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## A New Upper Bound for the Efficiency of a Block Design



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# Summary: <br> An upper bound for the efficieny of a block design, which in many cases is tighter than those reported by other authors, is derived. The bound is based on a lower bound for $E(1 / X)$ in terms of $E(X)$ and $\operatorname{var}(X)$ for a random variable $X$ on the unit interval. For the special case of a resolvable design, an improved bound is given. 

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Key Words and Phrases.
Block design, efficiency, optimal design, A-optimality.

1. Introduction and notation.

Consider a block design with $T$ treatments allocated to $B$ blocks of $k$ plots such that each treatment occurs on $r$ plots ( $\mathrm{Bk}=\mathrm{Tr}$ ). By $N$ we denote the $\mathrm{T} \times \mathrm{B}$ incidence matrix with elements $n_{t b}=$ the number of plots on block $b$ with treatment $t$. We assume $n_{t b} \leq 1$ (i.e. that the design is binary), and the $T \times T$ information matrix I - $(\mathrm{rk})^{-1} \mathrm{NN}^{*}$ is assumed to be af rank $\mathrm{T}-1$ (i.e. the design is assumed connected). By $\overline{\mathrm{e}}$ we denote the average of the $T-1$ non-zero eigenvalues of this matrix. The harmonic mean of the eigenvalues, $E=(T-1)\left(\Sigma e_{i}^{-1}\right)^{-1}$, is known as the efficiency. This quantity $E$ can be interpreted as the inverse proportion between the average contrast variance in the design under the usual additive (block + treatment) model and the average contrast variance in the simpler (hypothetical) model without block effects (but with the same error variance).

It is common statistical practice to select a design of maximal efficiency when the design constants $B, k, T$ and $r$ are given. Balanced incomplete block designs and certain partially balanced designs are known to be optimal in this sense, but no general solution to the problem of selecting a design of maximal efficiency is known. Hence, the availability of good upper bounds for the efficiency is important, because closeness to such a bound tells the designer of experiments that the design can not be significantly improved. The best bound available today seems to be that given by Jarrett (1983, theorem 5.3) and Fitzpatrick and Jarrett (1987). This bound is based on a lower bound for the harmonic mean of the non-zero eigenvalues $e_{i}$ in terms of the first three moments of a random variable which takes the values $\mathrm{e}_{1}, \ldots, \mathrm{e}_{\mathrm{T}-1}$ with equal probabilities. The first moment is given by the design constants ( $\overline{\mathrm{e}}=$ $(1-1 / k) /(1-1 / T))$, and for the variance $(T-1)^{-1} \sum\left(e_{i}-\bar{e}\right)^{2}$ the value

$$
\begin{equation*}
\mathrm{V}=\frac{\mathrm{T}}{\mathrm{k}^{2} \mathrm{r}^{2}}(\bar{\lambda}-[\bar{\lambda}])([\bar{\lambda}]+1-\bar{\lambda}) \tag{1.1}
\end{equation*}
$$

is substituted. Here, $\bar{\lambda}=\mathrm{r}(\mathrm{k}-1) /(\mathrm{T}-1)$ is the average of the $T(T-1) / 2$ concurrence counts $\lambda\left(t_{1}, t_{2}\right) \quad(=$ the number of blocks in which treatments $t_{1}$ and $t_{2}$ occur). This is the value taken by the variance in case of a regular graph design, i.e. a design with any concurrence count equal to either $[\bar{\lambda}]$ or $[\bar{\lambda}]+1$; since it is a commonly accepted conjecture (though never proved in general) that a design of maximal efficiency must be a regular graph design when a regular graph design exists, the substitution of this value for the variance is reasonable, and even for design constants which do not allow for a regular graph design, the bound derived by Jarrett and Fitzpatrick seems to be valid. Formally, however, the bound is only valid for regular graph designs. Having fixed the first and second moment, an upper bound for the efficiency can be derived from a lower bound for the third moment, which turns out to be related to the number of triangles in the variety-concurrence graph, see Paterson (1983). Lower bounds for this number of triangles can be given by combinatorial arguments, which then establish the final step. Bounds for the efficiency derived in this way are exact (i.e. equal to the efficiency $E$ ) if and only if the design is a regular graph design, the information matrix has at most two distinct eigenvalues and the number of triangles in the variety-concurrence graph equals the lower bound substituted for it.

In the present paper we establish a simpler upper bound for the efficiency, based on the first and second moment of the eigenvalue distribution, but not the third. This bound is exact for regular graph designs with exactly one (multiple) eigenvalue $\neq 1$. Since this condition on the eigenvalue distribution is obviously more restrictive than the corresponding restriction for the bound due to Fitzpatrick and Jarrett, one would expect the simpler second moment bound to be less precise than the bound taking three moments
into account. However, this heuristic argument ignores the final step, which requires a good lower bound for the number of triangles in the variety-concurrence graph. This is not always so easy, and this is probably the reason why the second moment bound frequently turns out to be tighter than that of Jarrett and Fitzpatrick.
2. The second moment bound.

Theorem 2.1. For any binary design,

$$
\begin{equation*}
E \leq \frac{\overline{\mathrm{e}}(1-\overline{\mathrm{e}})-V}{(1-\overline{\mathrm{e}})-V} \tag{2.1}
\end{equation*}
$$

where V is given by (1.1).

Proof. Let var(e) denote the variance in the distribution of the eigenvalues, i.e. $\operatorname{var}(\mathrm{e})=(\mathrm{T}-1)^{-1} \Sigma\left(\mathrm{e}_{\mathrm{i}}-\overline{\mathrm{e}}\right)^{2}$. Similarly, let $\operatorname{var}(\lambda)$ denote the variance in the distribution of the $T(T-1) / 2$ concurrence counts. Then it is wellknown that $\operatorname{var}(\mathrm{e})=\left(\mathrm{T} /(\mathrm{rk})^{2}\right) \operatorname{var}(\lambda)$. A lower bound for $\operatorname{var}(\lambda)$ is easily seen to be $\operatorname{var}(\lambda) \geq$ $(\bar{\lambda}-[\bar{\lambda}])([\bar{\lambda}]+1-\bar{\lambda})$, with equality if and only if the design is a regular graph design. Hence $\operatorname{var}(e) \geq V$, and since the right hand side of (2.1) is decreasing as a function of $V$, we are through if we can show that

$$
E \leq \frac{\overline{\mathrm{e}}(1-\overline{\mathrm{e}})-\operatorname{var}(\mathrm{e})}{(1-\overline{\mathrm{e}})-\operatorname{var}(\mathrm{e})}
$$

But this follows immediately by application of the following lemma to a random variable $X$, uniformly distributed on the $\mathrm{T}-1$ eigenvalues $\mathbf{e}_{\mathbf{i}}$.

Lemma. Let X be a random variable on $(0,1]$. Then

$$
E(1 / X) \quad \geq \frac{(1-E X)-\operatorname{var}(X)}{(1-E X) E X-\operatorname{var}(X)}
$$

This inequality is closely related to results concerned with the moment problem and similar matters, see e.g. Karlin and Studden (1966), Lew (1976), Fitzpatrick and Jarrett (1986). We have not been able to find exactly this result, but a very elementary proof goes as follows. For any real $\alpha$, we obviously have

$$
E\left[\frac{1}{X}(X-\alpha)^{2}(1-X)\right] \geq 0
$$

Expand the product in powers of $X$, introduce $a=1 / \alpha$ and make some rearrangement of terms to obtain the inequality

$$
E(1 / X) \quad \geq \quad\left(E\left(X^{2}\right)-E X\right) a^{2}+2(1-E X) a+1
$$

which then holds for any $a$. For given $E X$ and $E\left(X^{2}\right)$, maximize the right hand side with respect to $a$ to obtain the tightest possible bound. This gives the desired result.

Notice that this proof is very similar to the proof given by Fitzpatrick and Jarrett (1987). In fact, application of exactly the same technique to the inequalities

$$
\mathrm{E}\left[\frac{1}{\mathrm{X}}(\mathrm{X}-\alpha)^{2}(\mathrm{X}-\beta)^{2}\right] \quad \geq \quad 0 \quad, \alpha \text { and } \beta \text { arbitrary }
$$

gives their inequality (2.6).
3. The dual design and an improved bound for resolvable designs.

The dual design is obtained by interchange of the roles taken by blocks and treatments. Thus, the dual design has $B$ treatments allocated to $T$ blocks of size $r$, and its incidence matrix is $N^{*}$, the transpose of the original incidence matrix. Quantities related to the dual design will be denoted by an asterix. It is wellknown that the efficiency of the dual design is related to the efficiency of
the original design by the equation

$$
\begin{equation*}
\left(\mathrm{E}^{-1}-1\right)(\mathrm{T}-1)=\left(\left(\mathrm{E}^{*}\right)^{-1}-1\right)(\mathrm{B}-1) \tag{3.1}
\end{equation*}
$$

which is an easy consequence of the fact that the spectral decompositions of $N N^{*}$ and $N^{*} N$ differ only by the multiplicity of the eigenvalue 0 . Since (3.1) establishes a monotone relationship between $E$ and $E^{*}$, any upper bound for the efficiency has a dual version which comes out by application of the same upper bound to the dual design, followed by substitution into (3.1). Very of ten, the dual bound differs from the direct bound. Typically, this happens in situations where a regular graph design does not exist. In such situations, the bound (2.1) is obviously too optimistic. By complicated combinatorial arguments the lower bound for $\operatorname{var}(\lambda)$ can be improved in these cases, but usually such improvements seem to be taken into account when the bound is computed from the dual design.

Resolvable designs are characterized by the property that s $=\mathrm{T} / \mathrm{k}=\mathrm{B} / \mathrm{r}$ is integer, together with the requirement that the blocks can be divided into groups of $s$ (called the replicates) such that each treatment appears on exactly one block in each group. For some sets of design constants, resolvability can only be obtained on the cost of efficiency. For the dual design, resolvability implies that some of the concurrence counts $\lambda^{*}\left(b_{1}, b_{2}\right) \quad(=$ the number of treatments common to blocks $b_{1}$ and $b_{2}$ ) must be zero, namely those corresponding to blocks in the same replicate. If, at the same time, some of the $\lambda^{*}$ must be greater than 1 in order to give the correct average, then the dual design can not be a regular graph design. This gives the following lower bound for $\operatorname{var}\left(\lambda^{*}\right)$. Arguing as in the proof of theorem 2.1, this can be used to give an improved bound for $E^{*}$, and thus (via 3.1) an upper bound for the efficiency $E$ which is valid for resolvable designs only.

Proposition 3.1. For a resolvable design with $k>s$,

$$
\operatorname{var}\left(\lambda^{*}\right) \geq p_{m} m^{2}+p_{m+1}(m+1)^{2}-\left(\lambda^{*}\right)^{2}
$$

where

$$
\begin{aligned}
& m=[\mathrm{k} / \mathrm{s}], \\
& \bar{\lambda}^{*}=\mathrm{k}(\mathrm{r}-1) /(\mathrm{B}-1), \\
& \mathrm{p}_{\mathrm{m}+1}=\left(\bar{\lambda}^{*}\right)-\mathrm{m}(1-(\mathrm{s}-1) /(\mathrm{B}-1)), \\
& p_{m}=1-\left((\mathrm{s}-1) /(\mathrm{B}-1)+\mathrm{p}_{\mathrm{m}+1}\right) .
\end{aligned}
$$

Proof. With an obvious notation, referring to $\lambda^{*}$ as a randomly selected concurrence count among the $B(B-1) / 2$ possible, the resolvability condition implies that

$$
p_{0}=P\left(\lambda^{*}=0\right) \quad \geq \quad r\binom{s}{2} /\binom{B}{2}=(s-1) /(B-1)
$$

since $r\binom{s}{2}$ of the $\binom{B}{2}$ concurrence counts in the dual design correspond to pairs of blocks within the same replicate. The average of the remaining concurrence counts must be

$$
\begin{aligned}
& \mathrm{E}\left(\lambda^{*} \mid \lambda^{*}>0\right)=\lambda^{*} /\left(1-\mathrm{p}_{0}\right) \geq \bar{\lambda}^{*} /(1-(\mathrm{s}-1) /(\mathrm{B}-1)) \\
& \quad=\frac{\mathrm{k}(\mathrm{r}-1)}{\mathrm{B}-1} / \frac{\mathrm{B}-\mathrm{s}}{\mathrm{~B}-1}=\frac{\mathrm{k}(\mathrm{r}-1)}{\mathrm{B}-\mathrm{s}}=\mathrm{k} / \mathrm{s} .
\end{aligned}
$$

When $k>s$, this implies that the dual design can not be a regular graph design, because at least one of these concurrence counts must be greater than 1 to give the average $\mathrm{k} / \mathrm{s}>1$. The smallest possible value of $\operatorname{var}\left(\lambda^{*}\right)$ is obtained when $p_{0}$ is exactly $(s-1) /(B-1)$ and all remaining concurrence counts are either $m=[k / s]$ or $m+1$. The corresponding point probabilities $p_{m}$ and $p_{m+1}$ are given by the equations

$$
p_{0}+p_{m}+p_{m+1}=1
$$

and

$$
p_{m} m+p_{m+1}(m+1)=\lambda^{*} \text {, }
$$

and the proposition follows after some straightforward calculations.
4. Discussion.

The bound given by Jarrett (1983) is tighter than the bounds reported e.g. in Williams and Patterson (1977), Paterson and Wild (1986), in the sense that if the same lower bound for the number of triangles is substituted then the formula (5.2) in Jarrett (1983) gives the best result. Two ways of constructing lower bounds for the number of triangles have been considered in this connection. Jarrett (1983) uses a bound which seems to be the same as the one reported by Williams and Patterson (1977); Fitzpatrick and Jarrett (1987) reference another bound, due to Paterson (1983), based on the number of 'intra-block triangles'. In this sense, we have two versions of the bound due to Jarrett and Fitzpatrick. We have computed these together with the second moment bound of the present paper in a large number of cases, and quite frequently the second moment bound came out as the smallest. We shall not bother the reader with statistical facts concerning the frequency of this event. The important thing is, after all, that the additional computation of this new bound makes an improvement. A ready-for-use program, which (among other things) computes these upper bounds is available from the author ('.EXE'-file for IBM PC or compatibles, coprocessor required).

Example. Jarrett (1983) discusses the case $B=54, \mathrm{k}=3, \mathrm{~T}=27$, $r=6$. He found the upper bound 0.6745 for the efficiency. With Paterson's lower bound for the number of triangles, the same formula gives 0.6728. The second moment bound (our theorem 2.1) is 0.6701 . A design of efficiency 0.6667 is known to exist.

## References.

Fitzpatrick, S. and Jarrett, R. G. (1986). Upper bounds for the harmonic mean, with an application to experimental design.

Austral. J. Statist 28, pp. 220-229.

Jarrett, R. G. (1977). Bounds for the efficency factor of block designs.

Biometrika 64, pp. 67-72.

Jarrett, R. G. (1983). Definitions and properties for m-concurrence designs.
J. R. Statist. Soc. B 45, pp. 1-10.

Karlin, S. J. and Studden, W. J. (1966). Tchebycheff Systems With Applications In Analysis and Statistics.

Wiley.

Lew, R. A. (1976). Bounds on negative moments.
SIAM J. Appl Math. 30, pp. 728-731.

Paterson, L. J. (1983). Circuits and efficiency in incomplete block designs.

Biometrika 70, pp. 215-225.

Paterson, L. J. and Wild, P. (1986). Triangles and efficiency factors.

Biometrika 73, pp. 289-299.

Williams, E. R. and Patterson, H. D. (1977). Upper bounds for efficiency factors in block designs.

Austral. J. Statist. 19, pp. 194-201.

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