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HEADNOTE

The purpose of this paper is to give a systematic account of the maximum likelihood inference concerning cointegration vectors in non-stationary vector valued autoregressive time series with Gaussian errors. The hypothesis of r cointegration vectors is given a simple parametric formulation in terms of cointegration vectors and their weights. We then estimate and test linear hypotheses about these. We find that the asymptotic inference for the linear hypotheses can be performed by applying the usual $\chi^2$ test. We also give some very simple Wald test and their asymptotic properties. The methods are illustrated by data from the Danish and the Finnish economy on the demand for money.

Keywords: Cointegration, Error correction, maximum likelihood estimation, likelihood ratio test, vector autoregressive processes, money demand.
1. Introduction


The solution we propose to this problem is to start with a relatively simple model specifying a vectorvalued autoregressive process (VAR) with independent Gaussian errors, and formulate the hypothesis of reduced rank or the hypothesis of the existence of cointegration vectors in a simple parametric form which allows the application of the method of maximum likelihood and likelihood ratio tests. In this way we can derive estimates and test statistics for the hypothesis of of a given number of cointegration vectors, as well as estimates and tests for linear hypotheses about the cointegration vectors and their weights. We have also derived the asymptotic properties of these statistics, and illustrated the methods by money demand data from the Danish and Finnish economy.
We will consider the case where the observed data is a sequence of random vectors $X_t$ with components $(X_{1t}, \ldots, X_{pt})$ drawn sequentially from a $p$-dimensional Gaussian distribution with mean $\mu_t$ and variance matrix $\Lambda$, where $\mu_t$ depends linearly on the past $k$ values of the process.

Thus we consider the model

\[(1.1) \quad H_1: \quad X_t = \Pi_1 X_{t-1} + \cdots + \Pi_k X_{t-k} + \varepsilon_t, \quad (t = 1, \ldots, T)\]

where $\varepsilon_1, \ldots, \varepsilon_T$ are i.i.d. $N_p(0, \Lambda)$ and $X_{-k+1}, \ldots, X_0$ are fixed.

The unrestricted parameters $(\Pi_1, \ldots, \Pi_k, \Lambda)$ have to be estimated on the basis of $T$ observations from a vector autoregressive process.

In general economic time series are non-stationary processes, and VAR-systems like (1.1) have usually been expressed in first differenced form. Unless the difference operator is also applied to the error process and explicitly taken account of, differencing implies loss of information in the data. Using $\Delta = 1-L$, where $L$ is the lag operator it is convenient to rewrite the model (1.1) as

\[(1.2) \quad \Delta X_t = \Gamma_1 \Delta X_{t-1} + \cdots + \Gamma_{k-1} \Delta X_{t-k+1} - \Pi X_{t-k} + \varepsilon_t \]

where

\[\Gamma_i = -I + \Pi_1 + \cdots + \Pi_i, \quad (i = 1, \ldots, k-1)\]

and

\[(1.3) \quad \Pi = I - \Pi_1 - \cdots - \Pi_k.\]

Notice that model (1.2) is expressed as a traditional first differenced VAR-model except for the term $\Pi X_{t-k}$. It is the main purpose of this paper to investigate the coefficient matrix $\Pi$ as to the information it may convey concerning long-run information in the chosen data. Three cases can be considered:
(i) Rank(\(\Pi\)) = p, i.e. the matrix \(\Pi\) has full rank, indicating that the vector process \(X_t\) is stationary.

(ii) Rank(\(\Pi\)) = 0, i.e. the matrix \(\Pi\) is the null vector and (1.2) corresponds to a traditional differenced vector time series model.

(iii) \(0 < \text{rank}(\Pi) = r < p\) implying that there are \(p \times r\) matrices \(\alpha\) and \(\beta\) such that \(\Pi = \alpha \beta'\).

The cointegration vectors \(\beta\) have the property that \(\beta'X_t\) is stationary even though \(X_t\) is self is non-stationary, see Theorem 3.1 for a precise formulation. In this case (1.2) can be interpreted as an error correction model, see Engle and Granger (1987), Davidson (1986) or Johansen (1988a).

Thus the main hypothesis we shall consider here is

\[(1.4) \quad H_2: \Pi = \alpha \beta'.\]

where \(\alpha\) and \(\beta\) are \(p \times r\) matrices. This can also be formulated as the condition that \(\text{rank}(\Pi) \leq r\).

We shall further investigate linear hypotheses expressed in terms of the coefficients \(\alpha\) and \(\beta\).

We have chosen to illustrate the procedures by data from the Danish and Finnish economy on the demand for money. The relation \(m = f(y, p, c)\) expresses money demand \(m\) as a function of income \(y\), the price level \(p\) and the cost of holding money \(c\). Since price homogeneity was clearly accepted by the data the empirical analysis here will be on real money, real income and some proxies measuring the cost of holding money. All variables are expressed in logarithmic form, since multiplicative effects are
The two data sets differ both as to which variables are included and the length of the sample. More interestingly, however, the institutional relations in the two economies have been quite different in the sample period. In Denmark the market forces have been allowed to play much more freely than in Finland, where both interest rates and prices have been subject to regulation for most of the sample period. One would expect this to show up in the empirical results and so it does.

For the Danish data $m_2$ was chosen because the data available on quarterly basis has been collected using more homogeneous definitions for $m_2$ than for $m_1$. The cost of holding money was measured by the difference between the bank deposit rate, $i^d$, for interest bearing deposits (which are the main part of $m_2$) and the bond rate, $i^b$, which plays a very important role in the Danish economy. The two interest rates were included unrestrictedly in the analysis, but subsequently tested for equal coefficients with opposite signs. The inflation rate, $\Delta p$, was also included as a possible proxy for the cost of holding money, but since it did not enter significantly into the cointegration relation for money demand it was omitted from the present analysis.

For the Finnish data $m_1$ was chosen, since the $m_1$ cointegration relation was found to enter the demand for money equation more significantly and hence illustrated the methodology better. Since the interest rates have been regulated a good proxy for the actual costs of holding money is difficult to find. The inflation rate, $\Delta p$, is a natural candidate and therefore included in the data set. Moreover, the marginal rate of interest, $i^m$, of the Bank of Finland is included in spite of the fact...
that the marginal rate measures the restrictedness of money rather than the cost of holding money. It has however been chosen as a determinant of the Finnish money demand in other studies and therefore is also included here. All series are quarterly and the data is given in Appendix A.

The structure of the paper is the following. In Section 2 we give the derivation of the maximum likelihood estimators and likelihood ratio tests expressed in terms of eigenvectors and eigenvalues of suitable product moment matrices. The results are illustrated by the above mentioned data, and the interpretation of the results is discussed. In Section 3 we have then given a systematic account of the statistical and probabilistic results that are necessary to justify the analysis as described in section 2.

2. Maximum likelihood estimation and likelihood ratio tests of cointegration vectors

2.1. A survey of the various hypotheses and the initial analysis of the data

In the following we will use model (1.1) in the form (1.2):

\begin{equation}
\Delta X_t = \Gamma_1 \Delta X_{t-1} + \cdots + \Gamma_{k-1} \Delta X_{t-k+1} - \Pi X_{t-k} + \epsilon_t
\end{equation}

The reason for this is that the parameters

\((\Gamma_1, \ldots, \Gamma_{k-1}, \Pi, \lambda)\)

are variation independent and, since all the models we are interested in are expressed as restrictions on \(\Pi\), it is possible to maximize over all
the other parameters once and for all. The hypotheses we shall consider
and discuss in this section are given as follows

\[ H_2: \Pi = \alpha \beta', \]
\[ H_3^*: \Pi = \alpha \varphi' H' \text{ or } \beta = H \psi, \]
\[ H_3: \Pi = \Lambda \psi \beta' \text{ or } \alpha = \Lambda \psi, \]
\[ H_4: \Pi = \Lambda \psi \varphi' H' \text{ or } \beta = H \psi \text{ and } \alpha = \Lambda \psi. \]

The matrices \( A(p \times m) \) and \( H(p \times s) \) are known and define the restrictions on the parameters \( \alpha(p \times r) \) and \( \beta(p \times r) \). The restrictions reduce the parameters to \( \varphi(s \times r) \) and \( \psi(m \times r) \), where \( r \leq s \leq p \) and \( r \leq m \leq p \).

Note that \( H_4 = H_3^* \cap \tilde{H}_3 \) and that \( H_3^* \subset H_2 \) and \( \tilde{H}_3 \subset H_2 \). In fact all hypotheses are special cases of \( H_4 \) if we choose either \( A \) or \( H \) as the identity matrix. Thus it is to be expected that the analysis of these models are similar in nature, and this is what we hope to demonstrate in this section. The relation between the various hypotheses are illustrated in Figure 1.

[Figure 1.]

All these hypotheses are restrictions of the matrix \( \Pi \) which under \( H_1 \) contains \( p^2 \) parameters. Under the hypothesis \( H_2 \) there are \( pr + (p-r)r \) parameters which are further restricted to \( sr + (p-r)r \) under \( H_3^* \) and \( mr + (p-r)r \) under \( \tilde{H}_3 \). Finally \( mr + (s-r)r \) parameters remain under \( H_4 \). Note also that the parameters \( \alpha \) and \( \beta \) are not identified in the sense that given any choice of \( \alpha \) and \( \beta \), then for any non-singular matrix \( \xi(r \times r) \), the choice \( \alpha \xi \) and \( \beta(\xi')^{-1} \) will give the same matrix \( \Pi \), and hence determine the same probability distribution for the variables. One way of expres-
sing this is to say that what the data can determine is the space spanned by the columns in $\beta$, the cointegration space, and the space spanned by $\alpha$. It is of course possible to normalize $\beta$, for example by choosing one of the coefficients to unity, but this would be correct only if we know apriori that the corresponding variable enters the cointegration relation with a non-zero coefficient. Since we are here mainly interested in drawing conclusions about which cointegration vectors are present in the data and since the hypotheses we have formulated above do not depend on any normalization of $\beta$ we will avoid arbitrary normalizations, except in the case $r = 1$, where the interpretation is reasonably clear.

Note also that for each value of $r$ ($0 \leq r \leq p$) there is a corresponding hypothesis $H_2(r)$. The analysis which follows the procedures in Johansen (1988b) makes it possible to make inference about the value of $r$ by testing $H_2(r)$ in $H_1$.

We shall now turn to the maximum likelihood estimation of the parameters in the unrestricted model (2.1). These results are all well known but we shall give the formulae here mainly to establish the notation, since it will be useful for the discussion of the properties of the estimators and tests later. We shall not include fixed regressors in the model, but the theory can easily be modified to cover this case too. In the example we shall use a constant term and seasonal dummies.

For fixed value of $r$ the maximum likelihood estimation consists of a regression giving the normal equations

$$
(2.2) \quad \sum_{t=1}^{T} \Delta X_{t} \Delta X_{t-i} = \sum_{t=1}^{T} \Delta X_{t-j} \Delta X_{t-i-j} - \Pi_{i} \sum_{t=1}^{T} X_{t-k} \Delta X_{t-i}, \quad (i = 1, \ldots, k-1).
$$
Now introduce the notation for the product moment matrices 

\begin{align}
(2.3) \quad M_{ij} &= T^{-1} \sum_{t=1}^{T} AX_{t-1} \Delta X_{t-j} \quad (i,j = 0, \ldots, k-1) \\
(2.4) \quad M_{ki} &= T^{-1} \sum_{t=1}^{T} X_{t-k} \Delta X_{t-i} \quad (i = 0, \ldots, k-1) \\
\end{align}

and 

\begin{align}
(2.5) \quad M_{kk} &= T^{-1} \sum_{t=1}^{T} X_{t-k} X_{t-k}^\prime \\
\end{align}

and let \( M^\ast \) denote the matrix with elements \( M_{ij} \quad (i,j = 1, \ldots, k-1) \). Similarly let \( M_{kk}^\ast \) denote the matrix with elements \( M_{ij}^\ast \quad (j = 1, \ldots, k-1) \). Then (2.2) can be written as 

\begin{equation}
(2.6) \quad M_{0i} = \sum_{j=1}^{k-1} \Pi_{j} M_{ji} - \Pi M_{ki} \\
\end{equation}

and can be solved for \( \Pi \) to give 

\begin{equation}
(2.6) \quad \hat{\Pi}_{j} = \sum_{i=1}^{k-1} M_{0i} M_{ij} + \Pi \sum_{j=1}^{k-1} M_{kj} M_{ij} = M_{0i} M_{ij} + \Pi M_{kj} M_{ij} \\
\end{equation}

This leads to the definition of the residuals 

\begin{align}
(2.7) \quad R_{0t} &= AX_{t} - \sum_{j=1}^{k-1} M_{0j}^\ast \Delta X_{t-j} \quad \text{i.e. the residuals we would obtain by regressing } AX_{t} \text{ on } AX_{t-1}, \ldots, AX_{t-k+1} \\
(2.8) \quad R_{kt} &= X_{t-k} - \sum_{j=1}^{k-1} M_{kj}^\ast \Delta X_{t-j} \\
\end{align}

The concentrated likelihood function becomes 

\begin{equation}
(2.9) \quad |\Lambda|^{-T/2} \exp\{-\sum_{t=1}^{T} (R_{0t} + \Pi R_{kt})' \Lambda^{-1} (R_{0t} + \Pi R_{kt})/2\}. \\
\end{equation}

We can now express the estimates under the model \( H_1 \) by introducing the notation 

\begin{equation}
(2.10) \quad S_{ij} = T^{-1} \sum_{t=1}^{T} R_{it} R_{jt}' = M_{ij} - M_{ik} M_{kj}^{-1} M_{ij} \quad (i,j = 0,k) \\
\end{equation}

We shall formulate these well known results in
THEOREM 2.1: In the model

\[ H_1: \Delta X_t = \sum_{j=1}^{k-1} \Gamma_j \Delta X_{t-j} - \Pi X_{t-k} + \epsilon_t, \]

the parameters are estimated by ordinary least squares and we get

(2.11) \[ \hat{\Pi} = S_{0k} S_{kk}^{-1}, \]

and

(2.12) \[ \hat{\Lambda} = S_{00} - S_{0k} S_{kk}^{-1} S_{k0}, \]

(2.13) \[ L_{max}^{-2/T}(H_1) = |\hat{\Lambda}|. \]

The estimate of \( \Pi \) should then be inserted into (2.6) to get the estimate of \( \Gamma_j. \)

We shall now apply model (2.1) including constant term and seasonal dummies to the Danish and Finnish money demand data described in the introduction. It was found that for \( k = 2 \) the residuals for the Danish data clearly passed the test for being uncorrelated. For the Finnish data the test statistic for the residuals in the equation for \( \Delta y \) is almost significant. Looking at the autocorrelogram, see Appendix B, gives the impression that there is some seasonality left in the residuals, but since it is rather small we have chosen to ignore this. Accordingly the model (2.1) with \( k = 2 \) was fitted to both data sets. After conditioning on the two first data realizations the number of observations left for estimation was 46 in the Danish and 104 in the Finnish data. The estimates \( \Gamma_1, \Pi \) and \( \Lambda \) are given in Appendix B together with some of the details of the statistical analysis. Here we shall give only the estimates of \( \Pi. \)
2.2. **Derivation of the estimate of $\alpha$ and $\beta$ under the hypothesis $H = \alpha \beta'$**

as well the likelihood ratio test for this hypothesis

Now consider the model $H_2$, where $H = \alpha \beta'$. The estimation of $\Gamma_1, \ldots, \Gamma_{k-1}$ is the same as before leading to (2.9). For fixed $\beta$ it is now easy to estimate $\alpha$ and $A$ by regressing $R_{0t}$ on $\beta'X_{t-k}$ and obtain

\begin{align*}
\hat{\alpha}(\beta) &= -S_{0k}\beta(\beta'S_{kk}\beta)^{-1} \\
\hat{A}(\beta) &= S_{00} - S_{0k}\beta(\beta'S_{kk}\beta)^{-1}\beta'S_{k0}.
\end{align*}

Further we get

\begin{align*}
L_{\max}^{-2/T}(\beta) &= |\hat{A}(\beta)| = |S_{00} - S_{0k}\beta(\beta'S_{kk}\beta)^{-1}\beta'S_{k0}|.
\end{align*}

It was shown in Johansen (1988b) how one proceeds to estimate $\beta$ by applying the identity

\begin{align*}
|S_{00} - S_{0k}\beta(\beta'S_{kk}\beta)^{-1}\beta'S_{k0}| &= |S_{00}| |\beta'S_{kk}\beta - \beta'S_{k0}S_{00}^{-1}S_{0k}\beta|/|\beta'S_{kk}\beta|,
\end{align*}

which is easily minimized among all $p \times r$ matrices $\beta$. We shall formulate the results in

**THEOREM 2.2:** Under the hypothesis

$H_2$: $H = \alpha \beta'$.

the maximum likelihood estimator of $\beta$ is found by the following procedure: First solve the equation

\begin{align*}
|\lambda S_{kk} - S_{k0} S_{00}^{-1} S_{0k}| &= 0,
\end{align*}

giving the eigenvalues $\lambda_1 > \ldots > \lambda_p$ and eigenvectors $\hat{V} = (\hat{v}_1, \ldots, \hat{v}_p)$ normalized such that $\hat{V}'S_{kk}\hat{V} = I$. The choice of $\hat{\beta}$ is now
(2.19) \( \hat{\beta} = (\hat{\nu}_1, \ldots, \hat{\nu}_r) \),

which gives

(2.20) \( L_{\max}^{-2/T} (H_2) = |S_{00}|^{1/r} \Pi (1 - \hat{\lambda}_i) \).

The estimates of the other parameters are found by inserting \( \hat{\beta} \) into the above equations. In particular we find

(2.21) \( \hat{\alpha} = - S_{0k} \hat{\beta} \),

(2.22) \( \hat{\Pi} = - S_{0k} \hat{\beta}' \),

(2.23) \( \hat{\Lambda} = S_{00} - \hat{\alpha} \hat{\alpha}' \).

We can now immediately write down the likelihood ratio test statistic for the hypothesis \( H_2 \) in \( H_1 \), since \( H_1 \) is a special case of \( H_2 \) for the choice \( r = p \).

(2.24) \( -2 \ln(Q; H_2 | H_1) = -T \sum_{i=r+1}^{p} \ln(1 - \hat{\lambda}_i) \).

It was shown in Johansen (1988b) that the estimates of \( \Pi \) and \( \Lambda \) were consistent and that the asymptotic distribution of the test statistic can be tabulated by simulation.

Remark. Many computer packages contain procedures for solving the eigenvalue problem

\[ |\lambda I - A| = 0, \]

where \( A \) is symmetric. One can easily reduce (2.18) to this problem by first decomposing \( S_{kk} = CC' \) for some non-singular \( p \times p \) matrix \( C \). Now (2.18) is equivalent to

\[ |\lambda I - C^{-1} S_{kk}^{-1} S_{00}^{-1} S_{0k} C^{-1}| = 0, \]

which has the same eigenvalues \( \hat{\lambda}_1 > \ldots > \hat{\lambda}_p \) but eigenvectors \( e_1, \ldots, e_p \), where \( \hat{\nu}_i = C^{-1} e_i \).
Let us now apply these results to the Danish data. The matrices \( S_{00}, S_{20} \) and \( S_{22} \) are given in Appendix C together with \( S_{20} S_{00}^{-1} S_{02} \), and the eigenvalues and eigenvectors are given in Table II. Note that the calculation of all the eigenvectors allows one to estimate \( \alpha \) and \( \beta \) for any value of \( r \).

**[TABLE II]**

**[TABLE III]**

We shall now make inference about the number of cointegration vectors and we can here test a series of hypotheses as given in Table III. Consider for instance the hypothesis \( r \leq 1 \) versus the general alternative \( H_1 \). Here the test statistic is calculated as

\[
-2\ln(Q) = -T \{ \ln(1-\hat{\lambda}_2) + \ln(1-\hat{\lambda}_3) + \ln(1-\hat{\lambda}_4) \}
\]

\[
= -46 \{ \ln(1-.1940) + \ln(1-.1269) + \ln(1-.0138) \} = 16.80.
\]

A comparison with the 95\% quantile in the asymptotic distribution given as 23.8 in Table III, shows that this value is not significant. Hence there is no evidence in the Danish data that more than one cointegration relation exists. If on the other hand we test the hypothesis that \( r = 0 \) we find from Table III that this is strongly rejected, and hence we conclude that the data indicates that only one cointegration vector is found in the Danish data. This hypothesis will be maintained in the following.

From Table II we estimate the cointegrating relation as the first column in \( \hat{V} \). In this case it seems natural to normalize it by the coefficient of \( m_2 \) equal -1. Hence we divide the vector \( \hat{\beta} \) by 19.39. Then

\[
\hat{\beta}' = ( -1.00, +.96, -6.73, +5.42 ).
\]

This makes it straightforward to interpret the cointegration vector in
terms of an error correction mechanism measuring the excess demand for money, where the equilibrium relation is given by

\[ m_2 = 0.096y - 6.73b + 5.42d + \text{const.} \]

Similarly \( \hat{\alpha} \) is found as the first column in the matrix \(-S_{02}\hat{V}\), but now normalized by multiplying by 19.39,

\[ \hat{\alpha}' = (-0.317, -0.077, -0.006, +0.024). \]

The normalized coefficients of \( \alpha \) can now be interpreted as the weights with which excess demand for money enters the four equations of our system. In this case it is natural to give them an economic meaning in terms of the average speed of adjustment towards the estimated equilibrium state, such that a low coefficient indicates slow adjustment and a high coefficient indicates rapid adjustment. It is seen that in the first equation which measures the changes in money balances, the average speed of adjustment is approximately 0.32, whereas in the remaining three equations the adjustment coefficients are very low. This observation will be followed by a formal test in section 2.4.

For the Finnish data we find from Table III that at least 2 but possibly 3 cointegration vectors are present. We shall assume in the following that 3 cointegration relations exist, but keep in mind that this conclusion is based on rather vague evidence.

We then find \( \hat{\beta} \) as the first three columns of \( \hat{V} \) from Table II and \( \hat{\alpha} \) as the corresponding columns of \(-S_{02}\hat{V}\).

The interpretation of \( \hat{\beta} \) and \( \hat{\alpha} \) is not straightforward in this case. A thorough understanding of the economic problem seems to be mandatory in
order to illucidate the role of the various cointegration vectors estimated by \( \hat{\beta} \). We shall indicate here that a preliminary and heuristic interpretation is possible by considering the estimates in Table II. First note that for the three first eigenvectors we find, that the first two coefficients are equal with opposite sign: \( \hat{\beta}_{12} = -\hat{\beta}_{11} \) \( i = 1,2,3 \), and that \( \hat{\beta}_2 \) is approximately proportional to \((0,0,0,1)\). Thus it follows that \( \hat{\beta}_1, \hat{\beta}_2 \) and \( \hat{\beta}_3 \) can be approximately represented as linear combinations of the vectors \((-1,1,0,0), (0,0,0,1), \) and \((0,0,1,0)\). This would imply that \( \Delta m, \Delta y, \Delta m \) and \( \Delta p \) are stationary, and hence that the only interesting cointegration we have found is between \( m_1 \) and \( y \).

We shall formulate this finding as a precise hypothesis about a linear restriction on \( \beta \) and test this hypothesis in the next section.

This completes the investigation of the model \( H_2 \) in \( H_1 \) and we shall now turn to the model \( H_3^* \) in \( H_2 \).

2.3. Linear hypotheses concerning \( \beta \). Estimation and test

The model \( H_3^* : \beta = H\varphi \) is a formulation of a linear restriction on the cointegration vectors. The hypothesis specifies the same restriction on all the cointegration vectors. The reason for this is the following: If we have two cointegration vectors in which \( m \) and \( y \), say, enter then any linear combination of these relations will also be a cointegrating relation. Thus it will in general always be possible to find some relation which has, say, equal coefficients with opposite sign to \( m \) and \( y \), corresponding to a long-run elasticity of 1. This is clearly not interesting, and only if all rows of \( \beta \) show the same unit elasticity is it
meaningful to say that we have found a unit elasticity.

Under \( H^*_3 \) we have the restriction \( \beta = H \varphi \) where \( H \) is \((p \times s)\) but that means that the estimation of \( \Gamma_1, \ldots, \Gamma_{k-1} \) and \( \alpha \) and \( \Lambda \) are as just described for fixed \( \beta = H \varphi \), and \( \varphi \) now has to be chosen to minimize

\[
(2.25) \quad |\varphi' H^r S_{kk} H \varphi - \varphi' H^r S_{k0} S_{00}^{-1} S_{0k} H \varphi| / |\varphi' H^r S_{kk} H \varphi|
\]

over the set of all \( s \times r \) matrices \( \varphi \). This problem has the same kind of solution as above and we can formulate the results in

**THEOREM 2.3:** Under the hypothesis

\[ H^*_3: \quad \beta = H \varphi, \]

we can find the maximum likelihood estimator of \( \beta \) as follows: First we solve

\[
(2.26) \quad |\lambda H^r S_{kk} H_1 - H^r S_{k0} S_{00}^{-1} S_{0k} H_1| = 0,
\]

to give \( \lambda^*_1 > \ldots > \lambda^*_s \) and \( V^* = (v^*_1, \ldots, v^*_s) \) normalized by \( V^* H^r S_{kk} H^* V^* = I. \)

We now choose

\[
(2.27) \quad \varphi^* = (v^*_1, \ldots, v^*_r) \quad \text{and} \quad \beta^* = H \varphi^*,
\]

and find the estimates of \( \alpha, \Pi, \Lambda \) and \( \Gamma_j \) from (2.21), (2.22), (2.23) and (2.6). The maximized likelihood becomes

\[
(2.28) \quad L_{\text{max}}^{-2/T}(H^*_3) = |S_{00}| \prod_{i=1}^r (1-\lambda^*_i),
\]

which gives the likelihood ratio test of the hypothesis \( H^*_3 \) in \( H_2 \) as

\[
(2.29) \quad -2 \ln(Q; H^*_3 | H_2) = T \sum_{i=1}^r \ln((1-\lambda^*_i)/(1-\hat{\lambda}_i)).
\]

The asymptotic distribution of this statistic was shown in Johansen (1988b) to be \( \chi^2 \) with \( r(p-s) \) degrees of freedom.
Now consider the hypothesis that there is proportionality between money and income as the transactions demand for money would predict. Then the coefficients to money and income would be equal with opposite sign.

For the Danish data we have found one cointegration vector, and the restriction can, in the above matrix formulation, be expressed as

$$\beta = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ \varphi = \begin{bmatrix} \beta_{12} \\ \beta_3 \\ \beta_4 \end{bmatrix}.$$ 

We shall then solve (2.26) which now gives three eigenvalues and eigenvectors, see Table IV.

The test of $H_3$ in $H_2$ consists of comparing $\lambda_1^*$ and $\hat{\lambda}_1$ by the test

$$-2\ln(Q) = T\{\ln(1-\lambda_1^*) - \ln(1-\lambda_1)\} = 46\{\ln(1-.4999) - \ln(1-.5004)\} = .046.$$ 

The asymptotic distribution of this quantity is given by the $\chi^2$ distribution with degrees of freedom $r(p-s) = 1(4-3) = 1$. It is clearly not significant, and we can thus accept the hypothesis that for the Danish data the coefficients to $m_2$ and $y$ are equal with opposite sign.

In the Finnish data we have found three cointegration vectors and the hypothesis about proportionality between money and income can be formulated as

$$\beta = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ \varphi = \begin{bmatrix} \beta_{1.12} & \beta_{2.12} & \beta_{3.12} \\ \beta_{1.3} & \beta_{2.3} & \beta_{3.3} \\ \beta_{1.4} & \beta_{2.4} & \beta_{3.4} \end{bmatrix}.$$
We find the three eigenvalues from Table IV, and the test becomes

\[-2\ln(Q) = T\{\ln(1-\lambda_1^*)+\ln(1-\lambda_2^*)+\ln(1-\lambda_3^*)-\ln(1-\lambda_1^*)-\ln(1-\lambda_2^*)-\ln(1-\lambda_3^*)}\} = 3.82,\]

which should be compared with $\chi^2_{95}(r(p-s)) = \chi^2_{95}(3(4-3)) = 7.81$. Thus we accept the hypothesis of equal coefficients with opposite sign for $m_1$ and $y$.

For the Danish data we also want to test the hypothesis that the coefficients for the bond interest rate and the deposit interest rate are equal with opposite sign. This would imply that the cost of holding money can be measured as the difference between the bond yield and the yield from holding money in bank deposits. This hypothesis $H^**_2$ is formulated as

$$
\beta = \begin{bmatrix}
1 & 0 \\
-1 & 0 \\
0 & 1 \\
0 & -1
\end{bmatrix} \varphi, \quad \varphi = \begin{bmatrix}
\beta_{12} \\
\beta_{34}
\end{bmatrix}
$$

Under this hypothesis we get from (2.25) two eigenvalues $\lambda_1^* = .4828$ and $\lambda_2^* = .0473$. The first eigenvector is $v_1^* = (-16.92, -127.71)$ and the corresponding $\alpha^* = (-16.24, -4.02, +.08, +1.64) \times 10^{-3}$. The test for the hypothesis is given by

\[-2\ln(Q) = 46\ln\{(1-.4838) - \ln(1-.4999)\} = 1.46\]

which should be compared with the $\chi^2$ quantiles with $r(s_1-s_2) = 1(4-3) = 1$ degree of freedom. Again this is not significant and we conclude the analysis of the cointegration vectors for the Danish demand for money by the estimate

$$
\beta^* = (-1.00, +1.00, -7.55, +7.55),
$$

where we have normalized it by the coefficient to $m_2$ (16.92). The corresponding estimate of $\alpha$ is multiplied by 16.92 and is given by
It is also possible to test these hypotheses using the Wald test (3.41) given in Corollary 3.17, which are easily calculated from Table II. For the Danish data we first test that $m_2$ and $y$ have equal coefficients with opposite sign, i.e. sum to zero. We choose $K' = (1,1,0,0)$ and calculate the statistic

$$U_1 = \frac{46^{1/2}(-19.39+18.61)}{\{(1/5004-1)[(-14.77+25.04)^2+(10.27-26.05)^2+(-12.39+1.29)^2]\}^{1/2}} = -244$$

The second test that the coefficients to the bond interest rate and the deposit interest rate are equal with opposite sign is calculated in the same way:

$$U_2 = \frac{46^{1/2}(-130.40+105.05)}{\{(1/5004-1)[(-17.44-73.35)^2+(29.81-83.90)^2+(.54-24.94)^2]\}^{1/2}} = -1.56$$

Both these statistics are asymptotically normalized Gaussian and the values found are hence not significant.

The test for the Finnish data can likewise be performed using the Wald test (3.40), We can formulate the hypothesis in question as

$$K'\beta = (1,1,0,0)\beta = 0.$$  

Then we find from Table II

$$K'\hat{\gamma}K = (1.38 + 2.22)^2 = 12.96$$

and

$$K'\hat{\beta}(\hat{D}^{-1}-I)^{-1}\hat{\beta}'K = \left(-2.93+2.86\right)^2 + \left(4.58-6.06\right)^2 + \left(-11.13+10.24\right)^2 \approx .83$$
and the test statistic becomes

\[ 104 \times 0.834/12.96 = 6.22 \]

which should also be compared with the \( \chi^2 \) quantile with 3 degrees of freedom as before.

Note that with the restriction of proportionality we now have three cointegration vectors restricted to a three dimensional space defined by the restriction that \( m_1 \) and \( y \) have equal coefficients with opposite sign. Thus the hypothesis \( H_2^{*} \) is really the hypothesis of a complete specification of \( sp(\beta) \). In this space we can choose to present the results in any basis we want and it seems natural to consider the three variables \( m_1 - y, i^m \) and \( \Delta p \). Thus the conclusion about the Finnish data is that in fact the two last variables \( i^m \) and \( \Delta p \) are already stationary, and the two first \( y \) and \( m_1 \) are cointegrated.

Notice that the Wald test in all cases gives a value of the test statistic which is larger than the value for the likelihood ratio test statistic. This just emphasises the fact that we are relying on asymptotic results and a careful study of the small sample properties is necessary.

We shall conclude this section by giving the final estimate of \( \Pi \) as obtained under the various restrictions we have accepted above.

**[TABLE V]**

### 2.4. Test and estimation of restrictions on \( \alpha \)

Let us now turn to the hypothesis \( \tilde{H}_3 \) where \( \alpha \) is restricted by \( \alpha = A\psi \) in the model \( H_2 \). Here \( A \) is a \((p \times m)\) matrix. It is convenient to introduce
$B(p \times (p-m))$ such that $B'A = 0, B'B = I$. Then the hypothesis $H_3$ is given by

$$B'a = 0.$$  (2.30)

We shall now turn to the concentrated likelihood function (2.9) and express it in the variables given by

$$A'(R_0 + a\beta'R_{kt}) = A'R_0 + A'A\beta'R_{kt},$$  (2.31)

$$B'(R_0 + a\beta'R_{kt}) = B'R_0.$$  (2.32)

In the following we shall factor out that part of the likelihood function which depends on $B'R_0$ since it does not contain the parameters $\psi$ and $\beta$.

Again to save notation we shall define $A_{aa} = A'AA, A_{ab} = A'AB, S_{ak,b} = S_{ak} - S_{ab}S_{bb}^{-1}S_{bk} = A'S_{kk} - A'S_{00}B(B'S_{00}B)^{-1}B'S_{0k}$, etc.

We now get a factor corresponding to the marginal distribution of $B'R_0$ given by

$$|\Lambda_{bb}|^{-T/2} \exp \left\{ - \sum_{t=1}^{T} (B'R_0)^{-1}(B'R_0)^{-1}/2 \right\},$$  (2.33)

which gives the estimate

$$\tilde{\Lambda}_{bb} = S_{bb} = B'S_{00}B,$$

and the maximized likelihood function

$$L_{\text{max}} = |S_{bb}|.$$  (2.35)

The other factor corresponds to the conditional distribution of $A'R_0$ and $R_{kt}$ conditional on $B'R_0$ and is given by

$$|A'|^{T/2} |A_{aa,b}|^{-T/2} \exp \left\{ - \sum_{t=1}^{T} (A'R_0 + A'A\beta'R_{kt} - A_{ab}A_{bb}^{-1}B'R_0)^{-1} \right\}.$$

It is a well known result from the theory of the multivariate normal distribution that the parameters $\Lambda_{bb}, A_{ab}A_{bb}^{-1}$ and $\Lambda_{aa,b}$ are variation independent and hence that the estimate of $\Lambda_{ab}A_{bb}^{-1}$ is found by regression
for fixed $\psi$ and $\beta$ giving

$$\tilde{A}_{ab}^{-1}\tilde{b}^{\top}(\psi, \beta) = (S_{ab} + A'A\psi\beta'S_{kb})S_{bb}^{-1},$$

and new residuals defined by

$$\tilde{R}_{at} = A'R_{0t} - S_{ab}S_{bb}^{-1}B'R_{0t},$$
$$\tilde{R}_{kt} = R_{kt} - S_{kb}(S_{bb}^{-1}B'R_{0t}).$$

In terms of $\tilde{R}_{at}$ and $\tilde{R}_{kt}$ the concentrated likelihood function now has the form (2.9) which means that the estimation of $\beta$ follows as before, and we can formulate

**THEOREM 2.4:** Under the hypothesis

$$\tilde{H}_{3}: \quad \alpha = A\psi,$$

the maximum likelihood estimator of $\beta$ is found as follows: First solve the equation

$$|A'S_{hk,b} - S_{ha,b}S_{aa,b}^{-1}S_{ak,b}| = 0,$$

giving $\tilde{\lambda}_1 > \ldots > \tilde{\lambda}_m > \tilde{\lambda}_{m+1} = \ldots = \tilde{\lambda}_p = 0$ and $\tilde{V} = (\tilde{v}_1, \ldots, \tilde{v}_p)$ normalized such that $\tilde{V}'S_{hk,b}\tilde{V} = I$.

Now take

$$\tilde{\beta} = (\tilde{v}_1, \ldots, \tilde{v}_p),$$

which gives the estimates

$$\tilde{\psi} = -(A'A)^{-1}S_{ak,b}\tilde{\beta}$$

and

$$\tilde{\alpha} = A\tilde{\psi} = -A(A'A)^{-1}A'(S_{0k} - S_{00}B'S_{00}B)^{-1}B'S_{0k}\tilde{\beta},$$

$$\tilde{\lambda}_{aa,b} = S_{aa,b} - \tilde{\psi}\tilde{\psi}' ,$$

and the maximized likelihood function

$$L_{\text{max}}^{-2/T}(\tilde{H}_{3}) = |A'A|^{-1}|S_{bb}| |S_{aa,b}| \prod_{i=1}^{r} (1 - \tilde{\lambda}_i) = |S_{00}| \prod_{i=1}^{r} (1 - \tilde{\lambda}_i).$$

The estimate of $A$ can be found from (2.34), (2.37) and (2.41), and still
\( \Gamma_j \) is estimated from (2.6).

The likelihood ratio test statistic of \( \tilde{H}_3 \) in \( H_2 \) is

\[
(2.44) \quad -2\ln(Q; \tilde{H}_3 | H_2) = T \sum_{i=1}^r \ln \left\{ \frac{(1-\tilde{\lambda}_i)}{(1-\lambda_i)} \right\}.
\]

The asymptotic distribution of this test statistic is found in Theorem 3.14 and is given by a \( \chi^2 \) distribution with \( r(p-m) \) degrees of freedom.

The following very simple corollary is useful for explaining the role of single equation analysis:

**COROLLARY 2.5:** If \( m = r = 1 \) then the estimate of \( \beta \) is found as the coefficients of \( X_{t-k} \) in the regression of \( A'AX_t \) on \( X_{t-k}, B'X_t, \) and \( AX_{t-1}, \ldots, AX_{t-k+1} \).

**PROOF:** It suffices to notice that when \( m = r = 1 \) then only one cointegration vector has to estimated. It is seen from (2.38) that since the matrix \( S^{-1}_{ka,b} S^{-1}_{a,b} \) is singular and in fact of rank 1, then only 1 eigenvalue is non zero, and the corresponding eigenvector is proportional to \( S^{-1}_{kk,b} S^{-1}_{ka,b} \) which is exactly the regression coefficient to \( R_{kt} \) we would get by regressing \( A'R_{O_t} \) on \( B'R_{O_t} \) and \( R_{kt} \). This can of course be seen directly from (2.36) since \( A'A_\phi \) is 1x1 and can be absorbed into \( \beta \), which shows that \( \beta \) is given by the regression as described.

In particular if \( A' = (1,0,0,0) \) then the least squares regression of the difference of the first variable on the remaining variables and their differences will be the maximum likelihood estimator.

We shall now apply these results to the Danish data. We want to test the hypothesis that the cointegration relation only enters the first equation. Thus we let
and find the eigenvalues of (2.46), see below, since we maintain the hypothesis that \( \beta \) has the form that was found in the previous section. Thus there are two eigenvalues \( \tilde{\lambda}_1 = 0.4200 \) and \( \tilde{\lambda}_2 = 0 \), giving an eigenvector corresponding to \( \tilde{\lambda}_1 \) of \((-18.56, -135.68)\), and a corresponding estimate of \( \alpha \), \( \tilde{\alpha} = (-13.21, 0.00, 0.00, 0.00) \times 10^{-3} \).

The test statistic for this hypothesis about \( \alpha \) is then given by
\[
-2\ln(Q) = T\{ln(1-\tilde{\lambda}_1)-ln(1-\tilde{\lambda}_1^\mathbf{M})\} = 46\{ln(1-0.4200)-ln(1-0.4828)\} = 5.17.
\]
This should be compared with the 95% quantile \( \chi^2_{.95}(r(p-m)) = \chi^2_{.95}(1(4-1)) = 7.81 \). On the basis of this we accept the hypothesis about \( \alpha \) and conclude the analysis of the Danish data by giving the estimate of
\[
\tilde{\beta}' = (-1.00, +1.00, -7.31, +7.31),
\]
\[
\tilde{\alpha}' = (-.25, 0.00, 0.00, 0.00),
\]
from which the estimate of \( \Pi \) can be constructed. Thus we have reduced the 16 parameters in the matrix \( \Pi \) which describes the long-run relations in the data to just 2 parameters.

This completes the estimation and test of \( H_3^\mathbf{M} \) and \( \tilde{H}_3 \) concerning restrictions on \( \alpha \) and \( \beta \) respectively.

The conclusion is thus the following: By correcting \( AX_t \) and \( X_{t-k} \) for \( AX_{t-1}, \ldots, AX_{t-k+1} \) we obtain the product moment matrices from the residuals:

\[
A = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\]
Under the hypothesis \( H_2 : \Pi = \alpha \beta' \) the estimates of \( \beta \) and \( \alpha \) are related to the canonical variates between \( R_{0t} \) and \( R_{kt} \), see Anderson (1984). The estimate of \( \beta \) is given as the eigenvectors of (2.18) corresponding to the \( r \) largest eigenvalues, i.e. the choice of \( \hat{\beta} \) is the choice of the \( r \) linear combinations which have the largest correlation with the stationary process \( \Delta X_t \).

If \( \beta = H \varphi \) we note that \( \beta'R_{kt} = \varphi'H'R_{kt} \) which leads to solving (2.25) where \( R_{kt} \) has been replaced by \( H'R_{kt} \). Thus restricting \( \beta \) to lie in \( \text{sp}(H) \) implies that the levels of the process should be transformed by \( H' \).

If \( \alpha = A \psi \) we shall solve (2.38), where we have conditioned on \( B'R_{0t} \). In other words if we assume that the equations for \( B'R_{0t} \) do not contain the parameter \( \alpha \), i.e. \( B'\alpha = 0 \), then we shall also correct for these before solving the eigenvalue problem.

It is now clear how one should solve the model \( H_4 = \tilde{H}_3 \cap H_3^* \), where restrictions have been imposed on \( \beta \) as well as on \( \alpha \), namely by solving the eigenvalue problem

\[
(2.46) \quad | \lambda H'S_{kk}.bH - H'S_{ka}.bS_{aa}.b^{-1}S_{ka}.b| = 0.
\]

This gives the final solution to the estimation problem of \( H_4 \). Notice how (2.46) contains the previous problems by choosing either \( H = I \) or \( A = I \) or both.

Note that a linear restriction on \( \beta \) implies a transformation of the process, and that a linear restriction on \( \alpha \) implies a conditioning. Thus

\[
(2.45) \quad \begin{bmatrix} S_{00} & S_{0k} \\ S_{k0} & S_{kk} \end{bmatrix}
\]

which form the basis for all subsequent analyses of \( \Pi \).
all the calculations can easily be performed starting with the product moment matrices $S_{ij}$ and using the usual operations of finding marginal (transformed) and conditional variances and then apply an eigenvalue routine.

The test statistic for the hypothesis $H_2$ has an asymptotic distribution that has to be tabulated by simulation, but all the hypotheses about $\beta$ and $\alpha$ lead to test statistics that are asymptotically distributed as $\chi^2$ with the appropriate degrees of freedom, and hence the usual tables can be applied.

3. The asymptotic properties of the estimators and the test statistics

3.1. Granger's representation theorem

When we want to investigate the probabilistic properties of the estimates and the test statistics we have to make more precise assumptions about the process. The basic assumption is that for

$$\Pi(z) = I - \Pi_1 z - \ldots - \Pi_k z^k$$

we have that $|\Pi(z)| = 0$ implies that either $|z| > 1$ or $z = 1$, which guarantees that the non-stationarity of $X_t$ can be removed by differencing.

Now write the model defined by (3.1), see (1.2), as

$$\Pi X_t + \Pi_1 (L) AX_t = e_t.$$  
where $\Pi = I - \Pi_1 - \ldots - \Pi_k = \Pi(1)$.

The first result that we want to prove is the fundamental result about error correction models of order 1 and their structure. The basic
result is due to Granger (1981), see Engle and Granger (1987) or Johansen (1988a), but we shall give a very simple proof here.

**THEOREM 3.1:** (Granger's representation theorem). If

\[(3.3) \quad \Pi = \alpha \beta'\]

for \(\alpha\) and \(\beta\) of dimension \(p \times r\) and rank \(r\) and if

\[(3.4) \quad \alpha_\perp \Pi(1) \beta_\perp\]

has full rank \(p - r\), where \(\beta_\perp\) and \(\alpha_\perp\) are \(p \times (p-r)\) of full rank such that \(\beta' \beta_\perp = 0\) and \(\alpha' \alpha_\perp = 0\), then

\[(3.5) \quad \Delta X_t \text{ is stationary}\]
\[(3.6) \quad X_t \text{ is non-stationary}\]
\[(3.7) \quad \beta' X_t \text{ is stationary}\]

and hence (3.2) can be interpreted as an error correction model. If we write the process in the moving average form \(\Delta X_t = C(L) \epsilon_t\) then the following representation holds:

\[(3.8) \quad C(1) = \beta_\perp (\alpha_\perp \Pi(1) \beta_\perp)^{-1} \alpha'_\perp.\]

Note that the relation between (3.3) and (3.8) shows that for this type of process there is a very nice symmetry between the singularity of the "impact" matrix \(\Pi\) for the autoregressive representation and the singularity of the "impact" matrix for the moving average representation, in the sense that what is the null space for \(C(1)'\) is the range space for \(\Pi\) and what is the range space for \(\Pi'\) is the null space for \(C(1)\). It is this symmetry that allows the results for this type of process to be exceptionally simple.
PROOF: If we multiply the equation (3.2) by $\alpha'$ and $\alpha'_\perp$ we get the equations

$$\alpha'\alpha'\beta'X_t + \alpha'\Pi_1(L)\Delta X_t = \alpha'\epsilon_t,$$
$$\alpha'_\perp\Pi_1(L)\Delta X_t = \alpha'_\perp\epsilon_t.$$

To discuss the properties of the process $X_t$ we shall solve the equations for $X_t$ and express it in terms of the $\epsilon_t$'s. We therefore introduce the variables $Z_t = (\beta'\beta)^{-1}\beta'X_t$ and $Y_t = (\beta'_\perp\beta'_\perp)^{-1}\beta'_\perp\Delta X_t$ as new variables, from which $\Delta X_t$ can be recovered:

$$\Delta X_t = \beta'_\perp Y_t + \beta\Delta Z_t.$$ 

This gives the equations

$$a'\alpha\beta'\beta'Z_t + a'\Pi_1(L)\beta\Delta Z_t + a'\Pi_1(L)\beta'_\perp Y_t = \alpha'\epsilon_t,$$
$$a'_\perp\Pi_1(L)\beta\Delta Z_t + a'_\perp\Pi_1(L)\beta'_\perp Y_t = \alpha'_\perp\epsilon_t.$$

The matrix function defining this new system consisting of $Z_t$ and $Y_t$ takes the form:

$$\tilde{A}(z) = \begin{pmatrix} a'\alpha'\beta + a'\Pi_1(z)\beta(1-z) & a'\Pi_1(z)\beta'_\perp \\ a'_\perp\Pi_1(z)\beta(1-z) & a'_\perp\Pi_1(z)\beta'_\perp \end{pmatrix}.$$

For $z = 1$ this has determinant

$$|a'\alpha| |\beta'\beta| |\alpha'_\perp\Pi_1(1)\beta'_\perp|$$

which is non-zero by assumption (3.3) and (3.4), hence $z = 1$ is not a root. For $z \neq 1$ we use the representation

$$\tilde{A}(z) = (a,a'_\perp)'\Pi(z)(\beta,\beta'_\perp(1-z)^{-1}),$$

which gives the determinant as

$$|\tilde{A}(z)| = |(a,a'_\perp)| |\Pi(z)| |(\beta,\beta'_\perp)| (1-z)^{-(p-r)},$$

which shows that all roots of $|\tilde{A}(z)| = 0$ are outside the unit disk, by assumption (3.1).

This shows that the system defined by (3.9) and (3.10) is invertible and that $Y_t$ and $Z_t$ are stationary processes, and hence that $\Delta X_t$ is
stationary. This proves (3.5) and (3.7). By summation of \( AX_t \) we find that \( X_t \) contains the non-stationary component \( \beta' \sum_{s=0}^{t} Y_s \), which proves (3.6).

From the representation of the processes \( Z_t \) and \( Y_t \) we can get a representation of \( AX_t \) by multiplying by the matrix \( (\beta A, \beta \beta') \). Hence

\[
C(L) = (\beta A, \beta \beta')A(L)^{-1}(\alpha, \alpha_\parallel).
\]

For \( L = 1 \) we get (3.8).

In the following when we discuss the limiting distributions we shall throughout assume the conditions of Theorem 3.1. We shall have to repeat some of the results about the process \( X_t \) given in Johansen (1988b). Since we assume that \( AX_t \) is stationary we define the covariance function

\[
\psi(i) = \text{Cov}(AX_t, AX_{t+1}).
\]

Then we define

\[
\mu_{ij} = \text{Cov}(AX_{t-1}, AX_{t-j}) = \psi(i-j), \\
\mu_{ki} = \sum_{j=k-1}^{\infty} \psi(j) = \lim_{t} \text{Cov}(X_{t-k}, AX_{t-1})
\]

and

\[
\mu_{kk} = -\sum_{j=-\infty}^{\infty} |j| \psi(j)
\]

with the interpretation that

\[
\lim_{t} \text{Var}(\beta'X_{t-k}) = \beta'\mu_{kk}\beta.
\]

Finally we define

\[
\Sigma_{ij} = \mu_{ij} - \mu_{i}^{-1}\mu_{\parallel}, \quad i,j = 0,k,
\]

where \( \mu_{\parallel} \) is the matrix with entries \( \mu_{ij}, \quad i,j = 1, \ldots, k-1. \)
3.2. A summary of technical asymptotic results

The following technical results are given in Johansen (1988b) based on the results by Phillips and Durlauf (1986). We let \( \mathbf{W} \) be a Brownian motion in \( p \) dimensions with covariance matrix \( \Lambda \), and let \( \mathbf{C} = \mathbf{C}(1) \), see (3.8).

**Lemma 3.2:** As \( T \to \infty \) we have

\[
T^{-1/2} \mathbf{X}_T \to \mathbf{C} \mathbf{W}(t)
\]

a.s.

\[
\begin{align*}
M_{ij} &\to \mu_{ij}, \ i,j = 0, \ldots, k-1 \\
M_{kt} &\to \mu_{kt} + \int_0^t \mathbf{C} f(w) d\mathbf{W}(t)
\end{align*}
\]

and

\[
\beta' M_{kk} \beta \to \beta' \mu_{kk} \beta
\]

while

\[
T^{-1} M_{kk} \to \int_0^t \mathbf{C} f(w) \mathbf{W}(u)' d\mathbf{C}'
\]

The relations between \( \Sigma_{ij} \), \( \alpha \) and \( \beta \) are given in the next Lemma.

**Lemma 3.3:** The following relations hold

\[
(3.11) \quad \Sigma_{00} = -\alpha \beta' \Sigma_{k0} + \Lambda,
\]

\[
(3.12) \quad \Sigma_{0k} \beta = -\alpha \beta' \Sigma_{kk} \beta,
\]

and hence

\[
(3.13) \quad \Sigma_{00} = \alpha (\beta' \Sigma_{kk} \beta) \alpha' + \Lambda.
\]

These relations imply that
The properties of $S_{ij}$ are given in Lemma 3.4.

**Lemma 3.4:** For $T \to \infty$ it holds that

\[ S_{00} \to \Sigma_{00}, \]

\[ a.s. \]

\[ \beta' S_{k0} \to \beta' \Sigma_{k0} \]

and

\[ \beta' S_{kk} \beta \to \beta' \Sigma_{kk} \beta, \]

whereas if $\gamma$ is not in the span of $\beta$ then

\[ T^{-1/2} \gamma' S_{kk} \gamma \to \gamma' C W(u) W'(u) du C' \gamma, \]

and if $\delta' \alpha = 0$ then

\[ \delta' S_{0k} \to \delta' \int_0^1 W(u) W'(u) du C'. \]

Finally a technical result about the asymptotic behaviour of some of the quantities that enter into the calculations later.

**Lemma 3.5:** For $T \to \infty$ we have for $\delta' \alpha = 0$

\[ \alpha' S_{0k}^{-1} S_{kk}^{-1} S_{k0} \alpha \to \alpha' \Sigma_{0k}^{-1} \beta (\beta' \Sigma_{kk}^{-1} \beta)^{-1} \beta' \Sigma_{k0}^{-1} \alpha = \alpha'(\Sigma_{00} - \Lambda) \alpha \]

\[ T^{1/2} \alpha' S_{0k}^{-1} S_{kk}^{-1} S_{k0} \delta = \alpha' \Sigma_{0k}^{-1} \beta (\beta' \Sigma_{kk}^{-1} \beta)^{-1} \left( T^{1/2} \beta' S_{k0} \delta \right) + o_p(1) \]

\[ T^{1/2} \beta' S_{k0} \delta \xrightarrow{w} N_{r \times (p-r)}(0, \beta' \Sigma_{kk} \beta \otimes \delta' \Lambda \delta) \]

Hence the limit distribution of (3.17) is Gaussian with mean zero and variance matrix given by $\alpha'(\Sigma_{00} - \Lambda) \alpha \otimes \delta' \Lambda \delta$. 

(3.14) \[(a' \Sigma_{00}^{-1} a)^{-1} a' \Sigma_{00}^{-1} = (a' \Lambda^{-1} a)^{-1} a' \Lambda^{-1} \]
PROOF: Let \((\beta, \gamma)\) be \((p \times p)\) and of full rank, then
\[
S_{0k}'S_{kk}^{-1}S_{k0} = (S_{0k}'S_{0k})^{-1}(\beta'S_{kk} \beta + \beta'S_{kk} \gamma)^{-1}\begin{bmatrix}
\gamma'\beta' & \beta' \\
\gamma' & \gamma'
\end{bmatrix}^{-1}
\]
\[
= (S_{0k}'S_{0k})^{-1}A_T^{-1}\begin{bmatrix}
\beta'S_{kk} \beta & \beta'S_{kk} \gamma \\
\gamma'\beta' & \gamma'
\end{bmatrix}
\]
where \(A_T = (\gamma'S_{kk})^{-1/2} \to 0\), for \(T \to \infty\), see Lemma 3.4. From Lemma 3.3 and 3.4 it follows that
\[
S_{0k}'S_{kk}^{-1}S_{k0} \to \Sigma_{0k}'\beta(\Sigma_{kk}'\beta)^{-1}\Sigma_{k0} = \alpha'(\Sigma_{00} - \Lambda)\alpha,
\]
which proves (3.16). Now use the above representation for \(T^{1/2}\alpha'S_{0k}'S_{kk}^{-1}S_{k0}\delta\), and we get the it has the same limit distribution as
\[
(a'\Sigma_{0k}\beta, 0)\begin{bmatrix}
(\beta'\Sigma_{kk}'\beta)^{-1} & 0 \\
0 & I
\end{bmatrix}T^{1/2}\Sigma_{k0}\delta = (a'\Sigma_{0k}\beta)(\beta'\Sigma_{kk}'\beta)^{-1}T^{1/2}\beta'S_{k0}\delta.
\]
This proves (3.17). The asymptotic distribution of \(T^{1/2}\beta'S_{k0}\delta\) is found as follows: If we multiply (2.1) by \(\Delta X_{t-1}^{-1}\) and sum over \(t\) we get
\[
M_{0i} = \sum_{j=1}^{k-1} M_{ji} - \sum_{j=1}^{k-1} T^{-1} \delta \sum_{t=0}^{T} \epsilon_t \Delta X_t^{-1} i = 0, \ldots, k-1,
\]
and when multiplying by \(X_{t-k}^{-1}\) and summing over \(t\) we get
\[
M_{0k} = \sum_{j=1}^{k-1} M_{jk} - \sum_{j=1}^{k-1} T^{-1} \sum_{t=0}^{T} \epsilon_t X_t^{-1}.
\]
Now solve the equations (3.19), \(i = 1, \ldots, k-1\) for \(\Gamma_j\) and insert into (3.20) and use the definition of \(S_{ij}\), then we get
\[
S_{0k}' + \Pi S_{kk}' = \sum_{t=1}^{T} \sum_{j=1}^{k-1} \Delta X_{t-k}^{-1} \{X_{t-k}^{-1} \delta \sum_{j=1}^{k-1} M_{i+j}^{*} M_{k}^{-1}\} = U_T
\]
Since \(M_{ij} \to \mu_{ij}\) and \(\beta'M_{kj} \to \beta'\mu_{kj}\), it follows that \(U_T\) has the same limit distribution as
From the central limit theorem for martingales it follows that $\sqrt{t}V_T$ is asymptotically Gaussian with mean zero and variance matrix $\Lambda \beta^\top \Sigma_{kk} \beta$. Now multiply by $\delta$ and the second term on the left hand side of (3.21) vanishes since $\delta' \alpha = 0$, and hence (3.18) is proved.

Finally we can use Lemma 3.3 to see that
\[
a' \Sigma_{0k} \beta (\beta' \Sigma_{kk} \beta)^{-1} \beta' \Sigma_{0k} \alpha = a' a \beta' \Sigma_{kk} \beta a' \alpha = a'(\Sigma_{00} - \Lambda) a.
\]

We shall next give some results about the eigenvalues of (2.18) taken from Johansen (1988b) Lemma 4 and 6.

**Lemma 3.6:** The ordered eigenvalues of the equation
\[
|\lambda S_{kk} - S_{k0} S_{00}^{-1} S_{0k}| = 0
\]
converge in probability to $(\lambda_1, \ldots, \lambda_r, 0, \ldots, 0)$, where $\lambda_1, \ldots, \lambda_r$ are the ordered eigenvalues of the equation
\[
|\lambda \beta' \Sigma_{kk} \beta - \beta' \Sigma_{0k} S_{00}^{-1} \Sigma_{0k} \beta| = 0.
\]
Further we have that $\hat{T}_{r+1} \ldots, \hat{T}_p$ converge in distribution to the ordered eigenvalues of the equation
\[
\frac{1}{\lambda} \int BB' du - \frac{1}{\lambda} \int BdB' \int dB'B' = 0
\]
where $B$ is a Brownian motion in $p-r$ dimensions with variance matrix $I$.

The following result is well known from the theory of canonical correlations and express the duality there exists between the two sets of canonical variates, see Anderson (1984).
LEMMA 3.7: Let the partitioned matrix
\[
\begin{bmatrix}
A & B \\
B' & C
\end{bmatrix}
\]
be symmetric positive definite, where \(A\) is \(p \times p\) and \(C\) is \(m \times m\). It then holds that
\[
\begin{vmatrix}
\mu A & B \\
B' & \mu C
\end{vmatrix} = \mu^{p-m} |A| |\mu^2 C - B'A^{-1}B| = \mu^{m-p} |C| |\mu^2 A - BC^{-1}B'|
\]
such that the positive solutions of
\[
|\lambda C - B'A^{-1}B'| = 0
\]
and
\[
|\lambda A - BC^{-1}B'| = 0
\]
are identical. If
\[
\lambda Cx = B'A^{-1}Bx
\]
then
\[
\lambda Ay = BC^{-1}B'y
\]
where \(y = A^{-1}Bx\) and \(x = C^{-1}B'y\) and vice versa.

3.3. Asymptotic results about the estimators

Since the parameter \(\beta\) is not identified we can not expect to get a reasonable estimator for \(\beta\). We can however normalize the estimator so that the asymptotic properties can be formulated in a way that is useful for deriving further results. Choose \(\gamma (p \times (p-r))\) of full rank such that \(\beta' \gamma = 0\), then we can decompose \(\hat{\beta} = \beta \hat{b} + \gamma \hat{g}\), where \(\hat{b} = (\beta' \beta)^{-1} \beta \hat{\beta}\). The following result was proved in Johansen (1988b) except for a simplified expression for the variance which follows from Lemma 3.3.
PROPOSITION 3.8: Under the hypothesis $H_2 : \Pi = \alpha\beta'$ the maximum likelihood estimator $\hat{\beta}$ has the representation

$$(3.22) \; T[\hat{\beta}\Sigma^{-1} - \beta] = \gamma(\gamma'S_{kk}\gamma'T)^{-1}\gamma'(T^{-1} \sum_{t=1}^{T} X_t e_t'^t)\Lambda^{-1}a(a'\Lambda^{-1}a)^{-1} + o_p(1),$$

which converges in distribution to

$$\frac{1}{\gamma} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left[ \gamma UU'du \right]^{-1} \gamma UU'du',$$

where $U$ and $V$ are independent Brownian motions, see Lemma 3.4, given by

$$U = \gamma'CW,$$

$$V = (\alpha'\Lambda^{-1}a)^{-1}\alpha'\Lambda^{-1}W,$$

such that the variance of $V$ is given by $\text{Var}(V) = (\alpha'\Lambda^{-1}a)^{-1}$. The expression for the variance is found as follows. In Johansen (1988b) it was shown that (3.22) holds with $\Lambda^{-1}a(\alpha'\Lambda^{-1}a)^{-1}$ replaced by

$$\Sigma_{00}^{-1}\Sigma_{0k}\beta(\beta'\Sigma_{kk}^{-1}\Sigma_{0k}^{-1}\Sigma_{00}^{-1}\beta)^{-1}\beta'\Sigma_{kk}^{-1}\beta.$$  

Now apply (3.12) to replace $\beta'\Sigma_{kk}^{-1}\beta$ by an expression in $\alpha$ and we find

$$\Sigma_{00}^{-1}a(a'\Sigma_{00}^{-1}a)^{-1}$$

which by (3.14) equals the expression in (3.22) from which the variance follows easily.

Note that the limiting distribution for fixed $U$ is Gaussian with mean zero and variance

$$\frac{1}{\gamma} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left[ \gamma UU'du \right]^{-1} \gamma UU'du' \otimes (\alpha'\Lambda^{-1}a)^{-1}.$$  

Thus the limiting distribution of that part of $\hat{\beta}$ which is orthogonal to $\beta$ is a mixture of Gaussian distributions.

Next we shall find an asymptotic representation for $\hat{\alpha}$ as well as the asymptotic distribution. The estimate of $\alpha$ has to be normalized in a
way similar to that of $\beta$.

**PROPOSITION 3.9:** Under the hypothesis $H_2: \Pi = \alpha \beta'$ the estimator $\hat{\alpha}$ has the representation

$$T^{1/2}(\hat{\alpha} - \alpha) = T^{1/2}(-S_{0k}^\beta \beta' S_{kk}^\beta)^{-1} - \alpha + o_p(1)$$

which converges weakly to a Gaussian distribution of dimension $p \times r$ with mean zero and variance matrix $\Lambda \otimes (\beta' S_{kk}^\beta)^{-1}$.

**PROOF:** The definition of $\hat{\alpha}$ is

$$\hat{\alpha} = -S_{0k}^\beta (\beta' S_{kk}^\beta)^{-1}$$

and by (3.20) we can replace $\beta$ by $\beta$, which shows

$$\hat{\alpha} = -S_{0k}^\beta (\beta' S_{kk}^\beta)^{-1} + o_p(T^{-1})$$

and hence the representation (3.23)

$$T^{1/2}(\hat{\alpha} - \alpha) = -T^{1/2}(S_{0k}^\beta (\beta' S_{kk}^\beta)^{-1} + \alpha) + o_p(T^{-1/2})$$

From (3.21) we find

$$T^{1/2}(\hat{\alpha} - \alpha) = -T^{1/2}U_T(\beta' S_{kk}^\beta)^{-1} + o_p(1)$$

which proves Proposition 3.9.

Note that the asymptotic distribution of $\hat{\alpha}$ is not influenced by the asymptotic variance of $\hat{\beta}^{-1}$, since $\hat{\beta}^{-1}$ converges so fast to $\beta$, see Stock (1987) for a similar result for regression estimates of cointegration vectors.

**COROLLARY 3.10:** Under the hypothesis $H_1: \Pi = \alpha \beta'$ the asymptotic distribution of $T^{1/2}(\hat{\Pi} - \Pi)$ is Gaussian in $p \times r$ dimensions with mean 0 and a variance matrix given by

$$\Lambda \otimes (\beta' S_{kk}^\beta)^{-1}$$

which is consistently estimated by
We have found the estimate \( \hat{\alpha} = -S_{0k}\hat{\beta} \) and its asymptotic properties but it is convenient to have a representation of \( \hat{\alpha} \) as the solution to an eigenvalue problem, since this will give a function that \( \hat{\alpha} \) maximizes and the asymptotic properties of \( \hat{\alpha} \) and in particular the test statistics can be found by the usual methods. The idea is to maximize with respect to \( \Lambda, \beta, \) as well as the matrices \( \Gamma_j \), and find the likelihood profile with respect to \( \alpha \). We introduce \( \delta \) such that \( \delta \) is px(p-r). \((\alpha, \delta)\) has full rank and \( \delta'\alpha = 0 \) i.e. \( \text{sp}(\delta) = \text{sp}(\alpha) \perp \delta = \alpha \perp \).

We then find as in section 2, see (2.9), that
\[
L_{\text{max}}^{-2/T(\alpha, \beta)} = \left| S_{00} + \alpha \beta' S_{k0} + S_{0k} \beta' \alpha' \right|
\]
and hence for fixed \( \alpha \) and \( \delta \)
\[
|\alpha'\alpha| |\delta'\delta| \min_{\beta} L_{\text{max}}^{-2/T(\alpha, \beta)} = \\
\min_{\beta} \left| \begin{array}{cccc}
\alpha' & S_{00} & + & \beta' S_{k0} \\
\delta' & S_{0k} & + & \delta' S_{00}
\end{array} \right|
\]

In the last expression we have replaced \( \beta'\alpha \alpha \) by \( \beta \) since this gives the same minimum. To simplify the notation we introduce \( S_{\alpha\alpha} = \alpha' S_{00} \alpha \), \( S_{\alpha k} = \alpha' S_{0k} \) and \( S_{\alpha \delta} = S_{\alpha \alpha} - S_{\alpha \delta} S_{\delta \delta} S_{\delta \alpha} \) etc., then
\[
|\alpha'\alpha| |\delta'\delta| \min_{\beta} L_{\text{max}}^{-2/T(\alpha, \beta)} = \\
|S_{\delta \delta}| \min_{\beta} \left| S_{\alpha\alpha} + \beta' S_{k\alpha} \delta + S_{ak} \beta + \beta' S_{kk} \beta \right|
\]
This expression is minimized by
\[
\hat{\beta}(\alpha) = -S_{kk}^{-1} S_{k\alpha} \delta
\]
giving a minimum value of
\[
(S_{00} - \alpha' \alpha') \Theta(\beta' \beta').
\]
Now we use the identity

\[ |s_{\delta\delta}||s_{\alpha\delta} - s_{\alpha\delta}^{-1}s_{k\delta}k\delta| = |s_{\delta\delta}||s_{\alpha\delta}||. \]

which gives the representation

\[ (3.24) \quad L_{\text{max}}^{-2/T}(\alpha) = |s_{\delta\delta}||s_{\alpha\delta}||s_{k\delta}k\delta|/|s_{\alpha\delta}^{-1}s_{k\delta}k\delta| |. \]

Thus \( \hat{\delta} \) and thereby \( \hat{\alpha} \) can be determined by solving the eigenvalue problem

\[ (3.25) \quad |\lambda s_{\delta\delta} - s_{\delta\delta}^{-1}s_{k\delta}k\delta| = 0, \]

and choosing \( \hat{\delta} = (\hat{u}_{r+1}, \ldots, \hat{u}_n) \), corresponding to the \( p-r \) smallest eigenvalues.

We can now apply Lemma 3.7 to see that the solutions of \( (3.25) \) are the same as the solutions of \( (2.18) \), and that we can choose \( \hat{u}_1 \)

\[ = s_{\delta\delta}^{-1}s_{k\delta}k\delta v_1. \]

From \( (u_1, \ldots, u_r)'s_{\delta\delta} = 0 \) it follows that we can choose \( \hat{\alpha} = -s_{\delta\delta}^{-1}(u_1, \ldots, u_r) = -s_{\delta\delta}^{-1}v_1. \)

Thus we get the same solution as in section 2 but now as the solution of an optimization problem, a representation which will be convenient in the following.

We shall now find the properties of \( \hat{\delta} \). We let \( \hat{\delta} = \alpha + 2\hat{\delta} \), then \( d \)

\[ = (\delta\delta)^{-1}\delta\delta, \]

and we have

**Proposition 3.11** For \( T \to \infty \) we find the representation

\[ (3.26) \quad T^{1/2}(\hat{\delta}d^{-1} - \delta) = \alpha(\beta'k_0\alpha)^{-1}(T^{1/2}\beta'k_0\delta) + o_p(1) \]

which converges weakly to a Gaussian distribution with mean zero and variance matrix \( \alpha(a'(k_0^{-1}a)^{-1}a') \otimes \delta'\delta. \)
PROOF: The normalization \( \hat{\delta}'S_{00}\hat{\delta} = I \) implies that \( \hat{\delta}, \hat{a} \) and \( \hat{d} \) are bounded in probability. Now consider for \( i = r+1, \ldots, p \) the equations

\[
\lambda_i \alpha'S_{00}\alpha_i + \lambda_i \alpha'S_{00}\delta_i = \alpha'S_{0k}\delta_{k}^{-1}S_{k0}\alpha_i + \alpha'S_{0k}\delta_{k}^{-1}S_{k0}\delta_i.
\]

By Lemma 3.5 we have \( \lambda_i \in O_p(T^{-1}) \) and Lemma 3.6 shows that the coefficient to \( \delta_i \) is of the order \( O_p(T^{-1/2}) \), which implies that \( \hat{\alpha}_i \in O_p(T^{-1/2}) \). Thus \( \hat{\delta}'S_{00}\hat{\delta} = I \) implies that \( \hat{\delta}'S_{00}\hat{\delta} \overset{P}{\rightarrow} I \) and hence that \( |\hat{d}| \) is bounded away from zero and finally that \( \hat{d}^{-1} \) is bounded in probability. From (3.27) we then find that

\[
\hat{\delta}_d^{-1} - \delta = \alpha\hat{\alpha}_d^{-1} = \alpha(\alpha'S_{0k}\delta_{k}^{-1}S_{k0}\alpha)^{-1}(\alpha'S_{0k}\delta_{k}^{-1}S_{k0}\delta) + o_p(1)
\]

which by Lemma 3.5 has the required representation and limit distribution.

3.4. The asymptotic distribution of the likelihood ratio test statistics

We shall now find the limiting distributions of the likelihood ratio test statistics for the various hypotheses discussed in Section 2.

The tests of \( H_2 \) in \( H_1 \) and \( H_3 \) in \( H_2 \) were discussed in Johansen (1988b). The following results were obtained:

THEOREM 3.12: Under the hypothesis \( H_2 : \Pi = \alpha\beta' \) the statistic

\[-2\ln(Q|H_2|H_1)\]

has a limit distribution which can be expressed in terms of a \( p-r \) dimensional Brownian motion \( B \) with i.i.d components as

\[
tr\{fBB'(fBB'du)^{-1}gBdB'\}.
\]
This distribution was tabulated by simulation, and approximated by a \( \chi^2 \) distribution of the form \((.85 - .58/f)\chi^2(f)\) with \( f = 2(p-r)^2 \). Further it was found that under the hypothesis \( H_3^* : \beta = H \) the statistic \(-2\ln(Q;H_3^*|H_2)\) is asymptotically distributed as \( \chi^2 \) with \( f = (p-s)r \) degrees of freedom.

The reason for getting a strange limit distribution for the first statistic is that it involves the \( p-r \) smallest eigenvalues of \((2.18)\) which are associated with the non-stationary part of the process.

We shall now as a preliminary result consider the test of a simple hypothesis concerning \( \alpha \) or \( \delta \).

**PROPOSITION 3.13:** The test statistic \(-2\ln(Q;\delta|H_2)\) of a simple hypothesis concerning \( \delta \) has the representation

\[
(3.29) \quad -2\ln(Q;\delta|H_2) = \text{Tr}\{(\delta'\Lambda_0\delta)^{-1}\delta'\Sigma_{0k}\beta(\beta'\Sigma_{kh}\beta)^{-1}\beta'\Sigma_{k0}\delta\} + o_p(1).
\]

and is asymptotically distributed as \( \chi^2 \) with \( r(p-r) \) degrees of freedom.

**PROOF:** From the expression \((3.24)\) for the likelihood profile in \( \alpha \) we get the expression

\[
-2\ln(Q) = 2\ln\{|\delta'S_{00,k}\delta|/|\delta'S_{00}\delta|\} - 2\ln\{|\delta'S_{00,k}\delta|/|\delta'S_{00}\delta|\}
\]

which by a Taylors expansion equals

\[
\text{Tr}\{(\delta'S_{00}\delta)^{-1}(\delta-\delta)'S_{00}(\delta-\delta) - (\delta'S_{00,k}\hat{\delta})^{-1}(\delta-\delta)'S_{00,k}(\delta-\delta)\} + o_p(T(\hat{\delta}-\delta)^3).
\]

Now replace \( \hat{\delta} \) by \( \hat{\delta}^* = \hat{\delta}d^{-1} \), see \((3.26)\), which is also a maximizing point for the likelihood function, and which converges in probability to \( \delta \). We
then find that
\[ \hat{\delta}^*S_{00,k}^a = \hat{\delta}^*(S_{00} - S_{0k}S_{kk}^{-1}S_{k0})\hat{\delta}^* \rightarrow \delta'\Sigma_{00}\delta = \delta'\Lambda\delta, \]
and
\[ \hat{\delta}^*S_{00,0}^a \rightarrow \delta'\Sigma_{00}\delta = \delta'\Lambda\delta, \]
since \( \beta'\Sigma_{k0}\delta = \beta'\Sigma_{kk}\beta'\delta = 0 \). We thus get the representation
\[ -2\ln(Q) = \text{tr}\{((\delta'\Lambda\delta)^{-1}T^{1/2}(\delta - \delta')\Sigma_{0k}\beta(\beta'\Sigma_{kk}\beta)^{-1}\beta'\Sigma_{k0}T^{1/2}(\delta - \delta))\} + o_p(1). \]
From the representation of \( \hat{\delta}^* - \delta \) in (3.26) we then get
\[ -2\ln(Q) = \text{tr}\{(\delta'\Lambda\delta)^{-1}T^{1/2}\delta'\Sigma_{0k}\beta(\beta'\Sigma_{kk}\beta)^{-1}\beta'\Sigma_{k0}T^{1/2}\delta\} + o_p(1) \]
which by Lemma 3.5 is asymptotically distributed as \( \chi^2 \) with \( (p-r)r \) degrees of freedom.

Consider now finally the composite hypothesis
\[ \tilde{H}_3: \alpha = A\psi. \]
where \( A \) is \( pxm \) of full rank. The restriction on \( \alpha \) can be formulated as \( \text{sp}(\alpha) \subset \text{sp}(A) \) which corresponds to the requirement that \( \text{sp}(\hat{\delta}) \) should contain the vectors in \( \text{sp}(A)^\perp \) the orthogonal complement of \( A \). Thus let \( B \) span this space, i.e. \( (A,B) \) has full rank, and \( A'B = 0 \), then we can formulate the hypothesis as
\[ \tilde{H}_3: \delta = (A\tau,B) \]
where \( \tau(m^*(m-r)) \) has to be estimated. We shall then solve the maximization problem, see (3.24)
\[
\begin{align*}
\max_{\tau} & \quad \tau'S_{aa,k}^\tau \quad \tau'S_{ab,k}^\tau \\
S_{ba,k}^\tau & \quad S_{bb,k}^\tau \\
\end{align*}
\]
\[
|S_{bb,k}^\tau||S_{bb}^{\tau=1}\max_{\tau} |\tau'(S_{aa,k} - S_{ab,k}S_{bb,k}^{\tau=1}S_{ba.k})|/|\tau'S_{aa.b}| = .
\]
\[
|S_{bb,k}^\tau||S_{bb}^{\tau=1}\max_{\tau} |\tau'(S_{aa.b} - S_{ak,b}S_{kk,b}^{\tau=1}S_{ka.b})|/|\tau'S_{aa.b}|.
\]
Hence we shall solve the equation
\[(3.30) \quad |\lambda S_{aa,b} - S_{ak,b}S_{kk,b}^{-1}S_{ka,b}| = 0,\]
giving the eigenvalues \(\tilde{\lambda}_1 > \ldots > \tilde{\lambda}_m > 0\) and eigenvectors \(\tilde{u}_1, \ldots, \tilde{u}_m\). We then choose \(\tilde{\tau} = (\tilde{u}_{r+1}, \ldots, \tilde{u}_m)\) corresponding to the \(m-r\) smallest eigenvalues. It follows from Lemma 3.7, that (3.30) and (2.38) have the same positive solutions. The eigenvectors are related by
\[\tilde{u}_i = S_{aa,b}^{-1}S_{ak,b}^{-1}\tilde{v}_i, \quad i = 1, \ldots, m.\]

From the relation
\[(\tilde{u}_1, \ldots, \tilde{u}_r)'A'S_{00,b}A(\tilde{u}_{r+1}, \ldots, \tilde{u}_m) = 0,\]

it follows that we can take
\[\tilde{\alpha} = -A'(A'A)^{-1}A'S_{00,b}A(\tilde{u}_1, \ldots, \tilde{u}_r) = -A'(A'A)^{-1}S_{ak,b}\tilde{\beta},\]
see (2.40). Thus we get the same solution as in section 2.

**THEOREM 3.14:** The likelihood ratio test of the hypothesis
\[H_3: A\beta = 0 \quad \text{or} \quad B'\alpha = 0\]
where \(A(p \times m)\) and \(B(p \times (m-r))\) are of full rank, and \(A'B = 0\) is denoted by \(-2\ln(Q;H_3|H_2)\), see (2.44), and is asymptotically distributed as \(\chi^2(r(p-m))\).

**PROOF:** Let us formulate the hypotheses in terms of \(\delta\) and use the result of Proposition 3.13 to get
\[(3.31) \quad -2\ln(Q;H_3|H_2) = \text{tr}\{(\delta'\Delta\delta)^{-1}\Omega^{1/2}\delta'\Sigma_{0k}\beta'\Sigma_{kk}\beta^{-1}\beta'\Sigma_{k0}\delta\Omega^{1/2}\}
- \text{tr}\{(\tau'A_{aa,b}\tau)^{-1}\Omega^{1/2}\tau'\Sigma_{ak,b}\beta'\Sigma_{kk,b}\beta^{-1}\beta'\Sigma_{ka,b}\tau\Omega^{1/2}\} + o_p(1).\]
where $\delta$ has the form $(\Lambda_r, B)$. Now $\alpha'B = 0$ implies that $\beta'\Sigma_k^0B = -\beta'\Sigma_{kk}\alpha'B = 0$ and hence that $\beta'\Sigma_{kk}\beta = \beta'(\Sigma_{kk} - \Sigma_k^0B(\Sigma_{kk}B)^{-1}B'^T\Sigma_k^0\beta = \beta'\Sigma_{kk}\beta$. From the form of $\delta$ we get that the first term contains the factors

$$
\begin{bmatrix}
\tau'S_{ak}\beta \\
S_{bk}\beta
\end{bmatrix}'
\begin{bmatrix}
\tau'A_{aa} \tau'\Lambda_{ab} \\
\Lambda_{ba} \tau'\Lambda_{bb}
\end{bmatrix}^{-1}
\begin{bmatrix}
\tau'S_{ak}\beta \\
S_{bk}\beta
\end{bmatrix} = \\
\beta'S_{ka,b}(\tau'\Lambda_{aa,b})^{-1}\tau'S_{ak}\beta + \beta'S_{kb,bb}^{-1}S_{bk}\beta.
$$

The first term of (3.32) cancels the second term in (3.31) and we get the representation

$$
-2\ln(Q; H_3 | H_2) = \text{Tr}((\beta'\Sigma_{kk}\beta)^{-1}B'S_{kb,bb}^{-1}S_{bk}\beta) + o_p(1),
$$

which by Lemma 3.5 is asymptotically distributed as $\chi^2$ with $r(p-m)$ degrees of freedom since $B'\alpha = 0$.

5. Wald tests for hypotheses about $\alpha$ and $\beta$

We shall consider Wald tests which are very easy to calculate once the eigenvectors and eigenvalues have been calculated under the hypothesis $H_2$. Let us first consider a test for the hypothesis concerning $\alpha$ and let us express it as

$$
\tilde{H}_3 : B'\alpha = 0.
$$

A Wald test can be constructed by suitably normalizing the statistic $B'\alpha$.

THEOREM 3.15: Under the hypothesis $\tilde{H}_3 : B'\alpha = 0$ where $B$ is $p \times (p-m)$ of full rank the asymptotic distribution of

$$
\text{Tr}((B'(S_{00} - \hat{\alpha}\hat{\alpha}')B)^{-1}(B'\hat{\alpha}\hat{\alpha}'B))
$$

is $\chi^2$ with $(p-m)r$ degrees of freedom.
PROOF: In view of the results of Proposition 3.9 we can consider
\[ T_{\text{tr}}\{A_{bb}^{-1}B^{\prime}\hat{\alpha}\hat{b}^{\prime}(\beta^{\prime}\Sigma_{kk}\beta)\hat{b}\hat{\alpha}B\} \]
which is asymptotically distributed as $\chi^2$ with $r(p-m)$ degrees of freedom.
To apply the test we need consistent estimates for the variance matrices, and we thus insert the consistent estimates
\[ \hat{\beta}^{\prime}S_{kk}\hat{\beta} = b^{\prime}-1\hat{\beta}^{\prime}S_{kk}\hat{\beta}^{-1} = b^{\prime}-1^{-1} \]
and
\[ B^{\prime}(S_{00} - \hat{\alpha}\hat{\alpha}^\prime)B, \]
and the result follows. If we apply the result (3.23) we get the representation
\[ B^{\prime}(\hat{\alpha}\hat{b} - \alpha) = - S_{bk}\beta^{\prime}S_{kk}\beta^{-1} + O_p(T^{-1/2}) \]
and we see that the Wald test is just the quadratic approximation derived to the likelihood ratio test in (3.33)

One can derive a different expression for this statistic by using the alternative derivation of $\hat{\alpha} = - S_{00}\hat{u}$, where $\hat{u}$ consists of the $r$ eigenvectors of (3.25) corresponding to the $r$ largest eigenvalues. Since $\hat{u}\hat{u}^{\prime} + \hat{\delta}\hat{\delta}^{\prime} = S_{00}^{-1}$ we find the expression
\[ T_{\text{tr}}\{(B^{\prime}S_{00}\hat{\delta}\hat{\delta}^{\prime}S_{00}B)^{-1}(B^{\prime}S_{00}\hat{u}\hat{u}^{\prime}S_{00}B)\} \]
Let us next consider the hypothesis $H_3^\times$ but expressed as
\[ H_3^\times: K^{\prime}\beta = 0, \]
where $K$ is $p \times (p-s)$ of full rank. This suggests a Wald test on the statistic $K^{\prime}\hat{\beta}$ and the problem is again how to normalize it.
THEOREM 3.16: Under the hypothesis $H^*_3: K\beta = 0$, where $K$ is $p \times (p-s)$ of full rank, the asymptotic distribution of

\[(3.36) \quad \text{Tr}\{(K'\hat{\Pi}'(S_{00} - \hat{\alpha}\hat{\alpha}')^{-1}K')(K'\gamma\gamma'K)^{-1}\}\]

is $\chi^2$ with $(p-s)r$ degrees of freedom.

PROOF: The idea of this test is to note that the limiting distribution of $K'\hat{\beta}b^{-1}$ is a mixture of Gaussian distributions, see Proposition 3.8, hence the asymptotic distribution of

\[(3.37) \quad \frac{1}{(K'\gamma(JUU'du)^{-1}\gamma'K)^{-1/2}}(TK'\hat{\beta}b^{-1})(\alpha'\Lambda^{-1}\alpha)^{1/2}\]

will be, for fixed $U$, a Gaussian random matrix of dimension $(p-s) \times r$ with mean zero and covariance matrix $I\otimes I$. Since this result is the same for all fixed $U$ it also holds unconditionally, from which we derive that

\[(3.38) \quad \text{tr}\{(K'\gamma(JUU'du)^{-1}\gamma'K)^{-1}(TK'\hat{\beta}b^{-1})(\alpha'\Lambda^{-1}\alpha)(\hat{b}^{-1}\hat{\beta}'K)\}\]

is asymptotically $\chi^2$ with $(p-s)r$ degrees of freedom. We shall now insert consistent estimates for the variance matrices. The first one follows from Proposition 3.9 and the consistency of $\hat{\Lambda} = S_{00} - \hat{\alpha}\hat{\alpha}'$:

$$\hat{\beta}a'(S_{00} - \hat{\alpha}\hat{\alpha}')^{-1}\hat{\beta}b \rightarrow \alpha'\Lambda^{-1}\alpha.$$  

For the second we first apply Lemma 3.4 and the definition of $U$ in Proposition 3.8:

$$K'\gamma(S_{kk}/T\gamma)^{-1}\gamma'K \rightarrow K'\gamma(JUU'du)^{-1}\gamma'K.$$  

We shall now show how to estimate the left hand side from the data.

We first decompose $\hat{\gamma} = \hat{\gamma}c + \hat{\gamma}e$, and find from the equation

$$\hat{\gamma}'S_{00}S_{0k}\hat{\gamma}_1 = \hat{\lambda}_i \in O_p(T^{-1}), \ i = r+1, \ldots, p,$$

that $\hat{\gamma}_i$ and hence $\hat{c}_i$ and $\hat{e}_i \in O_p(T^{-1/2})$. Then the normalization
\[ I = \hat{\gamma}'S_{kk}\hat{\gamma} = \hat{e}'\beta'S_{kk}\hat{\beta} + \hat{c}'\gamma'S_{kk}\hat{\gamma} + \hat{c}'\gamma'S_{kk}\hat{\gamma} + \hat{c}'\gamma'S_{kk}\hat{\gamma} \]

shows that

\[ \hat{c}'\gamma'S_{kk}\hat{\gamma} \xrightarrow{p} I. \]  

(3.39)

Note that \( K'\hat{\gamma} = K'\hat{\gamma} \), such that

\[ TK'\gamma(S_{kk})^{-1}\gamma'K = TK'\gamma(c'\gamma'S_{kk}\hat{\gamma})^{-1}\gamma'K. \]

By (3.39) this has the same limit distribution as \( TK'\hat{\gamma}\gamma'K \). Combining these results we obtain Theorem 3.16.

Note that the test looks like a Wald test on \( \hat{\mu}K \) but one has not normalized it by the asymptotic variance of \( \hat{\mu}K \) since, as is seen from Corollary 3.10, this will be zero when \( K'\beta = 0 \).

An alternative form of the test statistic which is very easy to calculate is

\[ \text{Tr}((K'\hat{\beta}(D^{-1} - I)^{-1}\hat{\mu}'K)(K'\hat{\gamma}'\gamma')^{-1}) \]

where \( D = \text{diag}(\lambda_1, \ldots, \lambda_r) \). This is seen from the identity

\[ \beta'\Sigma_{kk}\beta - \beta'\Sigma_{kk}\beta = (a'\Lambda^{-1}a)^{-1} \]

by inserting the estimates.

We shall apply it to the special case when \( r = 1 \) and \( K' = (k_1, \ldots, k_p) \), hence \( s = p-1 \), and we have only one cointegration relation where we want to test some linear constraint on the coefficients. We shall formulate the result as a Corollary.

**Corollary 3.17:** If only 1 cointegration vector \( \beta \) is present \((r = 1)\), and if we want to test the hypothesis

\[ K'\beta = 0 \]

then the test statistic
(3.41) \[ T^{1/2} K' \hat{\beta} / \left( (\hat{\lambda}_1^{-1} - 1) (K' \hat{\gamma}' K) \right)^{1/2} \]

is asymptotically normalized Gaussian. Here \( \hat{\lambda}_1 \) is the maximal eigenvalue and \( \hat{\beta} \) the corresponding eigenvector of the equation

\[ |\lambda S_{kk} - S_{k0} S_{00}^{-1} S_{0k}| = 0. \]

The remaining eigenvectors form \( \hat{\gamma} \).

The normalization \( \hat{\gamma}' S_{kk} \hat{\gamma} = I \) implies that \( \hat{\gamma}' \hat{\gamma} \) is of the order of \( T^{-1} \) which shows that \( \hat{\beta} \) is really normalized by \( T \).

Thus if there is only one cointegration vector \( \hat{\beta} \) one can think of the matrix \( (\hat{\lambda}^{-1} - 1) \hat{\gamma}' \) as giving an estimate of the asymptotic "variance" of \( \hat{\beta} \).

This result should be interpreted with care since \( K' \hat{\beta} \) is not asymptotically Gaussian and may not have an asymptotic variance but one can normalize a linear combination of the components of \( \hat{\beta} \) in such a way that it becomes asymptotically Gaussian.

A comparison with the proof of Theorem 4 (1987) shows that the representation of the asymptotic distribution of \(-2 \ln(Q; H_2^\infty | H_1)\) given there involves the same Brownian motions, such that the \( \chi^2 \) test suggested here is asymptotically the same as the likelihood ratio test. Only the introduction of \( \hat{\beta} \) makes the result more transparent.

As a final remark one should add, that we can of course test further restriction on \( \beta \) or \( \alpha \), i.e. in the hypothesis \( H_2^\infty: \beta = H \phi \) we can test the hypothesis \( H_2^\infty: \phi = H_1 \eta \) and derive the corresponding likelihood ratio test or Wald test. These will be asymptotically distributed as \( \chi^2 \) with the appropriate degrees of freedom corresponding to the loss of parameters in going from \( H_2^\infty \) to \( H_2^\infty \). We shall not formulate these results in detail here.

University of Copenhagen
REFERENCES


Discussion paper, LSE.


M2 (REAL DEMAND FOR MONEY)
QUARTERLY DATA FROM 1974:1 TO 1985:4

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Y" (REAL NATIONAL INCOME)
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I" (BOND INTEREST RATE)
QUARTERLY DATA FROM 1973:1 TO 1985:4

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I" (DEPOSITE INTEREST RATE)
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I" (DEPOSITE INTEREST RATE)
QUARTERLY DATA FROM 1973:1 TO 1985:4

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### Y (Nominal Income)

**Quarterly Data from 1958:1 to 1984:4**

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### P (THE PRICE LEVEL)
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### I" (THE MARGINAL INTEREST RATE)
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APPENDIX B Model (2.1) for the Danish data

\[ \Delta X_t = \Gamma_1 \Delta X_{t-1} - \Pi X_{t-2} + \text{const} + \sum_i k_i \eta_{it} + \epsilon_t \]

\[ \Gamma_1 = \begin{bmatrix} -0.46 & 0.02 & -1.18 & 0.02 \\ (0.25) & (0.22) & (0.60) & (0.82) \\ -0.01 & -0.45 & 0.23 & -1.21 \\ (0.28) & (0.25) & (0.67) & (0.93) \\ -0.05 & 0.08 & -0.29 & -0.08 \\ (0.07) & (0.07) & (0.18) & (0.25) \\ 0.05 & 0.02 & 0.39 & -0.05 \\ (0.05) & (0.04) & (0.11) & (0.16) \end{bmatrix} \]

\[ -\Pi = \begin{bmatrix} -0.25 & 0.14 & -1.98 & 1.77 \\ (0.12) & (0.17) & (0.55) & (0.64) \\ 0.09 & -0.31 & 0.16 & 0.22 \\ (0.13) & (0.19) & (0.63) & (0.72) \\ 0.00 & 0.00 & -0.02 & 0.17 \\ (0.04) & (0.05) & (0.17) & (0.20) \\ -0.00 & 0.01 & 0.14 & -0.33 \\ (0.02) & (0.04) & (0.11) & (0.13) \end{bmatrix} \]

Estimated correlations and variances of regression residuals.

\[ C_\epsilon = \begin{bmatrix} 0.028^2 & -0.031^2 & 0.009^2 & 0.006^2 \\ -0.39 & 0.03 & 0.11 & 0.19 \end{bmatrix} \]

Residual autocorrelations, \( r_i \) (i=1,..,8) and Box-Pierce Q-statistic, \( x^2 \)-distributed with 18 degrees of freedom.

\[
\begin{array}{cccccccc}
\text{r}_i & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & Q(18) \\
\hline
\Delta m1 & -0.03 & -0.09 & 0.12 & -0.08 & 0.11 & 0.02 & 0.10 & 0.06 & 7.8 \\
A y & -0.10 & -0.17 & 0.10 & -0.21 & 0.13 & -0.02 & 0.07 & -0.06 & 10.7 \\
\Delta m & 0.14 & 0.06 & 0.10 & -0.18 & 0.03 & -0.02 & -0.20 & 0.00 & 10.9 \\
\Delta^2 p & -0.02 & 0.01 & 0.00 & -0.10 & 0.00 & -0.24 & 0.04 & -0.07 & 7.5 \\
\end{array}
\]
Appendix B Model (2.1) for the Finnish data

\[ \Delta X_t = \Gamma_1 \Delta X_{t-1} + \Pi X_{t-2} + \text{const} + \sum \kappa_i q_{it} + \epsilon_t \]

\[
\begin{bmatrix}
-0.21 & -0.22 & -0.10 & -0.01 \\
(0.11) & (0.14) & (0.13) & (0.45) \\
-0.04 & -0.69 & -0.06 & -0.16 \\
(0.07) & (0.09) & (0.08) & (0.27) \\
-0.16 & -0.02 & -0.23 & 0.42 \\
(0.08) & (0.11) & (0.10) & (0.33) \\
0.01 & 0.02 & 0.02 & -0.82 \\
(0.03) & (0.03) & (0.03) & (0.11)
\end{bmatrix}
\]

\[
\begin{bmatrix}
-0.12 & 0.11 & -0.23 & -0.42 \\
(0.06) & (0.06) & (0.11) & (0.54) \\
0.02 & 0.04 & -0.14 & 0.21 \\
(0.04) & (0.04) & (0.07) & (0.33) \\
0.09 & 0.10 & -0.48 & 0.67 \\
(0.04) & (0.04) & (0.08) & (0.40) \\
0.00 & 0.01 & -0.00 & -0.48 \\
(0.01) & (0.01) & (0.03) & (0.13)
\end{bmatrix}
\]

Estimated correlations and variances of regression residuals.

\[
C_\epsilon = \begin{bmatrix}
0.05^2 & 0.31^2 & 0.037^2 & 0.012^2 \\
0.25 & 0.19 & -0.31 & -0.02
\end{bmatrix}
\]

Residual autocorrelations, \( r_i \) (i=1,...,8) and Box-Pierce Q-statistic, \( \chi^2 \)-distributed with 30 degrees of freedom.

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<td>-.01</td>
<td>-.17</td>
<td>.02</td>
<td>-.02</td>
<td>.01</td>
<td>-.05</td>
<td>.26</td>
<td>31.3</td>
</tr>
<tr>
<td>( \Delta y )</td>
<td>.01</td>
<td>-.23</td>
<td>-.15</td>
<td>.22</td>
<td>-.16</td>
<td>.03</td>
<td>-.02</td>
<td>.14</td>
<td>59.2</td>
</tr>
<tr>
<td>( \Delta i^m )</td>
<td>.01</td>
<td>-.01</td>
<td>-.05</td>
<td>-.05</td>
<td>-.04</td>
<td>.05</td>
<td>.10</td>
<td>-.08</td>
<td>26.0</td>
</tr>
<tr>
<td>( \Delta^2 p )</td>
<td>.04</td>
<td>-.02</td>
<td>-.14</td>
<td>.10</td>
<td>.08</td>
<td>-.04</td>
<td>.02</td>
<td>.02</td>
<td>24.6</td>
</tr>
</tbody>
</table>
APPENDIX C.

Product moment matrices from the regression of $\Delta X_t$ on $\Delta X_{t-1}$ and $X_{t-2}$, for Denmark (m2,y,ib,ib) and Finland (m1,y,im,Ap). The matrices are given in the form

$$\begin{bmatrix}
S_{00} & S_{01} & S_{02} \\
S_{10} & S_{11} & S_{12} \\
S_{20} & S_{21} & S_{22}
\end{bmatrix} \times 10^3$$

<table>
<thead>
<tr>
<th>DK</th>
<th>SF</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0.874$</td>
<td>$2.488$</td>
</tr>
<tr>
<td>$0.569$</td>
<td>$0.408$</td>
</tr>
<tr>
<td>$-0.067$</td>
<td>$-0.135$</td>
</tr>
<tr>
<td>$-0.051$</td>
<td>$-0.143$</td>
</tr>
<tr>
<td>$-0.190$</td>
<td>$-1.831$</td>
</tr>
<tr>
<td>$-0.360$</td>
<td>$-0.893$</td>
</tr>
<tr>
<td>$-0.143$</td>
<td>$-0.615$</td>
</tr>
<tr>
<td>$0.007$</td>
<td>$-0.068$</td>
</tr>
<tr>
<td>$0.192$</td>
<td>$4.155$</td>
</tr>
<tr>
<td>$0.093$</td>
<td>$3.934$</td>
</tr>
<tr>
<td>$-0.049$</td>
<td>$0.527$</td>
</tr>
<tr>
<td>$-0.041$</td>
<td>$0.222$</td>
</tr>
<tr>
<td>$-0.049$</td>
<td>$-0.041$</td>
</tr>
<tr>
<td>$-0.042$</td>
<td>$0.223$</td>
</tr>
<tr>
<td>$-0.067$</td>
<td>$0.046$</td>
</tr>
<tr>
<td>$-0.018$</td>
<td>$0.027$</td>
</tr>
</tbody>
</table>
TABLE I
The estimate of the unrestricted matrix $\Pi$ in model $H_1$ for the Danish and Finnish data.

<table>
<thead>
<tr>
<th></th>
<th>DK</th>
<th>SF</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_2$</td>
<td>.248 - .133</td>
<td>.119 - .110</td>
</tr>
<tr>
<td>$y$</td>
<td>-.095 .308</td>
<td>-.024 .036</td>
</tr>
<tr>
<td>$i^b$</td>
<td>.004 -.002</td>
<td>-.095 .104</td>
</tr>
<tr>
<td>$i^d$</td>
<td>.001 -.010</td>
<td>-.004 -.007</td>
</tr>
</tbody>
</table>

$$\begin{bmatrix}
\Pi
\end{bmatrix}_{\text{DK}} = 
\begin{bmatrix}
.248 & - .133 & 1.978 & -1.767 \\
-.095 & .308 & -.156 & -.231 \\
.004 & -.002 & .024 & -.174 \\
.001 & -.010 & -.144 & .333
\end{bmatrix}_{\text{SF}} = 
\begin{bmatrix}
.119 & - .110 & .227 & .418 \\
-.024 & .036 & .137 & .214 \\
-.095 & .104 & .479 & -.670 \\
-.004 & -.007 & .003 & .484
\end{bmatrix}$$
TABLE II

The eigenvalues $\hat{\lambda}$ and eigenvectors $\hat{V}$ as well as $-S_{02}^\gamma V$ for the Danish and Finnish data.

<table>
<thead>
<tr>
<th></th>
<th>DK</th>
<th>SF</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>eigenvalues $\hat{\lambda}$</td>
<td>eigenvalues $\hat{\lambda}$</td>
</tr>
<tr>
<td></td>
<td>(.5004 .1940 .1269 .0138)</td>
<td>(.3093 .2260 .0731 .0295)</td>
</tr>
<tr>
<td></td>
<td>eigenvectors $\hat{V}$</td>
<td>eigenvectors $\hat{V}$</td>
</tr>
<tr>
<td>y</td>
<td>18.61 25.04 -26.05 1.29</td>
<td>2.86 -6.06 10.24 2.22</td>
</tr>
<tr>
<td>iₜ</td>
<td>-130.40 -17.44 29.81 .54</td>
<td>20.79 -9.23 -2.99 -1.90</td>
</tr>
<tr>
<td>iₙ</td>
<td>105.05 -73.35 -83.90 -24.94</td>
<td>20.58 104.17 20.28 -21.62</td>
</tr>
</tbody>
</table>

$-S_{02}^\gamma V \times 10^3$

<table>
<thead>
<tr>
<th></th>
<th>DK</th>
<th>SF</th>
</tr>
</thead>
<tbody>
<tr>
<td>m2</td>
<td>-16.33 4.22 -2.59 -1.64</td>
<td>11.37 4.43 -11.68 1.86</td>
</tr>
<tr>
<td>y</td>
<td>-3.96 6.98 -8.00 -1.05</td>
<td>7.69 1.25 1.16 4.55</td>
</tr>
<tr>
<td>iₜ</td>
<td>-.33 -1.38 -1.48 .66</td>
<td>18.25 -10.23 -.66 -1.58</td>
</tr>
<tr>
<td>iₙ</td>
<td>1.22 -1.99 -.65 -1.16</td>
<td>1.89 3.79 1.31 -1.11</td>
</tr>
</tbody>
</table>
TABLE III

Test statistics for the hypothesis $H_2$ for various values of $r$ versus the general alternative $H_1$ for the Danish and Finnish data. The quantiles from the asymptotic distribution are taken from Johansen [9].

<table>
<thead>
<tr>
<th>$H_2$</th>
<th>$-2\ln(Q;DK)$</th>
<th>$-2\ln(Q;SF)$</th>
<th>95% quantile</th>
<th>90% quantile</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r \leq 3$</td>
<td>0.64</td>
<td>3.11</td>
<td>4.2</td>
<td>2.9</td>
</tr>
<tr>
<td>$r \leq 2$</td>
<td>6.88</td>
<td>11.01</td>
<td>12.0</td>
<td>10.3</td>
</tr>
<tr>
<td>$r \leq 1$</td>
<td>16.80</td>
<td>37.65</td>
<td>23.8</td>
<td>21.2</td>
</tr>
<tr>
<td>$r = 0$</td>
<td>48.72</td>
<td>76.14</td>
<td>38.6</td>
<td>35.6</td>
</tr>
</tbody>
</table>
TABLE IV

Eigenvalues $\lambda^*$ and eigenvectors $\varphi^*$ as well as $-S_{0k}H\varphi^*$ for the Danish and Finnish data under the restriction that money demand and national income have equal coefficients with opposite sign.

<table>
<thead>
<tr>
<th></th>
<th>DK</th>
<th>SF</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>eigenvales $\lambda^*$</td>
<td>eigenvales $\lambda^*$</td>
</tr>
<tr>
<td></td>
<td>( .4999 .1759 .0462 )</td>
<td>( .3093 .1994 .0704 )</td>
</tr>
<tr>
<td></td>
<td>eigenvectors $\varphi^*$</td>
<td>eigenvectors $\varphi^*$</td>
</tr>
<tr>
<td>m2-y</td>
<td>$\begin{bmatrix} -19.77 &amp; -8.96 &amp; 18.09 \end{bmatrix}$</td>
<td>$m1-y \begin{bmatrix} -2.95 &amp; 6.02 &amp; -10.30 \end{bmatrix}$</td>
</tr>
<tr>
<td>i^b</td>
<td>$\begin{bmatrix} -131.45 \ 0.05 \ 16.57 \end{bmatrix}$</td>
<td>$i^m \begin{bmatrix} 20.92 &amp; -8.67 &amp; -3.61 \end{bmatrix}$</td>
</tr>
<tr>
<td>i^d</td>
<td>$\begin{bmatrix} 106.00 &amp; -107.58 &amp; -6.46 \end{bmatrix}$</td>
<td>$\Delta p \begin{bmatrix} 18.56 &amp; 86.87 &amp; 17.05 \end{bmatrix}$</td>
</tr>
</tbody>
</table>

|        | $-S_{0k}H\varphi^* \times 10^3$ | $-S_{0k}H\varphi^* \times 10^3$ |
|        | m2                              | m1                              |
| y      | $\begin{bmatrix} -16.15 & 2.27 & -.66 \end{bmatrix}$ | $\begin{bmatrix} 11.36 & 6.26 & -10.70 \end{bmatrix}$ |
| i^b    | $\begin{bmatrix} -3.59 & 2.02 & -4.24 \end{bmatrix}$ | $\begin{bmatrix} 7.72 & 2.71 & 2.31 \end{bmatrix}$ |
| i^d    | $\begin{bmatrix} -0.33 & -1.85 & -1.04 \end{bmatrix}$ | $\begin{bmatrix} 18.40 & -9.78 & -1.30 \end{bmatrix}$ |
| $\Delta p$ | $\begin{bmatrix} 1.20 & -2.07 & .17 \end{bmatrix}$ | $\begin{bmatrix} 1.82 & 2.99 & 1.09 \end{bmatrix}$ |
TABLE V

The estimate of the restricted matrix \( \Pi \) in model \( H_2 \) for the Danish and Finnish data. For the Danish data \( r = 1 \) and the coefficients to \( m_2 \) and \( y \) are equal with opposite sign. The same holds for the coefficients to \( i^b \) and \( i^d \). For the Finnish data \( r = 3 \) and the coefficients of \( m_2 \) and \( y \) are equal with opposite sign.

<table>
<thead>
<tr>
<th></th>
<th>DK</th>
<th>SF</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m_2 )</td>
<td>( \begin{bmatrix} .275 &amp; -.275 &amp; 2.076 &amp; -2.076 \end{bmatrix} )</td>
<td>( \begin{bmatrix} .114 &amp; -.114 &amp; -.222 &amp; -.572 \end{bmatrix} )</td>
</tr>
<tr>
<td>( y )</td>
<td>( \begin{bmatrix} .068 &amp; -.068 &amp; .513 &amp; -.513 \end{bmatrix} )</td>
<td>( \begin{bmatrix} -.030 &amp; .030 &amp; -.130 &amp; -.417 \end{bmatrix} )</td>
</tr>
<tr>
<td>( i^b )</td>
<td>( \begin{bmatrix} .001 &amp; -.001 &amp; -.008 &amp; .008 \end{bmatrix} )</td>
<td>( \begin{bmatrix} -.100 &amp; .100 &amp; -.474 &amp; .531 \end{bmatrix} )</td>
</tr>
<tr>
<td>( i^d )</td>
<td>( \begin{bmatrix} -.028 &amp; .028 &amp; -.211 &amp; .211 \end{bmatrix} )</td>
<td>( \Delta p )</td>
</tr>
</tbody>
</table>
Figure 1 - Illustration of the relation between various hypotheses concerning the cointegration vectors

\[ H_1: \quad X_t = \sum_{i=1}^{k} \Pi_i X_{t-i} + \epsilon_t \]

\[ \Pi = I - \Pi_1 - \ldots - \Pi_k \]

\[ H_2: \quad \Pi = \alpha \beta^* \]

\[ \tilde{H}_3: \quad \alpha = \Lambda \psi \]

\[ \tilde{H}_3^*: \quad \beta = H \varphi \]

\[ H_4 = \tilde{H}_3^* \cap \tilde{H}_3 \]
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