{n} / 16$ to the variance. The last term vanishes as we shall now show using (3.3):

$$
\begin{gathered}
V\left(U_{i} \bar{U}_{i+1}, U_{j} \bar{U}_{j+1}\right)=E\left(U_{i} \bar{U}_{i+1} U_{j} \bar{U}_{j+1}\right)-E\left(U_{i} \bar{U}_{i+1}\right) E\left(U_{j} \bar{U}_{j+1}\right) \\
=1 / 16\left[1-2\left(-H_{i, i+1}+H_{i j}-H_{i, j+1}-H_{i+1, j}+H_{i+1, j+1}-H_{j, j+1}\right) / \pi\right] \\
-1 / 16\left(1+2 H_{i, i+1} / \pi\right)\left(1+2 H_{j, j+1} / \pi\right) \\
=-\left(H_{i j}-H_{i, j+1}-H_{i+1, j}+H_{i+1, j+1}\right) / 8 \pi
\end{gathered}
$$

Summing over $i$ and $j$ gives $O(1)$. The last relation $V\left(N_{+}, N_{+-}\right)=0$ is proved similarly.

Next we have to find a simple expression for statistics $N_{+-}^{*}$ and $N_{+}^{*}$, and for that we need the following

LEMMA 3. For $n \rightarrow \infty$ we find up to terms of order $O_{P}\left(n^{-1}\right)$

$$
\begin{array}{ll}
\mathrm{E}\left(\mathrm{~N}_{+} \mid \mathrm{Y}_{\mathrm{i}}\right)-\mathrm{E}\left(\mathrm{~N}_{+}\right) & =1\left\{\mathrm{Y}_{\mathrm{i}}>0\right\}-1 / 2-(2 \pi)^{-1 / 2} \sum_{\mathrm{j}=1}^{\mathrm{n}} \mathrm{H}_{\mathrm{ji}} \mathrm{Y}_{\mathrm{i}} \\
\mathrm{E}\left(\mathrm{~N}_{+-} \mid \mathrm{Y}_{\mathrm{i}}\right)-\mathrm{E}\left(\mathrm{~N}_{+-}\right) & =0  \tag{3.6}\\
\mathrm{E}\left(\mathrm{~N}_{+-} \mid \mathrm{Y}_{\mathrm{i}}, \mathrm{Y}_{\mathrm{i}+1}\right)-\mathrm{E}\left(\mathrm{~N}_{+-}\right)= \\
1\left\{\mathrm{Y}_{\mathrm{i}}>0, \mathrm{Y}_{\mathrm{i}+1}>0\right\}-1 / 4+1 / 2\left(-1\left\{\mathrm{Y}_{\mathrm{i}}>0\right\}+1\left\{\mathrm{Y}_{\mathrm{i}+1}<0\right\}\right)
\end{array}
$$

Proof. To prove (3.5) note that

$$
E\left(N_{+} \mid Y_{i}\right)=P\left(E_{i}>0 \mid Y_{i}\right)+\sum_{j \neq i}^{n} P\left(E_{j}>0 \mid Y_{i}\right)
$$

Since $V(E)=V(E, Y)=I-H$ we have

$$
E\left(E_{i} \mid Y_{i}\right)=\left(1-H_{i i}\right) Y_{i}, V\left(E_{i} \mid Y_{i}\right)=\left(1-H_{i i}\right) H_{i i}
$$

such that

$$
P\left(E_{i}>0 \mid Y_{i}\right)=\Phi\left\{\left(1-H_{i i}\right) Y_{i} /\left[\left(1-H_{i i}\right) H_{i i}\right]^{1 / 2}\right\} \rightarrow 1\left\{Y_{i}>0\right\}
$$

whereas

$$
E\left(E_{j} \mid Y_{i}\right)=-H_{j i} Y_{i}, V\left(E_{j} \mid Y_{i}\right)=1-H_{j j}-H_{i j}^{2}
$$

such that
$P\left(E_{j}>0 \mid Y_{i}\right)=\Phi\left\{-H_{j i} Y_{i} /\left[1-H_{j j}-H_{i j}\right]^{1 / 2}\right\}=1 / 2-(2 \pi)^{-1 / 2} H_{j i} Y_{i}+O_{P}\left(n^{-2}\right)$

Summing over j gives the relation (3.5). Similarly we find

$$
\begin{aligned}
& E\left(N_{+-} \mid Y_{i}\right)=P\left(E_{i}>0, E_{i+1}<0 \mid Y_{i}\right)+P\left(E_{i-1}>0, E_{i}<0 \mid Y_{i}\right) \\
& +\sum_{j \neq i, i-1}^{n-1} P\left(E_{j}>0, E_{j+1}<0 \mid Y_{i}\right) .
\end{aligned}
$$

Now we find from (3.3) and (3.4) using the notation $U_{i}=1\left\{E_{i}>0\right\}=1-\bar{U}_{i}$

$$
\begin{aligned}
& E\left(U_{i} \bar{U}_{i+1} \mid Y_{i}\right)-E\left(U_{i} \bar{U}_{i+1}\right)=1 / 2\left[1\left\{Y_{i}>0\right\}-1 / 2\right]+o_{P}\left(n^{-1}\right) \\
& E\left(U_{i-1} \bar{U}_{i} \mid Y_{i}\right)-E\left(U_{i-1} \bar{U}_{i}\right)=1 / 2\left[1\left\{Y_{i}<0\right\}-1 / 2\right]+o_{P}\left(n^{-1}\right)
\end{aligned}
$$

which cancel, and

$$
E\left(U_{j} \bar{U}_{j+1} \mid Y_{i}\right)-E\left(U_{j} \bar{U}_{j+1}\right) \rightarrow(2 \pi)^{-1 / 2}\left(H_{j i}-H_{j+1, i}\right) Y_{i} / 2+o_{P}\left(n^{-2}\right)
$$

summing over j we get

$$
E\left(N_{+-} \mid Y_{i}\right)-E\left(N_{+-}\right)=O_{P}\left(n^{-1}\right) .
$$

Finally we want to evaluate $E\left(N_{+-} \mid Y_{i}, Y_{i+1}\right)$, and for that we need the expressions

$$
\begin{aligned}
& E\left(U_{i} \bar{U}_{i+1} \mid Y_{i}, Y_{i+1}\right)-E\left(U_{i} \bar{U}_{i+1}\right) \rightarrow 1\left\{Y_{i}>0, Y_{i+1}<0\right\}-1 / 4 \\
& E\left(U_{i-1} \bar{U}_{i} \mid Y_{i}, Y_{i+1}\right)-E\left(U_{i-1} \bar{U}_{i}\right) \rightarrow 1 / 2\left[1\left\{Y_{i}<0\right\}-1 / 2\right] \\
& E\left(U_{i+1}, \bar{U}_{i+2} \mid Y_{i}, Y_{i+1}\right)-E\left(U_{i+1} \bar{U}_{i+2}\right) \rightarrow 1 / 2\left[1\left\{Y_{i+1}>0\right\}-1 / 2\right] \\
& E\left(U_{j} \bar{U}_{j+1} \mid Y_{i}, Y_{i+1}\right)-E\left(U_{j} \bar{U}_{j+1}\right)= \\
& (2 \pi)^{-1 / 2}\left[H_{j . j+1} Y_{i}+\left(H_{j, i+1}-H_{j+1 . i+1}\right) Y_{i+1}\right] / 2+o_{P}\left(n^{-2}\right)
\end{aligned}
$$

Now add these terms over $j \neq i-1, i, i+1$ and we get a term of the order $n^{-1}$ which proves (3.7).

We can now turn to the proof of Theorem 1 on the asymptotic normality of the statistics and the representations (2.4) and (2.5). The results of Lemma 3 give the representation

$$
N_{+-}^{*}=N_{+-}^{* *}+O_{P}(1)
$$

and

$$
\mathrm{N}_{+}^{*}=\mathrm{N}_{+}^{* *}+\mathrm{O}_{\mathrm{P}}(1)
$$

It is a simple exercise to show that the approximations to $\mathrm{N}_{+-}^{*}$ and $\mathrm{N}_{+}^{*}$ have variances and covariances that behave asymptotically like those of $\mathrm{N}_{+-}$and $\mathrm{N}_{+}$. One should really evaluate the variances of $\mathrm{N}_{+}^{*}$ and $\mathrm{N}_{+-}^{*}$ but that requires a somewhat more detailed analysis and shall not be given here. Now we can apply Lemma 2, and it remains to find the limiting distribution of the approximations to $\mathrm{N}_{+-}^{*}$ and $\mathrm{N}_{+}^{*}$. The relevant limit theorem for sums of m-dependent random variables can be found in Anderson (1971) p. 427.

We shall now investigate the power of the tests based upon $\mathrm{N}_{+}$and $N_{+-}$against misspecified linear models. That is, we shall assume that $Y$ is distributed as $N_{n}\left(\xi_{n}, \sigma^{2} I\right)$, see (2.11), such that $E$ is distributed as $N_{n}\left((I-H) \xi_{n}, \sigma^{2}(I-H)\right)$. We shall investigate the power function as $\delta_{n}=$ $(\mathrm{I}-\mathrm{H}) \xi_{\mathrm{n}}=\mathrm{h}_{\mathrm{n}}(\mathrm{I}-\mathrm{H}) \xi_{1} \rightarrow 0$ at a suitable rate.

It is no problem, although rather tedious, to go through the proof of Theorem 1 and check that in fact, under the sequence of models given by $\delta_{n} \rightarrow 0$, we can show that the statistics $N_{+}$and $N_{+-}$are asymptotically normally distributed, and that $V_{\xi_{n}}(N) / V_{0}(N) \rightarrow 1$, for any of the statistics. For any of the statistics considered we approximate as follows:

$$
\begin{gathered}
\beta\left(\xi_{\mathrm{n}}\right)=\mathrm{P}_{\xi_{\mathrm{n}}}\left\{\left|\mathrm{~N}-\mathrm{E}_{0}(\mathrm{~N})\right| \geq \mathrm{u}_{1-\alpha / 2} \mathrm{~V}_{0}(\mathrm{~N})^{1 / 2}\right\} \\
\simeq 1-\Phi\left\{\mathrm{u}_{1-\alpha / 2}\left(\mathrm{~V}_{0}(\mathrm{~N}) / \mathrm{V}_{\xi_{\mathrm{n}}}(\mathrm{~N})\right)^{1 / 2}+\left(\mathrm{E}_{0}(\mathrm{~N})-\mathrm{E}_{\xi_{\mathrm{n}}}(\mathrm{~N})\right) / \mathrm{V}_{\xi_{\mathrm{n}}}(\mathrm{~N})^{1 / 2}\right\} \\
+\Phi\left\{-\mathrm{u}_{1-\alpha / 2}\left(\mathrm{~V}_{0}(/ \mathrm{N}) / \mathrm{V}_{\xi_{\mathrm{n}}}(\mathrm{~N})\right)^{1 / 2}+\left(\mathrm{E}_{0}(\mathrm{~N})-\mathrm{E}_{\xi_{\mathrm{n}}}(\mathrm{~N})\right) / \mathrm{v}_{\xi_{\mathrm{n}}}(\mathrm{~N})^{1 / 2}\right\},
\end{gathered}
$$

which has the form (2.12) for $c=\left(E_{0}(N)-E_{\xi_{n}}(N)\right) / V_{0}(N)^{1 / 2}$.
Thus we shall concentrate on the calculation of the difference in the expectations under the two models. To ease the notation we shall drop the subscript $n$ on $\xi$ and $\delta$. Consider first the statistic $N_{+}$. We find

$$
E_{\xi}\left(N_{+}\right)-E_{0}\left(N_{+}\right)=\sum_{i=1}^{n}\left\{P\left(E_{i}>0\right)-1 / 2\right\}
$$

and by Lemma 2

$$
\begin{aligned}
& P\left(E_{i}>0\right)-1 / 2=\Phi\left(\delta_{i} /\left(1-H_{i i}\right)^{1 / 2}\right)-1 / 2= \\
& \delta_{i} / \sqrt{2} \pi-\delta_{i}^{3} / 6 \sqrt{2} \pi-\delta_{i} H_{i i} / 2 \sqrt{2} \pi+O\left(\delta_{i}^{4}, \delta_{i}^{3} n^{-1}, \delta_{i} n^{-2}\right)
\end{aligned}
$$

$$
\text { If } \mathrm{n}_{\mathrm{i}=1}^{-1} \mathrm{n}\left((\mathrm{I}-\mathrm{H}) \xi_{1}\right)_{\mathrm{i}} \rightarrow \mathrm{a}_{1} \neq 0 \text {, then the dominating term is }
$$

$$
\sum_{\mathrm{i}=1}^{\mathrm{n}} \delta_{\mathrm{i}} /(2 \pi)^{1 / 2} \simeq \mathrm{~h}_{\mathrm{n}} \mathrm{n}^{-1} \mathrm{a}_{1} /(2 \pi)^{1 / 2}
$$

For this to balance with $\mathrm{V}_{0}\left(\mathrm{~N}_{+}\right)^{1 / 2} \in \mathrm{O}\left(\mathrm{n}^{1 / 2}\right)$, one must choose $\mathrm{h}_{\mathrm{n}} \epsilon$ $O\left(\mathrm{n}^{-1 / 2}\right)$ and then (2.12) follows with the proper choice of c .

If $\sum_{i=1}^{n} \delta_{i}=0$ and $n_{i=1}^{-1} \sum_{i}^{n} \delta_{i}^{3} \rightarrow a_{2} \neq 0$ then the dominating term is

$$
\sum_{\mathrm{i}=1}^{\mathrm{n}} \delta_{\mathrm{i}}^{3} / 6(2 \pi)^{1 / 2} \simeq \mathrm{~h}_{\mathrm{n}}^{3} \mathrm{na}_{2} / 6(2 \pi)^{1 / 2}
$$

For this to balance with $\mathrm{n}^{1 / 2}$ we get $\mathrm{h}_{\mathrm{n}} \in \mathrm{O}\left(\mathrm{n}^{-1 / 6}\right)$ which shows the next result. Finally we find for $\mathrm{N}_{+-}$the evaluations

$$
\mathrm{E}_{\xi}\left(\mathrm{N}_{+-}\right)-\mathrm{E}_{0}\left(\mathrm{~N}_{+-}\right)=\sum_{\mathrm{i}=1}^{\mathrm{n}-1}\left(\mathrm{P}\left(\mathrm{E}_{\mathrm{i}}>0, \mathrm{E}_{\mathrm{i}+1}<0\right)-1 / 4\right)
$$

and from Lemma 2

$$
\begin{aligned}
& P\left(E_{i}>0, E_{i+1}<0\right)-1 / 4= \\
& {\left[\Phi\left(\delta_{i} /\left(1-H_{i i}\right)^{1 / 2}\right) \Phi\left(\left(-\delta_{i+1} /\left(1-H_{i+1 . i+1}\right)^{1 / 2}\right)-1 / 4\right](1+\right.}
\end{aligned}
$$

$\left.2 \mathrm{H}_{\mathrm{i} . \mathrm{i}+1} / \pi\right)$
where the leading terms are

$$
\left(\delta_{i}-\delta_{i+1}\right) / 2 \sqrt{2} \bar{\pi}-\delta_{i} \delta_{i+1} / 2 \pi
$$

The first cancels when summed over $i$ and the second gives the required result for $\mathrm{N}_{+-}$.

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