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## On Non-Null Distributions <br> Connected with Testing that a Real Normal Distribution Is Complex



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## Summary

The real zonal polynomials are used to obtain a series expansion for the density of the non-null distribution of the maximal invariant corresponding to testing that the covariance matrix of a 2 m -dimensional real normal distribution has complex structure.

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Abbreviated title: Non-null distribution.

## 1. Introduction

In a paper by Andersson, Brøns and Jensen (1983) ten fundamental tests concerning the structure of covariance matrices in multivariate analysis are treated. Each of the ten problems is invariant under a group of linear transformations and the maximal invariant statistic is obtained in terms of eigenvalues of matrices with certain structures. A series expansion for the density of the distribution of the maximal invariant under the alternative hypothesis has been obtained for some of the ten problems by James (1964) and Constantine (1963) by use of zonal polynomials and hypergeometric functions; it concerns the tests for independence and the tests for identity of two sets of variates where the simultaneous covariance matrix has real or complex structure. The test that a $2 \mathrm{~m} \times 2 \mathrm{~m}$ covariance matrix with complex structure has real structure, which is also one of the ten problems, has been solved by Bertelsen (1987) using methods similar to those of James and Constantine. In this paper one of the remaining non-null distribution problems are solved in the same way ; it concerns the test that a $2 m \times 2 m$ covariance matrix with real structure has complex structure; this test was first considered by Andersson (1978).

Andersson and Perlman (1984) study the non-null distribution of the maximal invariant and we use their results as a starting point. The problem is the evaluation of a certain integral over a matrix group isomorphic to the group of non-singular m×m matrices with complex elements. The theory of group representations is used to define certain polynomials, and it turns out that these polynomials can be expressed in a simpel way by the real zonal polynomials. These polynomials are then used to obtain a series expansion for the integral.

## 2. The statistical problem

Let $x_{1}, \ldots, x_{N}$ be independent observations from a $2 m$-dimensional real normal distribution with mean 0 and unknown covariance matrix $\Sigma \in$ $\mathscr{H}^{+}(2 m, \mathbb{R})$, the set of all positive definite symmetric $2 \mathrm{~m} \times 2 \mathrm{~m}$ matrices. Andersson and Perlman (1984) have considered the problem of testing that $\Sigma$ is of complex structure, i.e. that $\Sigma$ belongs to $\mathscr{H}^{+}(\mathrm{m}, \mathbb{C})$ the set of all positive definite matrices of the form

$$
\Sigma=\left[\begin{array}{lr}
\Sigma(1) & -\Sigma(2)  \tag{1}\\
\Sigma(2) & \Sigma(1)
\end{array}\right]
$$

where $\Sigma(1)$ and $\Sigma(2)$ are $m \times m$ matrices. The problem may be expressed as that of testing

$$
\mathrm{H}_{0}: \Sigma \in \mathscr{H}^{+}(\mathrm{m}, \mathbb{C}) \text { against } \mathrm{H}_{1}: \Sigma \in \mathscr{H}^{+}(2 \mathrm{~m}, \mathbb{R}) .
$$

They reduce the testing problem by invariance to the maximal invariant statistics and the non-null distribution of this statistics is obtained in terms of an expression containing an integral, which is not evaluated. To define this integral let $\widetilde{G} L(m, \mathbb{C})$ denote the group of all non-singular $2 \mathrm{~m} \times 2 \mathrm{~m}$ matrices of the form

$$
M=\left[\begin{array}{rr}
M(1) & -M(2)  \tag{2}\\
M(2) & M(1)
\end{array}\right]
$$

where $M(1)$ and $M(2)$ are $m \times m$ matrices.

Let $\mathscr{U}(m, \mathbb{C})$ denote the group of orthogonal matrices in $\widetilde{G} L(m, \mathbb{C})$. Finally let $\mathscr{R}(m, \mathbb{C})$ be the set of all $2 m \times 2 m$ matrices of the form

$$
R=\left[\begin{array}{rr}
R(1) & R(2)  \tag{3}\\
R(2) & -R(2)
\end{array}\right]
$$

where $R(1)$ and $R(2)$ are $m \times m$ symmetric matrices.
For $R_{1}, R_{2} \in \mathscr{R}(m, \mathbb{C})$ the integral has the form

$$
\begin{equation*}
I\left(R_{1}, R_{2}\right)=\int_{\widetilde{G} L(m, \mathbb{C})} \varphi(M) \exp \left(-\frac{1}{2} \operatorname{tr}\left(R_{1} M R_{2} M^{\prime}\right)\right) d \beta(M) \tag{4}
\end{equation*}
$$

where

$$
\begin{equation*}
\varphi(\mathrm{M})=|\mathrm{M}|^{\mathrm{N}} \exp \left(-\frac{1}{2} \operatorname{tr} M M^{\prime}\right) \quad, \quad \mathrm{M} \in \tilde{\mathrm{G}} \mathrm{~L}(\mathrm{~m}, \mathbb{C}) \tag{5}
\end{equation*}
$$

and $\beta$ is a Haar measure on $\widetilde{G} L(m, \mathbb{C})$ normalized such that the integral of $\varphi(\mathrm{M})$ over $\tilde{\mathrm{G}} \mathrm{L}(\mathrm{m}, \mathbb{C})$ with respect to $\beta$ is 1 .

In the present paper we obtain an explicit expression for $I\left(R_{1}, R_{2}\right)$.

To simplify (4) we consider matrices in $\mathscr{R}(m, \mathbb{C})$ of a special form: for real numbers $\lambda_{1}, \ldots, \lambda_{m}$ let $\Lambda$ denote the matrix of the form

$$
\left[\begin{array}{lc}
\Lambda(1) & 0  \tag{6}\\
0 & -\Lambda(1)
\end{array}\right]
$$

where $\Lambda(1)=\operatorname{diag}\left(\lambda_{1},-, \lambda_{m}\right)$. For every $R \in \mathscr{R}(m, \mathbb{C})$ there exists a $U \in \mathscr{U}(\mathrm{~m}, \mathbb{C})$ such that $\mathrm{URU}^{\prime}=\Lambda$, where $\Lambda$ has the form (6) (see Andersson and Perlman (1984)). Using that $\beta$ is a Haar measure it follows that we only have to consider $I(\Lambda, \Gamma)$, where $\Lambda$ and $\Gamma$ have the form (6).

The next step is to express $I(\Lambda, \Gamma)$ by means of an integral over $\mathscr{T}_{+}(\mathrm{m}, \mathbb{C})$, where $\mathscr{T}_{+}(\mathrm{m}, \mathbb{C})$ denotes the group of $2 \mathrm{~m} \times 2 \mathrm{~m}$ matrices of the form

$$
\left[\begin{array}{rr}
\mathrm{T}(1) & -\mathrm{T}(2)  \tag{7}\\
\mathrm{T}(2) & \mathrm{T}(1)
\end{array}\right]
$$

where $T(1) \in \mathscr{T}_{+}(m, \mathbb{R})$, the group of upper triangular matrices with positive diagonal elements and $T(2)$ is upper diagonal with zero diagonal elements.
 the right Haar measure on $\mathscr{T}_{+}(m, \mathbb{C})$ given by

$$
\frac{1}{c} \prod_{i=1}^{m}\left(t_{1, i \mathrm{i}}\right)^{1-2 i_{d T}}
$$

where $t_{1, i i}$ is a diagonal element of $T(1)$, and

$$
\mathrm{c}=2^{-\mathrm{m}} \pi_{\pi}^{\frac{\mathrm{m}(\mathrm{~m}-1)}{2}} \prod_{\mathrm{i}=1}^{\mathrm{m}} \Gamma(\mathrm{~N}-\mathrm{i}+1)
$$

Then the integral of $\varphi(\mathrm{T})$ over $\mathscr{T}_{+}(\mathrm{m}, \mathbb{C})$ w.r.t. $\mu$ is 1.
Consider now the Iwasawa decomposition , i.e. the one-to-one and onto mapping given by

$$
\begin{equation*}
(\mathrm{U}, \mathrm{~T}) \rightarrow \mathrm{UT} \quad, \quad \mathscr{U}(\mathrm{~m}, \mathbb{C}) \times \mathscr{T}_{+}(\mathrm{m}, \mathbb{C}) \rightarrow \widetilde{\mathrm{G}} \mathrm{~L}(\mathrm{~m}, \mathbb{C}) \tag{8}
\end{equation*}
$$

By this mapping $\beta$ is the transformed measure of $\alpha \otimes \mu$ (see Bourbaki, Chap, 7-8 (1963)) . Using this we can write $I(\Lambda, \Gamma)$ as

$$
\begin{equation*}
\int_{\mathscr{T}_{+}(\mathrm{m}, \mathbb{C})} \varphi(\mathrm{T}) \int_{\mathscr{U}(\mathrm{m}, \mathbb{C})} \exp \left(\frac{1}{2} \operatorname{tr}\left(\Gamma U \mathrm{~T} \wedge \mathrm{~T}^{\prime} \mathrm{U}^{\prime}\right)\right) \mathrm{d} \alpha(\mathrm{U}) \mathrm{d} \mu(\mathrm{~T}) \tag{9}
\end{equation*}
$$

Expanding $\exp \left(\frac{1}{2} \operatorname{tr}\left(\Gamma \cup T \wedge T^{\prime} U^{\prime}\right)\right)$ as a power series the integral above can be expressed as an infinite sum of terms of the form

$$
\begin{equation*}
\frac{1}{\mathrm{k}!} \int_{\mathscr{T}_{+}(\mathrm{m}, \mathbb{C})} \varphi(\mathrm{T}) \int_{\mathscr{U ( \mathrm { m } , \mathbb { C } )}}\left(\frac{1}{2} \operatorname{tr}\left(\Gamma \mathrm{U} \wedge \mathrm{~T}^{\prime} \mathrm{U}^{\prime}\right)\right)^{\mathrm{k}} \mathrm{~d} \alpha(\mathrm{U}) \mathrm{d} \mu(\mathrm{~T}) \tag{10}
\end{equation*}
$$

## 3. The polynomials $D_{\bar{k}}$

As a function of $\lambda_{1},-, \lambda_{m}$ the integral (10) is a homogeneous symmetric polynomial of degree $k$. We shall show that is possible to obtain an explicit expression for the integral by selecting a suitable basis for the homogeneous symmetric polynomials of degree k. Since the integral over $\mathscr{U}(\mathrm{m}, \mathbb{C})$ contains the term $T \Lambda T^{\prime}$ it will be convenient to define these polynomials as functions of a matrix $\operatorname{Re} \mathscr{R}(\mathrm{m}, \mathbb{C})$, instead of just a matrix of the form (6).

First let $P_{m}(k)$ be the set of ordered sequences $\bar{k}=\left(k_{1},-, k_{m}\right)$, where $k_{i} \in \mathbb{N} \cup\{0\}, k_{1} \geq k_{2} \geq \ldots \geq k_{m}$ and $\sum_{i=1}^{m} k_{i}=k$. An element of $P_{m}(k) \quad$ is also called a partition of $k$ in at most m parts.

In the appendix it is shown how it is possible to define, for each $\overline{\mathrm{k}} \in \mathrm{P}_{\mathrm{m}}(\mathrm{k})$, a polynomial $\mathrm{D}_{\overline{\mathrm{k}}}$, which is a homogeneous polynomial in the different elements of $\operatorname{Re} \mathscr{R}(m, \mathbb{C})$ and that these polynomials have the following essential properties

$$
\begin{align*}
& \mathrm{D}_{\overline{\mathrm{k}}}\left(\mathrm{URU}^{\prime}\right)=\mathrm{D}_{\overline{\mathrm{k}}}(\mathrm{R}) \quad \text { for all } \mathrm{U} \in \mathscr{U}(\mathrm{~m}, \mathbb{C})  \tag{11}\\
& \int_{\mathscr{T}_{+}(\mathrm{m}, \mathbb{C})} \varphi(\mathrm{T}) \mathrm{D}_{\overline{\mathrm{k}}}\left(\mathrm{TRT}^{\prime}\right) \mathrm{d} \mu(\mathrm{~T})=\mathrm{d}(\mathrm{~N}, \overline{\mathrm{k}}) \mathrm{D}_{\overline{\mathrm{k}}}(\mathrm{R}) \tag{12}
\end{align*}
$$

where

$$
\begin{gather*}
\mathrm{d}(\mathrm{~N}, \overline{\mathrm{k}})=\frac{\prod_{\mathrm{i}=1}^{\mathrm{m}} \Gamma\left(\mathrm{~N}+2 \mathrm{k}_{\mathrm{i}}-\mathrm{i}+1\right)}{\prod_{\mathrm{i}=1}^{\mathrm{m}} \Gamma(\mathrm{~N}-\mathrm{i}+1)}  \tag{13}\\
\int_{\mathscr{( m , \mathbb { C }})}\left(\operatorname{tr}\left(\mathrm{R}_{1} \mathrm{UR}_{2} \mathrm{U}^{\prime}\right)\right)^{2 \mathrm{k}} \mathrm{~d} \alpha(\mathrm{U})=\sum_{\overline{\mathrm{k}} \in \mathrm{P}_{\mathrm{m}}(\mathrm{k})}^{\sum} \mathrm{c}(\overline{\mathrm{k}}) \mathrm{D}_{\overline{\mathrm{k}}}\left(\mathrm{R}_{1}\right) \mathrm{D}_{\bar{k}}\left(\mathrm{R}_{2}\right) \tag{14}
\end{gather*}
$$

and

$$
\begin{equation*}
\int_{\mathscr{U}(\mathrm{m}, \mathbb{C})}\left(\operatorname{tr}\left(\mathrm{R}_{1} U \mathrm{R}_{2} \mathrm{U}^{\prime}\right)\right)^{2 \mathrm{k}+1} \mathrm{~d} \alpha(\mathrm{U})=0 \tag{15}
\end{equation*}
$$

where $R_{1}, R_{2} \epsilon \mathscr{R}(m, \mathbb{C})$ and $\alpha$ is the normed Haar measure on $\mathscr{U}(\mathrm{m}, \mathbb{C})$. The coefficients $c(\bar{k})$ are found in the next section.

From (12) and (13) it follows (see the appendix) that when $T$ is distributed on $\mathscr{T}_{+}(m, \mathbb{C})$ such that $T^{\prime}$ has a complex Wishart ( $\left.I_{m}, m\right)$ distribution, then

$$
\begin{equation*}
\mathrm{E}_{\mathrm{T}} \mathrm{D}_{\mathrm{k}}\left(\mathrm{TRT}^{\prime}\right)=\mathrm{d}(\mathrm{~m}, \overline{\mathrm{k}}) \mathrm{D}_{\overline{\mathrm{k}}}(\mathrm{R}) \tag{16}
\end{equation*}
$$

From (11) it follows that $D_{\bar{k}}(\Lambda)$ is a homogeneous symmetric polynomial in $\lambda_{1}, \ldots, \lambda_{m}$. In the appendix it is shown that the term with highest weight is $\lambda_{1}^{2 k_{1}} \ldots \lambda_{m}^{2 k_{m}}$. In the next section we shall see, using the properties (11) and (12), that $D_{\bar{k}}(\Lambda)$ can be expressed by some well-known polynomials. It is then clear that (11)-(15) make it possible to evaluate the integral (10).

## 4. Evaluation of the integral

Consider $2 m \times 2 m$ matrices $X=\left[\begin{array}{cc}Y & -Z \\ Z & Y\end{array}\right]$, where $Y$ and $Z$ are $m \times m$ matrices with elements $y_{i j}$ and $z_{i j}$. Assume that all the elements $y_{i j}$ and $z_{i j}$ are independent standard normal variables and define the generating function $g$ by

$$
\begin{equation*}
\mathrm{g}(\vartheta, \Gamma, \Lambda)=\mathrm{E}_{\mathrm{X}} \exp \left(\vartheta \operatorname{tr}\left(\Gamma \mathrm{X} \Lambda \mathrm{X}^{\prime}\right)\right) \tag{17}
\end{equation*}
$$

for $\vartheta$ sufficiently small, $\Gamma$ and $\Lambda$ having the form (6).
It is seen that

$$
\begin{equation*}
\operatorname{tr}\left(\Gamma X \Lambda X^{\prime}\right)=\sum_{i=1}^{m} \sum_{j=1}^{m} 2 r_{i} \lambda_{j}\left(y_{i j}^{2}-z_{i j}^{2}\right) \tag{18}
\end{equation*}
$$

and using this we get that

$$
\begin{equation*}
\mathrm{g}(\vartheta, \Gamma, \Lambda)=\prod_{\mathrm{i}=1}^{\mathrm{m}} \prod_{\mathrm{j}=1}^{\mathrm{m}}\left(1-4 \vartheta^{2} \gamma_{\mathrm{i}}^{2} \lambda_{\mathrm{j}}^{2}\right)^{-\frac{1}{2}} \tag{19}
\end{equation*}
$$

From Takemura (1984, page 37-39) we get that

$$
\begin{equation*}
\mathrm{g}(\vartheta, \Gamma, \Lambda)=\sum_{\mathrm{k}=0}^{\infty} \frac{\left(2 \vartheta^{2}\right)^{\mathrm{k}}}{\mathrm{k}!} \sum_{\mathrm{k} \in \mathrm{P}_{\mathrm{m}}(\mathrm{k})}^{\sum} \delta(\overline{\mathrm{k}})^{-1} \mathrm{C}_{\overline{\mathrm{k}}}\left(\underline{\lambda}^{2}\right) \mathrm{C}_{\overline{\mathrm{k}}}\left(\underline{\gamma}^{2}\right) \tag{20}
\end{equation*}
$$

where $\underline{\lambda}^{2}=\left(\lambda_{1}^{2}, \ldots, \lambda_{m}^{2}\right), \underline{r}^{2}=\left(r_{1}^{2}, \ldots, r_{m}^{2}\right)$.
$\mathrm{C}_{\overline{\mathrm{k}}}$ are the real zonal polynomials normalised such that

$$
\begin{equation*}
\left(\lambda_{1}+\ldots+\lambda_{m}\right)^{k}=\sum_{k \in P_{m}(k)} C_{\bar{k}}(\underline{\lambda}) \tag{21}
\end{equation*}
$$

and

$$
\begin{equation*}
\delta(\bar{k})=\frac{2^{k} k!\prod_{i<j}^{m} \frac{\left(2 k_{i}-2 k_{j}-i+j\right)}{\prod_{i=1}^{m}}\left(2 k_{i}+m-i\right)!}{} \tag{22}
\end{equation*}
$$

We get another expression for $g$ by expanding (17) in a sum of terms of the form

$$
\begin{equation*}
(\mathrm{k}!)^{-1} \vartheta^{\mathrm{k}} \mathrm{E}_{\mathrm{X}}\left(\left(\operatorname{tr}\left(\Gamma \mathrm{X} \Lambda \mathrm{X}^{\prime}\right)\right)^{\mathrm{k}}\right) \tag{23}
\end{equation*}
$$

 $T$ is distributed on $\mathscr{T}_{+}(\mathrm{m}, \mathbb{C})$ such that $\mathrm{W}=\mathrm{TT}$. has the complex Wishart ( $I_{m}, m$ ) distribution. By (14) - (16) we get that $g(z, \Gamma, \Lambda)$ is a sum of terms of the form

$$
\begin{equation*}
(2 \mathrm{k})!^{-1} \vartheta^{2 \mathrm{k}}{\underset{\mathrm{k} \in \mathrm{P}_{\mathrm{m}}(\mathrm{k})}{\Sigma} \mathrm{d}(\mathrm{~m}, \overline{\mathrm{k}}) \mathrm{c}(\overline{\mathrm{k}}) \mathrm{D}_{\overline{\mathrm{k}}}(\underline{\lambda}) \mathrm{D}_{\overline{\mathrm{k}}}(\underline{\gamma}) . . . . .} \tag{24}
\end{equation*}
$$

Since the term of highest weight in both $D_{\bar{k}}(\underline{\lambda})$ and $C_{\bar{k}}\left(\underline{\lambda}^{2}\right)$ is of the form $\alpha \cdot \lambda_{1}^{2 k_{1}} \ldots \lambda_{m}^{2 k_{m}}$ it follows by comparing (20) with (24) that $D_{\bar{k}}(\underline{\lambda})$ and $\mathrm{C}_{\overline{\mathrm{k}}}\left(\underline{\lambda}^{2}\right)$ are proportional.

$$
\begin{equation*}
\mathrm{D}_{\overline{\mathrm{k}}}\left(\lambda_{1}, \ldots, \lambda_{\mathrm{m}}\right)=\alpha(\overline{\mathrm{k}}) \mathrm{C}_{\overline{\mathrm{k}}}\left(\lambda_{1}^{2}, \ldots, \lambda_{\mathrm{m}}^{2}\right) . \tag{25}
\end{equation*}
$$

Again by comparing (20) with (24)

$$
\begin{equation*}
\mathrm{c}(\overline{\mathrm{k}})=\frac{2^{\mathrm{k}}(2 \mathrm{k})!}{\mathrm{k}!} \frac{1}{\alpha(\overline{\mathrm{k}})^{2} \delta(\overline{\mathrm{k}}) \mathrm{d}(\mathrm{~m}, \overline{\mathrm{k}})} . \tag{26}
\end{equation*}
$$

We are now able to give a series expansion for $I(\Gamma, \Lambda)$ given by (4) Using (12)-(15) on (10) we get

Theorem Let $I(\Gamma, \Lambda)$ be given by (4) then

$$
\begin{equation*}
\mathrm{I}(\Gamma, \Lambda)=\sum_{\mathrm{k}=0}^{\infty} \frac{1}{(\mathrm{k}!)^{2}} \frac{1}{2^{2 \mathrm{k}}} \sum_{\mathrm{k} \in \mathrm{P}_{\mathrm{m}}(\mathrm{k})}^{\sum} \mathrm{q}(\overline{\mathrm{k}}) \mathrm{C}_{\overline{\mathrm{k}}}\left(\underline{\lambda}^{2}\right) \mathrm{C}_{\overline{\mathrm{k}}}\left(\underline{\gamma}^{2}\right) \tag{27}
\end{equation*}
$$

where the polynomials, $C_{\bar{k}}$, are the real zonal polynomials normalized by (21) and

$$
\begin{equation*}
q(\bar{k})=\frac{\prod_{i=1}^{m} \Gamma\left(N+2 k_{i}-i+1\right)}{\prod_{i=1}^{m} \Gamma(m-i+1)} \tag{28}
\end{equation*}
$$

■

Some manipulation with the coefficients $q(\bar{k})$ (see the appendix) show that

$$
\begin{equation*}
\mathrm{I}(\Gamma, \Lambda)={ }_{2} \mathrm{~F}_{1}\left(\frac{\mathrm{~N}}{2}, \frac{\mathrm{~N}}{2}+\frac{1}{2}, \frac{\mathrm{~m}}{2}+\frac{1}{2} ; \underline{\lambda}^{2}, \underline{r}^{2}\right) \tag{29}
\end{equation*}
$$

where ${ }_{2} \mathrm{~F}_{1}$ is the hypergeometric function as defined in James (1964), page 477 .
5. Appendix.

Definition of the polynomials $\mathrm{D}_{\overline{\mathrm{k}}}$.
Let $V(k)$ be the real vector space of homogeneous polynomials $f(R)$ of degree $k$ in the $m(m+1)$ different elements of $R \in \mathscr{R}(m, \mathbb{C})$.

For each $M \in \widetilde{G} L(m, \mathbb{C})$ a transformation $D(L)$ of $\mathscr{R}(m, \mathbb{C})$ (as a vector space) is defined by

$$
\begin{equation*}
D(M)(R)=M R M^{\prime} \tag{30}
\end{equation*}
$$

The transformation $D(L)$ defines a representation, $D$, of $\widetilde{G} L(m, \mathbb{C})$ on $\mathscr{R}(m, \mathbb{C})$. If we let $\operatorname{GL}(m, \mathbb{C})$ denote the group of non-singular $m \times m$ matrices with elements from $\mathbb{C}$ then (30) also defines a representation of $\operatorname{GL}(m, \mathbb{C}) \quad$ (using the group isomorphism $\widetilde{G} L(m, \mathbb{C}) \rightarrow \operatorname{GL}(m, \mathbb{C})$ see Andersson and Perlman (1984)). It is easy to see by considering the trace of $D(L)$ that $D$ is equivalent to the semi-rational representation $M_{2}(M) \otimes M_{2}(\bar{M})$ of $G L(m, \mathbb{C})$, where $M_{2}$ is the irreducible integral representation of $\operatorname{GL}(\mathrm{m}, \mathbb{C})$ corresponding to the partition $(2,0)$ of 2 (see Boerner (1955)).

For each $M \in \widetilde{G} L(m, \mathbb{C})$ a transformation $T(M)$ of $V(k)$ is defined by

$$
\begin{equation*}
f \rightarrow T(M) f \quad(T(M) f)(R)=f\left(M^{-1} R^{-1}{ }^{\prime}\right) . \tag{31}
\end{equation*}
$$

These transformations define a representation,$T$, of $\widetilde{G} L(m, \mathbb{C})$ on $\mathrm{V}(\mathrm{k})$, and again T can be considered as a representation of $\mathrm{GL}(\mathrm{m}, \mathbb{C})$ on $V(k)$.

For $\overline{\mathrm{k}} \in \mathrm{P}_{\mathrm{m}}(\mathrm{k})$ we let $2 \overline{\mathrm{k}}=\left(2 \mathrm{k}_{1},-, 2 \mathrm{k}_{\mathrm{m}}\right) \in \mathrm{P}_{\mathrm{m}}(2 \mathrm{k})$.
It follows from Thrall (1942, Lemma I page 377 and Theorem III page 378) that $T$ decomposes into the irreducible representations of $\operatorname{GL}(m, \mathbb{C})$
of the form $M_{2 p}-(M) \otimes M_{2 q}-(M)$, each of which is contained exactly once, and $\bar{p} \in P_{m}(p), \bar{q} \in P_{m}(q)$ with $p+q=k$. Let $V(\bar{p}, \bar{q})$ be the invariant irreducible subspace of $V(k)$ in which $M_{2-1}(M) \otimes M_{2}-(M)$ acts. In particular we have that

Lemma 2 By the representation $T$ defined by (31) $V(k)$ decomposes into a direct sum of irreducible invariant subspaces, none of which is equivalent.
$T$ with $M$ restricted to be orthogonal defines a representation of $\mathscr{U}(\mathrm{m}, \mathbb{C})$ on $V(\bar{p}, \bar{q})$ : by this representation $V(\bar{p}, \bar{q})$ decomposes into a direct sum of irreducible invariant subspaces $V(\bar{p}, \bar{q}, i), i=1,-n(\bar{p}, \bar{q})$. By comparing with the form of the similar representations corresponding to testing equality of two covariance matrices of complex form it follows that if and only if $\bar{p}=\bar{q}$ then exactly one of the subspaces $V(\bar{p}, \bar{q}, i)$, say $V(\bar{p}, \bar{p}, 1)$ has the following property: it is one-dimensional and the corresponding representation of $\mathscr{U}(\mathrm{m}, \mathbb{C})$ is the identity representation. Since we are only interested in spaces with this properrty we will now only consider the spaces $V(2 k)$. We remark that $f \in V(\bar{k}, \bar{k}, 1)$ implies that

$$
f\left(U R U^{\prime}\right)=f(R) \quad \text { for all } U \in \mathscr{U}(m, \mathbb{C})
$$

i.e. that $f$ is $\mathscr{U}(\mathrm{m}, \mathbb{C})$-invariant. We then have

Lemma 3 To each $\overline{\mathrm{k}} \in \mathrm{P}_{\mathrm{m}}(\mathrm{k})$ there exists a unique one dimensional subspace $V(\bar{k}, \bar{k}, 1)$ invariant under $\mathscr{U}(m, \mathbb{C})$ which is contained in one of the subspaces given by the decomposition of $\mathrm{V}(2 \mathrm{k})$ in Lemma 2. There are no other one dimensional subspaces invariant under $\mathscr{U}(\mathrm{m}, \mathbb{C})$ with this property.

By Lemma 1 a $\mathscr{U}(\mathrm{m}, \mathbb{C})$-invariant polynomial, f , is given by its values on matrices, $\Lambda$, of the form (6). Such a polynomial $f$ of $\Lambda$ is a homogeneous symmetric polynomial of degree 2 k in $\lambda_{1},-\lambda_{m}$.

Using a method similar to that of Constantine (1963 page 1271-1273) it can be shown that a polynomial, $f$, which generates $V(\bar{k}, \bar{k}, 1)$ has the form

$$
\mathrm{f}(\Lambda)=\mathrm{d}(\overline{\mathrm{k}}) \lambda_{1}^{2 \mathrm{k}_{1}} \ldots \lambda_{\mathrm{m}}^{2 \mathrm{k}_{\mathrm{m}}}+\text { terms of lower weight. }
$$

Definition $D_{\bar{k}}$ is the polynomial which generates $V(\bar{k}, \bar{k}, 1)$ normed such that the coefficient to the term with highest weight is 1 .

Proof (12)-(16) Consider the transformation, E, from $V(2 k)$ to $V(2 k)$ given by

$$
\begin{equation*}
E f(R)=\int_{\widetilde{G} L(\mathrm{~m}, \mathbb{C})} \varphi(\mathrm{M}) \mathrm{f}\left(\mathrm{MRM}^{\prime}\right) \mathrm{d} \beta(\mathrm{M}) \tag{32}
\end{equation*}
$$

Since $V(\bar{k}, \bar{k})$ is invariant under $T(M)$ for each $M$ it follows that Ef $\in V(\bar{k}, \bar{k})$ when $f \in V(\bar{k}, \bar{k})$. Using the invariance of $\beta$ it is seen that $E f$ is $थ(m, \mathbb{C})$-invariant. In particular we get that $E D_{\bar{k}}$ is proportional to $D_{\bar{k}}$, and note that since $D_{\bar{k}}$ is $\mathscr{U}(\mathrm{m}, \mathbb{C})$-invariant, we have by the Iwasawa-decomposition that $E D_{\bar{k}}(R)$ is given by the left side of (12).

To evaluate $d(N, \bar{k})$ we first remark that for $T \in \mathscr{T}_{+}(m, \mathbb{C})$ and $\Lambda$ of the form (6) the term of highest weight in $\mathrm{D}_{\overline{\mathrm{k}}}\left(\mathrm{T} \Lambda \mathrm{T}^{\prime}\right)$ becomes

$$
\begin{equation*}
\lambda_{1}^{2 \mathrm{k}_{1}} \ldots \lambda_{\mathrm{m}}^{2 \mathrm{k}_{\mathrm{m}}} \cdot \mathrm{~g}(\mathrm{~T}) \tag{33}
\end{equation*}
$$

where $\mathrm{g}(\mathrm{T})=\left(\mathrm{t}_{1,11}^{2}\right)^{2 \mathrm{k}_{1}-2 \mathrm{k}_{2}}\left(\mathrm{t}_{1,11}^{2} \mathrm{t}_{1,22}^{2}\right)^{2 \mathrm{k}_{2}-2 \mathrm{k}_{3}} \ldots\left(\left|\mathrm{~T}_{1}\right|^{2}\right)^{2 \mathrm{k}_{\mathrm{m}}} \quad$ (proceed as in Constantine (1963), page 1273).

By comparing the coefficients of the terms of highest weight on both sides of (12) we get

$$
\int_{\mathscr{T}_{+}(\mathrm{m}, \mathbb{C})} \varphi(\mathrm{T}) \mathrm{g}(\mathrm{~T}) \mathrm{d} \mu(\mathrm{~T})=\mathrm{d}(\mathrm{~N}, \overline{\mathrm{k}})
$$

and a direct calculations gives (13).

Now assume that $T$ is distributed on $\mathscr{T}_{+}(\mathrm{m}, \mathbb{C})$ such that $\mathrm{W}=\mathrm{TT}$ ' has a complex Wishart ( $I_{m}, m$ ) distribution, i.e. distribution with density

$$
\mathrm{a}(\mathrm{~m})^{-1} \exp \left(-\frac{1}{2} \operatorname{tr}(\mathrm{~W})\right) \mathrm{dW}
$$

where

$$
a(m)=\pi^{\frac{m(m-1)}{2}} \prod_{i=1}^{m} \Gamma(m+1-i)
$$

The mapping $\mathrm{T} \rightarrow \mathrm{TT}$ ' has the Jacobian

$$
2^{\mathrm{m}}|\mathrm{~T}|^{\mathrm{m}} \prod_{\mathrm{i}=1}^{\mathrm{m}} \mathrm{t}_{1, \mathrm{i} i}^{1-2 \mathrm{i}}
$$

and (16) follows using this and (12)-(13). An application of Schurs lemma (see Naimark and Stern (1982), page $26+58$ ) and the fact that

$$
\int_{U(\mathrm{~m}, \mathbb{C})} T(U) \mathrm{d} \alpha(\mathrm{U})=0
$$

when $T$ is an irreducible representation different from the identity representation of $\mathscr{U}(\mathrm{m}, \mathbb{C})$. give (14) and (15).

Proof of (29). Use the identities

$$
\begin{gather*}
\mathrm{C}_{\mathrm{k}}(1, \ldots, 1)=2^{2 \mathrm{k}_{\mathrm{k}}!} \frac{\prod_{\mathrm{i}=1}^{\mathrm{m}} \Gamma\left(\frac{1}{2} \mathrm{~m}-\frac{1}{2}(\mathrm{i}-1)+\mathrm{k}_{\mathrm{i}}\right)}{\prod_{\mathrm{i}=1}^{\mathrm{m}} \Gamma\left(\frac{1}{2} \mathrm{~m}-\frac{1}{2}(\mathrm{i}-1)\right)} \frac{\prod_{\mathrm{i}\langle\mathrm{j}}\left(2 \mathrm{k}_{\mathrm{i}}-2 \mathrm{k}_{\mathrm{j}}-\mathrm{i}+\mathrm{j}\right)}{\prod_{\mathrm{i}=1}^{m}\left(2 \mathrm{k}_{\mathrm{i}}+\mathrm{m}-\mathrm{i}\right)!}  \tag{34}\\
{[\mathrm{a}]_{2 \overline{\mathrm{k}}}=\left[\frac{\mathrm{a}}{2}\right]_{\bar{k}}\left[\frac{\mathrm{a}}{2}+\frac{1}{2}\right]_{\bar{k}}} \tag{35}
\end{gather*}
$$

where $a \in \mathbb{N}$, and $[a]_{\bar{k}},\left[\frac{a}{2}\right]_{\bar{k}}$ is defined as in (James 1964, page 477 and 487)

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