Søren Johansen   Katarina Juselius

Hypothesis Testing for Cointegration Vectors · with an Application to the Demand for Money in Denmark and Finland

Preprint
March
1988

2

Institute of Mathematical Statistics
University of Copenhagen

ISSN 0902-8846

Søren Johansen and Katarina Juselius

HYPOTHESIS TESTING FOR COINTEGRATION VECTORS - WITH AN APPLICATION TO THE DEMAND FOR MONEY IN DENMARK AND FINLAND

Preprint 1988 No. 2

INSTITUTE OF MATHEMATICAL STATISTICS
UNIVERSITY OF COPENHAGEN

March 1988

HYPOTHESIS TESTING FOR COINTEGRATION VECTORS -WITH AN APPLICATION TO THE DEMAND FOR MONEY IN DENMARK AND FINLAND

By Søren Johansen and Katarina Juselius

HEADNOTE

The purpose of this paper is to give a systematic account of the maximum likelihood inference concerning cointegration vectors in non-stationary vector valued autoregressive time series with Gaussian errors. The hypothesis of r cointegration vectors is given a simple parametric formulation in terms of cointegration vectors and their weights. We then estimate and test linear hypotheses about these. We find that the asymptotic inference for the linear hypotheses can be performed by applying the usual χ^2 test. We also give some very simple Wald test and their asymptotic properties. The methods are illustrated by data from the Danish and the Finnish economy on the demand for money.

Keywords: Cointegration, Error corection, maximum likelihood estimation, likelihood ratio test, vector autoregressive processes, money demand.

Søren Johansen
Institute of Mathematical Statistics
University of Copenhagen
Universitetsparken 5
DK-2100 Copenhagen Ø
Denmark

Katarina Juselius
Institute of Economics
University of Copenhagen
Studiestræde 6
DK-1455 Copenhagen K
Denmark

1. Introduction

Many papers have over the last few years been devoted to the estimation and test of long-run relations under the heading of *cointegration*, Granger (1981), Granger and Weiss (1983), Engle and Granger (1987), Stock (1987), Phillips and Ouliaris (1986), (1987), Johansen (1988b), *canonical analysis*, Box and Tiao (1981), Velu, Wichern and Reinsel (1987), Peña and Box (1987), *reduced rank regression*, Velu, Wichern, and Reinsel (1986), and Ahn and Reinsel (1987), *common trends*, Stock and Watson (1987), *regression with integrated regressors*, Phillips (1987), Phillips and Park (1986a), (1986b), (1987), as well as under the heading *testing for unit roots*, see for instance Sims, Stock, and Watson (1986). There is a special issue of Oxford Bulletin of Economics and Statistics (1986) dealing mainly with cointegration and a special issue of Journal of Economic Dynamics and Control (1988) dealing with the same problems.

The solution we propose to this problem is to start with a relatively simple model specifying a vectorvalued autoregressive process (VAR) with independent Gaussian errors, and formulate the hypothesis of reduced rank or the hypothesis of the existence of cointegration vectors in a simple parametric form which allows the application of the method of maximum likelihood and likelihood ratio tests. In this way we can derive estimates and test statistics for the hypothesis of of a given number of cointegration vectors, as well as estimates and tests for linear hypotheses about the cointegration vectors and their weights. We have also derived the asymptotic properties of these statistics, and illustrated the methods by money demand data from the Danish and Finnish economy.

We will consider the case where the observed data is a sequence af random vectors X_t with components $(X_{1t},\ldots,X_{pt})$ drawn sequentially from a p-dimensional Gaussian distribution with mean μ_t and variance matrix Λ, where μ_t depends linearly on the past k values of the process.

Thus we consider the model

$$(1.1) \qquad H_1: \quad X_t = \Pi_1 X_{t-1} + \ldots + \Pi_k X_{t-k} + \epsilon_t, \qquad (t = 1,\ldots,T)$$

where $\epsilon_1,\ldots,\epsilon_T$ are i.i.d. $N_p(0,\Lambda)$ and $X_{-k+1},\ldots,X_0$ are fixed.

The unrestricted parameters $(\Pi_1,\ldots,\Pi_k,\Lambda)$ have to be estimated on the basis of T observations from a vector autoregressive process.

In general economic time series are non-stationary processes, and VAR-systems like (1.1) have usually been expressed in first differenced form. Unless the difference operator is also applied to the error process and explicitly taken account of, differencing implies loss of information in the data. Using $\Delta = 1-L$, where L is the lag operator it is convenient to rewrite the model (1.1) as

$$(1.2) \qquad \Delta X_t = \Gamma_1 \Delta X_{t-1} + \ldots + \Gamma_{k-1} \Delta X_{t-k+1} - \Pi X_{t-k} + \epsilon_t$$

where

$$\Gamma_i = -I + \Pi_1 + \ldots + \Pi_i, \qquad (i = 1,\ldots,k-1)$$

and

$$(1.3) \qquad \Pi = I - \Pi_1 - \ldots - \Pi_k.$$

Notice that model (1.2) is expressed as a traditional first differenced VAR-model except for the term ΠX_{t-k}. It is the main purpose of this paper to investigate the coefficient matrix Π as to the information it may convey concerning long-run information in the chosen data. Three cases can be considered:

(i) Rank(Π) = p, i.e. the matrix Π has full rank, indicating that the vector process X_t is stationary.

(ii) Rank(Π) = 0, i.e. the matrix Π is the null vector and (1.2) corresponds to a traditional differenced vector time series model.

(iii) $0 < \text{rank}(\Pi) = r < p$ implying that there are p×r matrices α and β such that $\Pi = \alpha\beta'$.

The cointegration vectors β have the property that $\beta' X_t$ is stationary even though X_t is self is non-stationary, see Theorem 3.1 for a precise formulation. In this case (1.2) can be interpreted as an error correction model, see Engle and Granger (1987), Davidson (1986) or Johansen (1988a).

Thus the main hypothesis we shall consider here is

(1.4) $$H_2: \Pi = \alpha\beta',$$

where α and β are p×r matrices. This can also be formulated as the condition that rank(Π) $\leq$ r.

We shall further investigate linear hypotheses expressed in terms of the coefficients α and β.

We have chosen to illustrate the procedures by data from the Danish and Finnish economy on the demand for money. The relation m = f(y,p,c) expresses money demand m as a function of income y, the price level p and the cost of holding money c. Since price homogeneity was clearly accepted by the data the empirical analysis here will be on real money, real income and some proxies measuring the cost of holding money. All variables are expressed in logarithmic form, since multiplicative effects are assumed.

The two data sets differ both as to which variables are included and the length of the sample. More interestingly, however, the institutional relations in the two economies have been quite different in the sample period. In Denmark the market forces have been allowed to play much more freely than in Finland, where both interest rates and prices have been subject to regulation for most of the sample period. One would expect this to show up in the empirical results and so it does.

For the Danish data m2 was chosen because the data available on quarterly basis has been collected using more homogeneous definitions for m2 than for m1. The cost of holding money was measured by the difference between the bank deposit rate, i^d, for interest bearing deposits (which are the main part of m2) and the bond rate, i^b, which plays a very important role in the Danish economy. The two interest rates were included unrestrictedly in the analysis, but subsequently tested for equal coefficients with opposite signs. The inflation rate, Δp, was also included as a possible proxy for the cost of holding money, but since it did not enter significantly into the cointegration relation for money demand it was omitted from the present analysis.

For the Finnish data m1 was chosen, since the m1 cointegration relation was found to enter the demand for money equation more significantly and hence illustrated the methodology better. Since th interest rates have been regulated a good proxy for the actual costs of holding money is difficult to find. The inflation rate, Δp, is a natural candidate and therefore included in the data set. Moreover, the marginal rate of interest, i^m, of the Bank of Finland is included in spite of the fact that the marginal rate measures the restrictedness of money rather then the cost of holding money. It has however been chosen as a determinant of the Finnish money demand in other studies and therefore is also included here. All series are quarterly and the data is given in Appendix A.

The structure of the paper is the following. In Section 2 we give the derivation of the maximum likelihood estimators and likelihood ratio tests expressed in terms of eigenvectors and eigenvalues of suitable product moment matrices. The results are illustrated by the above mentioned data, and the interpretation of the results is discussed. In Section 3 we have then given a systematic account of the statistical and probabilistic results that are necessary to justify the analysis as described in section 2.

2. Maximum likelihood estimation and likelihood ratio tests of cointegration vectors

2.1. *A survey of the various hypotheses and the initial analysis of the data*

In the following we will use model (1.1) in the form (1.2):

(2.1) $$\Delta X_t = \Gamma_1 \Delta X_{t-1} + \dots + \Gamma_{k-1} \Delta X_{t-k+1} - \Pi X_{t-k} + \epsilon_t$$

The reason for this is that the parameters

$$(\Gamma_1, \dots, \Gamma_{k-1}, \Pi, \Lambda)$$

are variation independent and, since all the models we are interested in are expressed as restrictions on Π, it is possible to maximize over all the other parameters once and for all. The hypotheses we shall consider and discuss in this section are given as follows

$$H_2: \Pi = \alpha\beta',$$
$$H_3^*: \Pi = \alpha\varphi'H' \quad \text{or } \beta = H\varphi,$$
$$\tilde{H}_3: \Pi = A\psi\beta' \quad \text{or } \alpha = A\psi,$$
$$H_4: \Pi = A\psi\varphi'H' \quad \text{or } \beta = H\varphi \text{ and } \alpha = A\psi.$$

The matrices A(p×m) and H(p×s) are known and define the restrictions on the parameters α(p×r) and β(p×r). The restrictions reduce the parameters to φ(s×r) and ψ(m×r), where $r \leq s \leq p$ and $r \leq m \leq p$.

Note that $H_4 = H_3^* \cap \tilde{H}_3$ and that $H_3^* \subset H_2$ and $\tilde{H}_3 \subset H_2$. In fact all hypotheses are special cases of H_4 if we choose either A or H as the identity matrix. Thus it is to be expected that the analysis of these models are similar in nature, and this is what we hope to demonstrate in this section. The relation between the various hypotheses are illustrated in Figure 1.

[Figure 1.]

All these hypotheses are restrictions of the matrix Π which under H_1 contains p^2 parameters. Under the hypothesis H_2 there are pr + (p−r)r parameters which are further restricted to sr + (p−r)r under H_3^* and mr + (p−r)r under $\tilde{H}_3$. Finally mr + (s−r)r parameters remain under H_4. Note also that the parameters α and β are not identified in the sense that given any choice of α and β, then for any non-singular matrix ξ(r×r), the choice $\alpha\xi$ and $\beta(\xi')^{-1}$ will give the same matrix Π, and hence determine the same probability distribution for the variables. One way of expressing this is to say that what the data can determine is the space spanned by the columns in β, the cointegration space, and the space spanned by α. It is of course possible to normalize β, for example by choosing one of the coefficients to unity, but this would be correct only if we know apriori that the corresponding variable enters the cointegration relation with a non-zero coefficient. Since we are here mainly interested in drawing conclusions about which cointegration vectors are present in the data and since the hypotheses we have formulated above do not depend on any normalization of β we will avoid arbitrary normalizations, except in the case r = 1, where the interpretation is reasonably clear.

Note also that for each value of r ($0 \leq r \leq p$) there is a corresponding hypothesis $H_2(r)$. The analysis which follows the procedures in Johansen (1988b) makes it possible to make inference about the value of r by testing $H_2(r)$ in H_1.

We shall now turn to the maximum likelihood estimation of the parameters in the unrestricted model (2.1). These results are all well known but we shall give the formulae here mainly to establish the notation, since it will be useful for the discussion of the properties of the estimators and tests later. We shall not include fixed regeressors in the model, but the theory can easily be modified to cover this case too. In the example we shall use a constant term and seasonal dummies.

For fixed value of Π the maximum likelihood estimation consists of a regression giving the normal equations

$$(2.2)\quad \sum_{t=1}^{T} \Delta X_t \Delta X_{t-i}' = \sum_{j=1}^{k-1} \Gamma_j \sum_{t=1}^{T} \Delta X_{t-j} \Delta X_{t-i}' - \Pi \sum_{t=1}^{T} X_{t-k} \Delta X_{t-i}', \quad (i = 1,\dots,k-1).$$

Now introduce the notation for the product moment matrices

$$(2.3) \qquad M_{ij} = T^{-1} \sum_{t=1}^{T} \Delta X_{t-i} \Delta X_{t-j}', \qquad (i,j = 0,\dots,k-1)$$

$$(2.4) \qquad M_{ki} = T^{-1} \sum_{t=1}^{T} X_{t-k} \Delta X_{t-i}', \qquad (i = 0,\dots,k-1)$$

and

$$(2.5) \qquad M_{kk} = T^{-1} \sum_{t=1}^{T} X_{t-k} X_{t-k}',$$

and let M_{**} denote the matrix with elements M_{ij}, $(i,j, = 1,\dots,k-1)$. Similarly let M_{k*} denote the matrix with elements M_{kj}, $(j = 1,\dots,k-1)$. Then (2.2) can be written as

$$M_{0i} = \sum_{j=1}^{k-1} \Gamma_j M_{ji} - \Pi M_{ki}$$

and can be solved for Γ_j to give

$$(2.6) \qquad \hat{\Gamma}_j(\Pi) = \sum_{i=1}^{k-1} M_{0i} M^{ij} + \Pi \sum_{i=1}^{k-1} M_{ki} M^{ij} = M_{0*} M^{*j} + \Pi M_{k*} M^{*j}.$$

This leads to the definition of the residuals

$$(2.7) \qquad R_{0t} = \Delta X_t - \sum_{j=1}^{k-1} M_{0*} M^{*j} \Delta X_{t-j},$$

$$(2.8) \qquad R_{kt} = X_{t-k} - \sum_{j=1}^{k-1} M_{k*} M^{*j} \Delta X_{t-j},$$

i.e. the residuals we would obtain by regressing ΔX_t and X_{t-k} on $\Delta X_{t-1},\dots,\Delta X_{t-k+1}$.

The concentrated likelihood function becomes

$$(2.9) \qquad |\Lambda|^{-T/2} \exp\{- \sum_{t=1}^{T} (R_{0t} + \Pi R_{kt})' \Lambda^{-1} (R_{0t} + \Pi R_{kt})/2\}.$$

We can now express the estimates under the model H_1 by introducing the notation

$$(2.10) \qquad S_{ij} = T^{-1} \sum_{t=1}^{T} R_{it} R_{jt}' = M_{ij} - M_{i*} M_{**}^{-1} M_{*j} \qquad (i,j = 0,k).$$

We shall formulate these well known results in

THEOREM 2.1: *In the model*

$$H_1: \qquad \Delta X_t = \sum_{j=1}^{k-1} \Gamma_j \Delta X_{t-j} - \Pi X_{t-k} + \epsilon_t,$$

the parameters are estimated by ordinary least squares and we get

$$(2.11) \qquad \hat{\Pi} = - S_{0k} S_{kk}^{-1},$$

and

$$(2.12) \qquad \hat{\Lambda} = S_{00} - S_{0k} S_{kk}^{-1} S_{k0},$$

$$(2.13) \qquad L_{max}^{-2/T}(H_1) = |\hat{\Lambda}|.$$

The estimate of Π should then be inserted into (2.6) to get the estimate of Γ_j.

We shall now apply model (2.1) including constant term and seasonal dummies to the Danish and Finnish money demand data described in the introduction. It was found that for k = 2 the residuals for the Danish data clearly passed the test for being uncorrelated. For the Finnish data the test statistic for the residuals in the equation for Δy is almost significant. Looking at the autocorrelogram, see Appendix B, gives the impression that there is some seasonality left in the residuals, but since it is rather small we have chosen to ignore his. Accordingly the model (2.1) with k = 2 was fitted to both data sets. After conditioning on the two first data realizations the number of observations left for estimation was 46 in the Danish and 104 in the Finnish data. The estimates Γ_1,Π and Λ are given in Appendix B together with some of the details of the statistical analysis. Here we shall give only the estimates of Π.

[TABLE I]

2.2. *Derivation of the estimate of α and β under the hypothesis $\Pi = \alpha\beta'$ as well the likelihood ratio test for this hypothesis*

Now consider the model H_2, where $\Pi = \alpha\beta'$. The estimation of $\Gamma_1,\ldots,\Gamma_{k-1}$ is the same as before leading to (2.9). For fixed β it is now easy to estimate α and Λ by regressing R_{0t} on $\beta' X_{t-k}$ and obtain

(2.14) $$\hat{\alpha}(\beta) = -S_{0k}\beta(\beta' S_{kk}\beta)^{-1}$$

(2.15) $$\hat{\Lambda}(\beta) = S_{00} - S_{0k}\beta(\beta' S_{kk}\beta)^{-1}\beta' S_{k0}.$$

Further we get

(2.16) $$L_{max}^{-2/T}(\beta) = |\hat{\Lambda}(\beta)| = |S_{00} - S_{0k}\beta(\beta' S_{kk}\beta)^{-1}\beta' S_{k0}|.$$

It was shown in Johansen (1988b) how one proceeds to estimate β by applying the identity

(2.17) $$|S_{00} - S_{0k}\beta(\beta' S_{kk}\beta)^{-1}\beta' S_{k0}| = |S_{00}|\,|\beta' S_{kk}\beta - \beta' S_{k0}S_{00}^{-1}S_{0k}\beta|/|\beta' S_{kk}\beta|,$$

which is easily minimized among all p×r matrices β. We shall formulate the results in

THEOREM 2.2: *Under the hypothesis*

$$H_2: \qquad \Pi = \alpha\beta'.$$

the maximum likelihood estimator of β is found by the following procedure: First solve the equation

(2.18) $$|\lambda S_{kk} - S_{k0}\, S_{00}^{-1} S_{0k}| = 0,$$

giving the eigenvalues $\hat{\lambda}_1 > \ldots > \hat{\lambda}_p$ and eigenvectors $\hat{V} = (\hat{v}_1,\ldots,\hat{v}_p)$ normalized such that $\hat{V}' S_{kk}\hat{V} = I$. The choice of $\hat{\beta}$ is now

$$(2.19)\qquad \hat{\beta} = (\hat{v}_1,\dots,\hat{v}_r),$$

which gives

$$(2.20)\qquad L_{max}^{-2/T}(H_2) = |S_{00}| \prod_{i=1}^{r}(1-\hat{\lambda}_i).$$

The estimates of the other parameters are found by inserting $\hat{\beta}$ *into the above equations. In particular we find*

$$(2.21)\qquad \hat{\alpha} = - S_{0k}\hat{\beta},$$

$$(2.22)\qquad \hat{\Pi} = - S_{0k}\hat{\beta}\hat{\beta}',$$

$$(2.23)\qquad \hat{\Lambda} = S_{00} - \hat{\alpha}\hat{\alpha}',$$

We can now immediately write down the likelihood ratio test statistic for the hypothesis H_2 *in* H_1, *since* H_1 *is a special case of* H_2 *for the choice* $r = p$.

$$(2.24)\qquad -2\ln(Q;H_2|H_1) = -T\sum_{i=r+1}^{p} \ln(1-\hat{\lambda}_i).$$

It was shown in Johansen (1988b) that the estimates of Π and Λ were consistent and that the asymptotic distribution of the test statistic can be tabulated by simulation.

Remark. Many computer packages contain procedures for solving the eigenvalue problem

$$|\lambda I - A| = 0,$$

where A is symmetric. One can easily reduce (2.18) to this problem by first decomposing $S_{kk} = CC'$ for some non-singular p×p matrix C. Now (2.18) is equivalent to

$$|\lambda I - C^{-1}S_{k0}S_{00}^{-1}S_{0k}C'^{-1}| = 0,$$

which has the same eigenvalues $\hat{\lambda}_1 > \dots \hat{\lambda}_p$ but eigenvectors $e_1,\dots,e_p$, where $\hat{v}_i = C'^{-1}e_i$.

Let us now apply these results to the Danish data. The matrices S_{00}, S_{20} and S_{22} are given in Appendix C together with $S_{20}S_{00}^{-1}S_{02}$, and the eigenvalues and eigenvectors are given in Table II. Note that the calculation of all the eigenvectors allows one to estimate α and β for any value of r.

[TABLE II]

[TABLE III]

We shall now make inference about the number of cointegration vectors and we can here test a series of hypotheses as given in Table III. Consider for instance the hypothesis $r \leq 1$ versus the general alternative H_1. Here the test statistic is calculated as

$$-2\ln(Q) = -T\{\ln(1-\hat{\lambda}_2) + \ln(1-\hat{\lambda}_3) + \ln(1-\hat{\lambda}_4)\}$$

$$= -46\{\ln(1-.1940) + \ln(1-.1269) + \ln(1-.0138)\} = 16.80.$$

A comparison with the 95% quantile in the asymptotic distribution given as 23.8 in Table III, shows that this value is not significant. Hence there is no evidence in the Danish data that more than one cointegration relation exists. If on the other hand we test the hypothesis that $r = 0$ we find from Table III that this is strongly rejected, and hence we conclude that the data indicates that only one cointegration vector is found in the Danish data. This hypothesis will be maintained in the following.

From Table II we estimate the cointegrating relation as the first column in $\hat{V}$. In this case it seems natural to normalize it by the coefficient of m2 equal -1. Hence we divide the vector $\hat{\beta}$ by 19.39. Then

$$\hat{\beta}' = (\ -1.00,\ \ +.96,\ -\ 6.73,\ +5.42).$$

This makes it straightforward to interprete the cointegration vector in terms of an error correction mechanism measuring the exess demand for money, where the equilibrium relation is given by

$$m2 = .096y - 6.73i^{b} + 5.42i^{d} + \text{const.}$$

Similarly $\hat{\alpha}$ is found as the first column in the matrix $-S_{02}\hat{V}$, but now normalized by multiplying by 19.39,

$$\hat{\alpha}' = (-.317, -.077, -.006, +.024).$$

The normalized coefficients of α can now be interpreted as the weights with which exess demand for money enters the four equations of our system. In this case it is natural to give them an economic meaning in terms of the average speed of adjustment towards the estimated equilibrium state, such that a low coefficient indicates slow adjustment and a high coefficient indicates rapid adjustment. It is seen that in the first equation which measures the changes in money balances, the average speed of adjustment is approximately 0.32, whereas in the remai-ning three equations the adjustment coefficients are very low. This observation will be followed by a formal test in section 2.4.

For the Finnish data we find from Table III that at least 2 but possibly 3 cointegration vectors are present. We shall assume in the following that 3 cointegration relations exist, but keep in mind that this conclusion is based on rather vague evidence.

We then find $\hat{\beta}$ as the first three columns of $\hat{V}$ from Table II and $\hat{\alpha}$ as the corresponding columns of $-S_{02}\hat{V}$.

The interpretation of $\hat{\beta}$ and $\hat{\alpha}$ is not straightforward in this case. A thorough understanding of the economic problem seems to be mandatory in order to illucidate the role of the various cointegration vectors estimated by $\hat{\beta}$. We shall indicate here that a preliminary and heuristic interpretation is possible by considering the estimates in Table II. First note that for the three first eigenvectors we find, that the first two coefficients are equal with opposite sign: $\hat{\beta}_{i2} = -\hat{\beta}_{i1}$ $i = 1,2,3,$ and that $\hat{\beta}_2$ is approximately proportional to (0,0,0,1). Thus it follows that $\hat{\beta}_1, \hat{\beta}_2$ and $\hat{\beta}_3$ can be approximately represented as linear combinations of the vectors (-1,1,0,0), (0,0,0,1), and (0,0,1,0). This would imply that $m1 - y$, i^m and Δp are stationary, and hence that the only interesting cointegration we have found is between m1 and y.

We shall formulate this finding as a precise hypothesis about a linear restriction on β and test this hypothesis in the next section.

This completes the investigation of the model H_2 in H_1 and we shall now turn to the model H_3^* in H_2.

2.3. *Linear hypotheses concerning* β. *Estimation and test*

The model $H_3^* : \beta = H\varphi$ is a formulation of a linear restriction on the cointegration vectors. The hypothesis specifies the same restriction on all the cointegration vectors. The reason for this is the following: If we have two cointegration vectors in which m and y, say, enter then any linear combination of these relations will also be a cointegrating relation. Thus it will in general always be possible to find some relation which has, say, equal coefficients with opposite sign to m and y, corresponding to a long-run elasticity of 1. This is clearly not interesting, and only if all rows of β show the same unit elasticity is it meaningful to say that we have found a unit elasticity.

Under H_3^* we have the restriction $\beta = H\varphi$ where H is (p×s) but that means that the estimation of $\Gamma_1,\ldots,\Gamma_{k-1}$ and α and Λ are as just described for fixed $\beta = H\varphi$, and φ now has to be chosen to minimize

(2.25) $$|\varphi' H' S_{kk} H\varphi - \varphi' H' S_{k0} S_{00}^{-1} S_{0k} H\varphi| / |\varphi' H' S_{kk} H\varphi|$$

over the set of all s×r matrices φ. This problem has the same kind of solution as above and we can formulate the results in

THEOREM 2.3: *Under the hypothesis*

$$H_3^*: \quad \beta = H\varphi,$$

we can find the maximum likelihood estimator of β *as follows: First we solve*

(2.26) $$|\lambda H' S_{kk} H - H' S_{k0} S_{00}^{-1} S_{0k} H| = 0,$$

to give $\lambda_1^* > \ldots > \lambda_s^*$ *and* $V^* = (v_1^*,\ldots,v_s^*)$ *normalized by* $V^{*\prime} H' S_{kk} H V^* = I$.

We now choose

(2.27) $$\varphi^* = (v_1^*,\ldots,v_r^*) \text{ and } \beta^* = H\varphi^*,$$

and find the estimates of α, Π, Λ *and* Γ_j *from* (2.21),(2.22),(2.23) *and* (2.6). *The maximized likelihood becomes*

(2.28) $$L_{max}^{-2/T}(H_3^*) = |S_{00}| \prod_{i=1}^{r} (1-\lambda_i^*),$$

which gives the likelihood ratio test of the hypothesis H_3^* *in* H_2 *as*

(2.29) $$-2ln(Q;H_3^*|H_2) = T \sum_{i=1}^{r} ln\{(1-\lambda_i^*)/(1-\hat{\lambda}_i)\}.$$

The asymptotic distribution of this statistic was shown in Johansen (1988b) *to be* χ^2 *with* $r(p-s)$ *degrees of freedom.*

Now consider the hypothesis that there is proportionality between money and income as the transactions demand for money would predict. Then the coefficients to money and income would be equal with opposite sign.

For the Danish data we have found one cointegration vector, and the restriction can, in the above matrix formulation, be expressed as

$$\beta = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \varphi, \quad \varphi = \begin{bmatrix} \beta_{12} \\ \beta_3 \\ \beta_4 \end{bmatrix}.$$

We shall then solve (2.26) which now gives three eigenvalues and eigenvectors, see Table IV.

The test of H_3^* in H_2 consists of comparing λ_1^* and $\hat{\lambda}_1$ by the test $-2\ln(Q) = T\{\ln(1-\lambda_1^*) - \ln(1-\hat{\lambda}_1)\} = 46\{\ln(1-.4999) - \ln(1-.5004)\} = .046$. The asympotic distribution of this quantity is given by the χ^2 distribution with degrees of freedom $r(p-s) = 1(4-3) = 1$. It is clearly not significant, and we can thus accept the hypothesis that for the Danish data the coefficients to m2 and y are equal with opposite sign.

In the Finnish data we have found three cointegration vectors and the hypothesis about proportionality between money and income can be formulated as

$$\beta = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \varphi, \quad \varphi = \begin{bmatrix} \beta_{1.12} & \beta_{2.12} & \beta_{3.12} \\ \beta_{1.3} & \beta_{2.3} & \beta_{3.3} \\ \beta_{1.4} & \beta_{2.4} & \beta_{3.4} \end{bmatrix}.$$

We find the three eigenvalues from Table IV, and the test becomes

$$-2\ln(Q) = T\{\ln(1-\lambda_1^*)+\ln(1-\lambda_2^*)+\ln(1-\lambda_3^*)-\ln(1-\hat{\lambda}_1)-\ln(1-\hat{\lambda}_2)-\ln(1-\hat{\lambda}_3)\}= 3.82,$$

which should be compared with $\chi^2_{.95}(r(p-s)) = \chi^2_{.95}(3(4-3)) = 7.81$. Thus we accept the hypothesis of equal coefficients with opposite sign for m1 and y.

[TABLE IV]

For the Danish data we also want to test the hypothesis that the coefficients for the bond interest rate and the deposit interest rate are equal with opposite sign. This would imply that the cost of holding money can be measured as the difference between the bond yield and the yield from holding money in bank deposits. This hypothesis H_2^{**} is formulated as

$$\beta = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}\varphi, \quad \varphi = \begin{bmatrix} \beta_{12} \\ \beta_{34} \end{bmatrix}$$

Under this hypothesis we get from (2.25) two eigenvalues $\lambda_1^{**} = .4828$ and $\lambda_2^{**} = .0473$. The first eigenvector is $v_1^{**} = (-16.92, -127.71)$ and the corresponding $\alpha^{**} = (-16.24, \quad -4.02, +.08, +1.64)\times 10^{-3}$. The test for the hypothesis is given by

$$-2\ln(Q) = 46\ln\{(1-.4838) - \ln(1-.4999)\} = 1.46$$

which should be compared with the χ^2 quantiles with $r(s_1-s_2) = 1(4-3) = 1$ degree of freedom. Again this is not significant and we conclude the analysis of the cointegration vectors for the Danish demand for money by the estimate

$$\beta^{**} = (-1.00, +1.00, -7.55, +7.55),$$

where we have normalized it by the coefficient to m2 (16.92). The corresponding estimate of α is multiplied by 16.92 and is given by

$$\alpha^{**} = (-.275,-.068,.001,.028).$$

It is also possible to test these hypotheses using the Wald test (3.41) given in Corollary 3.17, which are easily calculated from Table II. For the Danish data we first test that m2 and y have equal coefficients with opposite sign, i.e. sum to zero. We choose K' = (1,1,0,0) and calculate the statistic

$$U_1 = \frac{46^{1/2}(-19.39+18.61)}{\{(1/.5004-1)[(-14.77+25.04)^2+(10.27-26.05)^2+(-12.39+1.29)^2]\}^{1/2}} = -.244$$

The second test that the coefficients to the bond interest rate and the deposit interest rate are equal with opposite sign is calculated in the same way:

$$U_2 = \frac{46^{1/2}(-130.40 + 105.05)}{\{(1/.5004-1)((-17.44-73.35)^2+(29.81-83.90)^2+(.54-24.94)^2\}^{1/2}} = -1.56$$

Both these statistics are asymptotically normalized Gaussian and the values found are hence not significant.

The test for the Finnish data can likewise be performed using the Wald test (3.40) We can formulate the hypothesis in question as

$$K'\beta = (1,1,0,0)\beta = 0.$$

Then we find from Table II

$$K'\hat{\gamma}\hat{\gamma}'K = (1.38 + 2.22)^2 = 12.96$$

and

$$K'\hat{\beta}(\hat{D}^{-1} - I)^{-1}\hat{\beta}'K = \frac{(-2.93+2.86)^2}{.3093^{-1}-1} + \frac{(4.58-6.06)^2}{.2260^{-1}-1} + \frac{(-11.13+10.24)^2}{.0731^{-1}-1} = .834$$

and the test statistic becomes

$$104 \times .834/12.96 = 6.22$$

which should also be compared with the χ^2 quantile with 3 degrees of freedom as before.

Note that with the restriction of proportionality we now have three cointegration vectors restricted to a three dimensional space defined by the restriction that m1 and y have equal coefficients with opposite sign. Thus the hypothesis H_2^* is really the hypothesis of a complete specification of $sp(\beta)$. In this space we can choose to present the results in any basis we want and it seems natural to consider the three variables m1 - y, i^m and Δp. Thus the conclusion about the Finnish data is that in fact the two last variables i^m and Δp are already stationary, and the two first y and m1 are cointegrated.

Notice that the Wald test in all cases gives a value of the test statistic which is larger than the value for the likelihood ratio test statistic. This just emphasises the fact that we are relying on asymptotic results and a careful study of the small sample properties is necessary.

We shall conclude this section by giving the final estimate of Π as obtained under the various restrictions we have accepted above.

[TABLE V]

2.4. *Test and estimation of restrictions on α*

Let us now turn to the hypothesis $\tilde{H}_3$ where α is restricted by $\alpha = A\psi$ in the model H_2. Here A is a (p×m) matrix. It is convenient to introduce

$B(p\times(p-m))$ such that $B'A = 0$, $B'B = I$. Then the hypothesis $\tilde{H}_3$ is given by

$$B'\alpha = 0. \tag{2.30}$$

We shall now turn to the concentrated likelihood function (2.9) and express it in the variables given by

$$A'(R_{0t} + \alpha\beta'R_{kt}) = A'R_{0t} + A'A\psi\beta'R_{kt}, \tag{2.31}$$

$$B'(R_{0t} + \alpha\beta'R_{kt}) = B'R_{0t}. \tag{2.32}$$

In the following we shall factor out that part of the likelihood function which depends on $B'R_{0t}$ since it does not contain the parameters ψ and β. Again to save notation we shall define $\Lambda_{aa} = A'\Lambda A$, $\Lambda_{ab} = A'\Lambda B$,$S_{ak.b} = S_{ak} - S_{ab}S_{bb}^{-1}S_{bk} = A'S_{kk} - A'S_{00}B(B'S_{00}B)^{-1}B'S_{0k}$, etc.

We now get a factor corresponding to the marginal distribution of $B'R_{0t}$ given by

$$|\Lambda_{bb}|^{-T/2}\exp\{-\sum_{t=1}^{T}(B'R_{0t})'\Lambda_{bb}^{-1}(B'R_{0t})/2\}, \tag{2.33}$$

which gives the estimate

$$\tilde{\Lambda}_{bb} = S_{bb} = B'S_{00}B, \tag{2.34}$$

and the maximized likelihood function

$$L_{max}^{-2/T} = |S_{bb}|. \tag{2.35}$$

The other factor corresponds to the conditional distribution of $A'R_{0t}$ and R_{kt} conditional on $B'R_{0t}$ and is given by

$$|A'A|^{T/2}|\Lambda_{aa.b}|^{-T/2}\exp\{-\sum_{t=1}^{T}(A'R_{0t} + A'A\psi\beta'R_{kt} - \Lambda_{ab}\Lambda_{bb}^{-1}B'R_{0t})' \tag{2.36}$$
$$\Lambda_{aa.b}^{-1}(A'R_{0t} + A'A\psi\beta'R_{kt} - \Lambda_{ab}\Lambda_{bb}^{-1}B'R_{0t})/2\}.$$

It is a well known result from the theory of the multivariate normal distribution that the parameters $\Lambda_{bb}, \Lambda_{ab}\Lambda_{bb}^{-1}$ and $\Lambda_{aa.b}$ are variation independent and hence that the estimate of $\Lambda_{ab}\Lambda_{bb}^{-1}$ is found by regression for fixed ψ and β giving

$$(2.37) \qquad \tilde{\Lambda}_{ab}\tilde{\Lambda}_{bb}^{-1}(\psi,\beta) = (S_{ab} + A'A\psi\beta'S_{kb})S_{bb}^{-1},$$

and new residuals defined by

$$\tilde{R}_{at} = A'R_{0t} - S_{ab}S_{bb}^{-1}B'R_{0t},$$
$$\tilde{R}_{kt} = R_{kt} - S_{kb}S_{bb}^{-1}B'R_{0t}.$$

In terms of $\tilde{R}_{at}$ and $\tilde{R}_{kt}$ the concentrated likelihood function now has the form (2.9) which means that the estimation of β follows as before, and we can formulate

THEOREM 2.4: *Under the hypothesis*

$$\tilde{H}_3: \qquad \alpha = A\psi ,$$

the maximum likelihood estimator of β *is found as follows: First solve the equation*

$$(2.38) \qquad |\lambda S_{kk.b} - S_{ka.b}S_{aa.b}^{-1}S_{ak.b}| = 0,$$

giving $\tilde{\lambda}_1 > \ldots > \tilde{\lambda}_m > \tilde{\lambda}_{m+1} = \ldots = \tilde{\lambda}_p = 0$ *and* $\tilde{V} = (\tilde{v}_1,\ldots,\tilde{v}_p)$ *normalized such that* $\tilde{V}'S_{kk.b}\tilde{V} = I$.

Now take

$$(2.39) \qquad \tilde{\beta} = (\tilde{v}_1,\ldots,\tilde{v}_r),$$

which gives the estimates

$$(2.40) \qquad \tilde{\psi} = -(A'A)^{-1}S_{ak.b}\tilde{\beta}$$

and

$$(2.41) \qquad \tilde{\alpha} = A\tilde{\psi} = -A(A'A)^{-1}A'(S_{0k} - S_{00}B(B'S_{00}B)^{-1}B'S_{0k})\tilde{\beta},$$

$$(2.42) \qquad \tilde{\Lambda}_{aa.b} = S_{aa.b} - \tilde{\psi}\tilde{\psi}',$$

and the maximized likelihood function

$$(2.43) \qquad L_{max}^{-2/T}(\tilde{H}_3) = |A'A|^{-1}|S_{bb}||S_{aa.b}|\prod_{i=1}^{r}(1-\tilde{\lambda}_i) = |S_{00}|\prod_{i=1}^{r}(1-\tilde{\lambda}_i).$$

The estimate of Λ *can be found from (2.34),(2.37) and (2.41), and still*

Γ_j is estimated from (2.6).

The likelihood ratio test statistic of $\tilde{H}_3$ in H_2 is

$$(2.44)\qquad -2\ln(Q;\tilde{H}_3|H_2) = T\sum_{i=1}^{r}\ln\{(1-\tilde{\lambda}_i)/(1-\hat{\lambda}_i)\}.$$

The asymptotic distribution of this test statistic is found in Theorem 3.14 and is given by a χ^2 distribution with $r(p-m)$ degrees of freedom.

The following very simple corollary is useful for explaining the role of single equation analysis:

COROLLARY 2.5: *If $m = r = 1$ then the estimate of β is found as the coefficients of X_{t-k} in the regression of $A'\Delta X_t$ on X_{t-k}, $B'X_t$, and $\Delta X_{t-1},\ldots,\Delta X_{t-k+1}$.*

PROOF: It suffices to notice that when $m = r = 1$ then only one cointegration vector has to estimated. It is seen from (2.38) that since the matrix $S_{ka.b}S_{aa.b}^{-1}S_{ak.b}$ is singular and in fact of rank 1, then only 1 eigenvalue is non zero, and the corresponding eigenvector is proportional to $S_{kk.b}^{-1}S_{ka.b}$, which is exactly the regression coefficient to R_{kt} we would get by regressing $A'R_{0t}$ on $B'R_{0t}$ and R_{kt}. This can of course be seen directly from (2.36) since $A'A\psi$ is 1×1 and can be absorbed into β, which shows that β is given by the regression as described.

In particular if $A' = (1,0,0,0)$ then the least squares regression of the difference of the first variable on the remaining variables and their differences will be the maximum likelihood estimator.

We shall now apply these results to the Danish data. We want to test the hypothesis that the cointegration relation only enters the first equation. Thus we let

$$A = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and find the eigenvalues of (2.46), see below, since we maintain the hypothesis that β has the form that was found in the previous section. Thus there are two eigenvalues $\tilde{\lambda}_1 = .4200$ and $\tilde{\lambda}_2 = 0$, giving an eigenvector corresponding to $\tilde{\lambda}_1$ of (-18.56, -135.68), and a corresponding estimate of α, $\tilde{\alpha} = (\ -13.21,\ 0.00,\ 0.00,\ 0.00)\times 10^{-3}$.

The test statistic for this hypothesis about α is then given by $-2\ln(Q) = T\{\ln(1-\tilde{\lambda}_1)-\ln(1-\lambda_1^*)\} = 46\{\ln(1-.4200)-\ln(1-.4828)\} = 5.17$. This should be compared with the 95% quantile $\chi^2_{.95}(r(p-m)) = \chi^2_{.95}(1(4-1)) = 7.81$. On the basis of this we accept the hypothesis about α and conclude the analysis of the Danish data by giving the estimate of

$$\tilde{\beta}' = (\ -1.00,\ +1.00\ ,\ -7.31,\ +7.31),$$

$$\tilde{\alpha}' = (\ -.25,\ 0.00\ ,\ 0.00\ ,\ 0.00),$$

from which the estimate of Π can be constructed. Thus we have reduced the 16 parameters in the matrix Π which describes the long-run relations in the data to just 2 parameters.

This completes the estimation and test of H_3^* and $\tilde{H}_3$ concerning restrictions on α and β respectively.

The conclusion is thus the following: By correcting ΔX_t and X_{t-k} for $\Delta X_{t-1},\ldots,\Delta X_{t-k+1}$ we obtain the product moment matrices from the residuals:

$$(2.45) \qquad \begin{bmatrix} S_{00} & S_{0k} \\ S_{k0} & S_{kk} \end{bmatrix}$$

which form the basis for all subsequent analyses of Π.

Under the hypothesis $H_2 : \Pi = \alpha\beta'$ the estimates of β and α are related to the canonical variates between R_{0t} and R_{kt}, see Anderson (1984), the estimate of β is given as the eigenvectors of (2.18) corresponding to the r largest eigenvalues, i.e. the choice of $\hat{\beta}$ is the choice of the r linear combinations which have the largest correlation with the stationary process ΔX_t.

If $\beta = H\varphi$ we note that $\beta' R_{kt} = \varphi' H' R_{kt}$ which leads to solving (2.25) where R_{kt} has been replaced by $H' R_{kt}$. Thus restricting β to lie in sp(H) implies that the levels of the process should be transformed by H'.

If $\alpha = A\psi$ we shall solve (2.38), where we have conditioned on $B' R_{0t}$. In other words if we assume that the equations for $B' R_{0t}$ do not contain the parameter α, i.e. $B'\alpha = 0$, then we shall also correct for these before solving the eigenvalue problem.

It is now clear how one should solve the model $H_4 = \tilde{H}_3 \cap H_3^*$, where restrictions have been imposed on β as well as on α, namely by solving the eigenvalue problem

$$(2.46) \qquad |\lambda H' S_{kk.b} H - H' S_{ka.b} S_{aa.b}^{-1} S_{ka.b} H| = 0.$$

This gives the final solution to the estimation problem of H_4. Notice how (2.46) contains the previous problems by choosing either $H = I$ or $A = I$ or both.

Note that a linear restriction on β implies a transformation of the process, and that a linear restriction on α implies a conditioning. Thus all the calculations can easily be performed starting with the product moment matrices S_{ij} and using the usual operations of finding marginal (transformed) and conditional variances and then apply an eigenvalue routine.

The test statistic for the hypothesis H_2 has an asymptotic distribution that has to be tabulated by simulation, but all the hypotheses about β and α lead to test statistics that are asymptotically distributed as χ^2 with the appropriate degrees of freedom, and hence the usual tables can be applied.

3. The asymptotic properties of the estimators and the test statistics

3.1. *Grangers representation theorem*

When we want to investigate the probabilistic properties of the estimates and the test statistics we have to make more precise assumptions about the process. The basic assumption is that for

$$\Pi(z) = I - \Pi_1 z - \ldots - \Pi_k z^k \tag{3.1}$$

we have that $|\Pi(z)| = 0$ implies that either $|z| > 1$ or $z = 1$, which guarantees that the non-stationarity of X_t can be removed by differencing.

Now write the model defined by (3.1), see (1.2), as

$$\Pi X_t + \Pi_1(L)\Delta X_t = \epsilon_t. \tag{3.2}$$

where $\Pi = I - \Pi_1 - \ldots - \Pi_k = \Pi(1)$.

The first result that we want to prove is the fundamental result about error correction models of order 1 and their structure. The basic result is due to Granger (1981), see Engle and Granger (1987) or Johansen (1988a), but we shall give a very simple proof here.

THEOREM 3.1: *(Grangers representation theorem). If*

$$\Pi = \alpha\beta' \tag{3.3}$$

for α and β of dimension $p \times r$ and rank r and if

$$\alpha_\perp' \Pi_1(1)\beta_\perp \tag{3.4}$$

has full rank $p - r$, where $\beta_\perp$ and $\alpha_\perp$ are $p \times (p-r)$ of full rank such that $\beta'\beta_\perp = 0$ and $\alpha'\alpha_\perp = 0$, then

$$\Delta X_t \text{ is stationary} \tag{3.5}$$

$$X_t \text{ is non-stationary} \tag{3.6}$$

$$\beta' X_t \text{ is stationary} \tag{3.7}$$

and hence (3.2) can be interpreted as an error correction model. If we write the process in the moving average form $\Delta X_t = C(L)\epsilon_t$ then the following representation holds:

$$C(1) = \beta_\perp(\alpha_\perp' \Pi_1(1)\beta_\perp)^{-1}\alpha_\perp'. \tag{3.8}$$

Note that the relation between (3.3) and (3.8) shows that for this type of process there is a very nice symmetry between the singularity of the "impact" matrix Π for the autoregressive representation and the singularity of the "impact" matrix for the moving average representation, in the sense that what is the null space for $C(1)'$ is the range space for Π and what is the range space for Π' is the null space for $C(1)$. It is this symmetry that allows the results for this type of process to be exceptionally simple.

PROOF: If we multiply the equation (3.2) by α' and $\alpha_\perp'$ we get the equations

$$\alpha'\alpha\beta' X_t + \alpha'\Pi_1(L)\Delta X_t = \alpha'\epsilon_t,$$

$$\alpha_\perp'\Pi_1(L)\Delta X_t = \alpha_\perp'\epsilon_t.$$

To discuss the properties of the process X_t we shall solve the equations for X_t and express it in terms of the ϵ_t's. We therefore introduce the variables $Z_t = (\beta'\beta)^{-1}\beta' X_t$ and $Y_t = (\beta_\perp'\beta_\perp)^{-1}\beta_\perp'\Delta X_t$ as new variables, from which ΔX_t can be recovered:

$$\Delta X_t = \beta_\perp Y_t + \beta\Delta Z_t.$$

This gives the equations

$$\alpha'\alpha\beta'\beta Z_t + \alpha'\Pi_1(L)\beta\Delta Z_t + \alpha'\Pi_1(L)\beta_\perp Y_t = \alpha'\epsilon_t \tag{3.9}$$

$$\alpha_\perp'\Pi_1(L)\beta\Delta Z_t + \alpha_\perp'\Pi_1(L)\beta_\perp Y_t = \alpha_\perp'\epsilon_t. \tag{3.10}$$

The matrix function defining this new system consisting of Z_t and Y_t takes the form:

$$\tilde{A}(z) = \begin{bmatrix} \alpha'\alpha\beta'\beta + \alpha'\Pi_1(z)\beta(1-z) & \alpha'\Pi_1(z)\beta_\perp \\ \alpha_\perp'\Pi_1(z)\beta(1-z) & \alpha_\perp'\Pi_1(z)\beta_\perp \end{bmatrix}.$$

For z = 1 this has determinant

$$|\alpha'\alpha|\,|\beta'\beta|\,|\alpha_\perp'\Pi_1(1)\beta_\perp|$$

which is non-zero by assumption (3.3) and (3.4), hence $z = 1$ is not a root. For $z \neq 1$ we use the representation

$$\tilde{A}(z) = (\alpha,\alpha_\perp)'\Pi(z)(\beta,\beta_\perp(1-z)^{-1}),$$

which gives the determinant as

$$|\tilde{A}(z)| = |(\alpha,\alpha_\perp)|\,|\Pi(z)|\,|(\beta,\beta_\perp)|\,(1-z)^{-(p-r)},$$

which shows that all roots of $|\tilde{A}(z)| = 0$ are outside the unit disk, by assumption (3.1).

This shows that the system defined by (3.9) and (3.10) is invertible and that Y_t and Z_t are stationary processes, and hence that ΔX_t is stationary. This proves (3.5) and (3.7). By summation of ΔX_t we find that X_t contains the non-stationary component $\beta_\perp' \sum_{s=0}^{t} Y_s$, which proves (3.6).

From the representation of the processes Z_t and Y_t we can get a representation of ΔX_t by multiplying by the matrix $(\beta\Lambda, \beta_\perp)$. Hence

$$C(L) = (\beta\Lambda, \beta_\perp)\tilde{A}(L)^{-1}(\alpha, \alpha_\perp)'$$

For L = 1 we get (3.8).

In the following when we discuss the limiting distributions we shall throughout assume the conditions of Theorem 3.1. We shall have to repeat some of the results about the process X_t given in Johansen (1988b). Since we assume that ΔX_t is stationary we define the covariance function

$$\psi(i) = \mathrm{Cov}(\Delta X_t, \Delta X_{t+i}).$$

Then we define

$$\mu_{ij} = \mathrm{Cov}(\Delta X_{t-i}, \Delta X_{t-j}) = \psi(i-j),$$

$$\mu_{ki} = \sum_{j=k-i}^{\infty} \psi(j) = \lim_t \mathrm{Cov}(X_{t-k}, \Delta X_{t-i})$$

and

$$\mu_{kk} = -\sum_{j=-\infty}^{\infty} |j|\psi(j)$$

with the interpretation that

$$\lim_t \mathrm{Var}(\beta' X_{t-k}) = \beta' \mu_{kk} \beta.$$

Finally we define

$$\Sigma_{ij} = \mu_{ij} - \mu_{i*}\mu_{**}^{-1}\mu_{*j}, \quad i,j = 0,k,$$

where μ_{**} is the matrix with entries μ_{ij}, $i,j = 1,\ldots,k-1$.

3.2. *A summary of technical asymptotic results*

The following technical results are given in Johansen (1988b) based on the results by Phillips and Durlauf (1986). We let W be a Brownian motion in p dimensions with covariance matrix Λ, and let $C = C(1)$, see (3.8).

LEMMA 3.2: *As $T \to \infty$ we have*

$$T^{-1/2}X_{[Tt]} \xrightarrow{w} CW(t)$$

$$M_{ij} \xrightarrow{a.s.} \mu_{ij}, \quad i,j = 0,\dots,k-1$$

$$M_{ki} \xrightarrow{w} \mu_{ki} + C\int_0^1 WdW'C', \quad i = 0,\dots,k-1$$

and

$$\beta' M_{kk}\beta \xrightarrow{a.s.} \beta'\mu_{kk}\beta$$

while

$$T^{-1}M_{kk} \xrightarrow{w} C\int_0^1 W(u)W'(u)du C'.$$

The relations between Σ_{ij}, α and β are given in the next Lemma.

LEMMA 3.3: *The following relations hold*

(3.11) $$\Sigma_{00} = -\alpha\beta'\Sigma_{k0} + \Lambda,$$

(3.12) $$\Sigma_{0k}\beta = -\alpha\beta'\Sigma_{kk}\beta,$$

and hence

(3.13) $$\Sigma_{00} = \alpha(\beta'\Sigma_{kk}\beta)\alpha' + \Lambda.$$

These relations imply that

$$(3.14) \qquad (\alpha'\Sigma_{00}^{-1}\alpha)^{-1}\alpha'\Sigma_{00}^{-1} = (\alpha'\Lambda^{-1}\alpha)^{-1}\alpha'\Lambda^{-1}.$$

The properties of S_{ij} are given in Lemma 3.4.

LEMMA 3.4: *For $T \to \infty$ it holds that*

$$S_{00} \overset{a.s.}{\to} \Sigma_{00},$$

$$\beta' S_{k0} \overset{a.s.}{\to} \beta'\Sigma_{k0}$$

and

$$\beta' S_{kk}\beta \overset{a.s.}{\to} \beta'\Sigma_{kk}\beta,$$

whereas if γ is not in the span of β then

$$T^{-1}\gamma' S_{kk}\gamma \overset{w}{\to} \gamma' C\int_0^1 W(u)W'(u)du C'\gamma,$$

and if $\delta'\alpha = 0$ then

$$(3.15) \qquad \delta' S_{0k} \overset{w}{\to} \delta'\int_0^1 dWW'C'.$$

Finally a technical result about the asymptotic behaviour of some of the quantities that enter into the calculations later.

LEMMA 3.5: *For $T \to \infty$ we have for $\delta'\alpha = 0$*

$$(3.16) \qquad \alpha' S_{0k}S_{kk}^{-1}S_{k0}\alpha \overset{P}{\to} \alpha'\Sigma_{0k}\beta(\beta'\Sigma_{kk}\beta)^{-1}\beta'\Sigma_{k0}\alpha = \alpha'(\Sigma_{00} - \Lambda)\alpha$$

$$(3.17) \qquad T^{1/2}\alpha' S_{0k}S_{kk}^{-1}S_{k0}\delta = \alpha'\Sigma_{k0}\beta(\beta'\Sigma_{kk}\beta)^{-1}(T^{1/2}\beta' S_{k0}\delta) + o_P(1)$$

$$(3.18) \qquad T^{1/2}\beta' S_{k0}\delta \overset{w}{\to} N_{r\times(p-r)}(0,\beta'\Sigma_{kk}\beta \otimes \delta'\Lambda\delta)$$

Hence the limit distribution of (3.17) is Gaussian with mean zero and variance matrix given by $\alpha'(\Sigma_{00} - \Lambda)\alpha \otimes \delta'\Lambda\delta$.

PROOF: Let (β,γ) be $(p\times p)$ and of full rank, then

$$S_{0k}S_{kk}^{-1}S_{k0} = (S_{0k}\beta, S_{0k}\gamma)\begin{bmatrix} \beta' S_{kk}\beta & \beta' S_{kk}\gamma \\ \gamma' S_{kk}\beta & \gamma' S_{kk}\gamma \end{bmatrix}^{-1}\begin{bmatrix} \beta' S_{k0} \\ \gamma' S_{k0}\end{bmatrix}$$

$$= (S_{0k}\beta, S_{0k}\gamma A_T)\begin{bmatrix} \beta' S_{kk}\beta & \beta' S_{kk}\gamma A_T \\ A_T\gamma' S_{kk}\beta & I \end{bmatrix}^{-1}\begin{bmatrix} \beta' S_{k0} \\ A_T\gamma' S_{k0}\end{bmatrix},$$

where $A_T = (\gamma' S_{kk}\gamma)^{-1/2} \xrightarrow{P} 0$, for $T \to \infty$, see Lemma 3.4. From Lemma 3.3 and 3.4 it follows that

$$S_{0k}S_{kk}^{-1}S_{k0} \xrightarrow{P} \Sigma_{0k}\beta(\beta'\Sigma_{kk}\beta)^{-1}\beta'\Sigma_{k0} = \alpha'(\Sigma_{00} - \Lambda)\alpha,$$

which proves (3.16). Now use the above representation for $T^{1/2}\alpha' S_{0k}S_{kk}^{-1}S_{k0}\delta$, and we get the it has the same limit distribution as

$$(\alpha'\Sigma_{0k}\beta, 0)\begin{bmatrix} (\beta'\Sigma_{kk}\beta)^{-1} & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} T^{1/2}\ \beta' S_{k0}\delta \\ T^{1/2}A_T\gamma' S_{k0}\delta \end{bmatrix} =$$

$$(\alpha'\Sigma_{0k}\beta)(\beta'\Sigma_{kk}\beta)^{-1}T^{1/2}\beta' S_{k0}\delta.$$

This proves (3.17). The asymptotic distribution of $T^{1/2}\beta' S_{k0}\delta$ is found as follows: If we multiply (2.1) by $\Delta X_{t-i}'$ and sum over t we get

$$\text{(3.19)} \qquad M_{0i} = \sum_{j=1}^{k-1} \Gamma_j M_{ji} - \Pi M_{ki} + T^{-1}\sum_{t=1}^{T} \epsilon_t \Delta X_{t-i}', \quad i = 0,\dots,k-1,$$

and when multiplying by X_{t-k}' and summing over t we get

$$\text{(3.20)} \qquad M_{0k} = \sum_{j=1}^{k-1} \Gamma_j M_{jk} - \Pi M_{kk} + T^{-1}\sum_{t=1}^{T} \epsilon_t X_{t-k}'.$$

Now solve the equations (3.19), $i = 1,\dots,k-1$ for Γ_j and insert into (3.20) and use the definition of S_{ij}, then we get

$$\text{(3.21)} \qquad S_{0k}\beta + \Pi S_{kk}\beta = T^{-1}\sum_{t=1}^{T} \epsilon_t\{X_{t-k}' - \sum_{i=1}^{k-1} \Delta X_{t-i}' M^{i*}M_{*k}\}\beta = U_T$$

Since $M_{ij} \xrightarrow{P} \mu_{ij}$ and $\beta' M_{kj} \xrightarrow{P} \beta'\mu_{kj}$, it follows that U_T has the same limit distribution as

$$V_T = T^{-1}\sum_{t=1}^{T} \epsilon_t\{(\beta' X_{t-k})' - \sum_{i=1}^{k-1} \Delta X_{t-i}'\mu^{i*}\mu_{*k}\beta\}.$$

From the central limit theorem for martingales it follows that $T^{1/2}V_T$ is asymptotically Gaussian with mean zero and variance matrix $\Lambda\otimes\beta'\Sigma_{kk}\beta$. Now multiply by δ and the second term on the left hand side of (3.21) vanishes since $\delta'\alpha = 0$, and hence (3.18) is proved.

Finally we can use Lemma 3.3 to see that

$$\alpha'\Sigma_{0k}\beta(\beta'\Sigma_{kk}\beta)^{-1}\beta'\Sigma_{k0}\alpha = \alpha'\alpha\beta'\Sigma_{kk}\beta\alpha'\alpha = \alpha'(\Sigma_{00} - \Lambda)\alpha.$$

We shall next give some results about the eigenvalues of (2.18) taken from Johansen (1988b) Lemma 4 and 6.

LEMMA 3.6: *The ordered eigenvalues of the equation*

$$|\lambda S_{kk} - S_{k0}S_{00}^{-1}S_{0k}| = 0$$

converge in probability to $(\lambda_1,\ldots,\lambda_r,0,\ldots,0)$, *where* $\lambda_1,\ldots,\lambda_r$ *are the ordered eigenvalues of the equation*

$$|\lambda\beta'\Sigma_{kk}\beta - \beta'\Sigma_{k0}\Sigma_{00}^{-1}\Sigma_{0k}\beta| = 0.$$

Further we have that $T\hat{\lambda}_{r+1},\ldots,T\hat{\lambda}_p$ *converge in distribution to the ordered eigenvalues of the equation*

$$|\lambda\int_0^1 BB'du - \int_0^1 BdB'\int_0^1 dBB'| = 0$$

where B *is a Brownian motion in* p-r *dimensions with variance matrix* I.

The following result is well known from the theory of canonical correlations and express the duality there exists between the two sets of canonical variates, see Anderson (1984).

LEMMA 3.7: *Let the partitioned matrix*

$$\begin{bmatrix} A & B \\ B' & C \end{bmatrix}$$

be symmetric positive definite, where A *is* $p\times p$ *and* C *is* $m\times m$. *It then holds that*

$$\begin{vmatrix} \mu A & B \\ B' & \mu C \end{vmatrix} = \mu^{p-m}|A|\,|\mu^2 C - B'A^{-1}B| = \mu^{m-p}|C|\,|\mu^2 A - BC^{-1}B'|$$

such that the positive solutions of

$$|\lambda C - B'A^{-1}B'| = 0$$

and

$$|\lambda A - BC^{-1}B'| = 0$$

are identical. If

$$\lambda Cx = B'A^{-1}Bx$$

then

$$\lambda Ay = BC^{-1}B'y$$

where $y = A^{-1}Bx$ *and* $x = C^{-1}B'y$ *and vice versa.*

3.3. *Asymptotic results about the estimators*

Since the parameter β is not identified we can not expect to get a reasonable estimator for β. We can however normalize the estimator so that the asymptotic properties can be formulated in a way that is useful for deriving further results. Choose γ $(p\times(p-r))$ of full rank such that $\beta'\gamma = 0$, then we can decompose $\hat{\beta} = \beta\hat{b} + \gamma\hat{g}$, where $\hat{b} = (\beta'\beta)^{-1}\beta'\hat{\beta}$. The following result was proved in Johansen (1988b) except for a simplified expression for the variance which follows from Lemma 3.3.

PROPOSITION 3.8: *Under the hypothesis* $H_2 : \Pi = \alpha\beta'$ *the maximum likelihood estimator* $\hat{\beta}$ *has the representation*

$$(3.22)\quad T[\hat{\beta}\hat{b}^{-1} - \beta] = \gamma(\gamma' S_{kk}\gamma/T)^{-1}\gamma'(T^{-1}\sum_{t=1}^{T} X_{t-k}\epsilon_t')\Lambda^{-1}\alpha(\alpha'\Lambda^{-1}\alpha)^{-1} + o_P(1),$$

which converges in distribution to

$$\gamma(\int_0^1 UU'du)^{-1}\int_0^1 UdV',$$

where U *and* V *are independent Brownian motions, see Lemma 3.4, given by*

$$U = \gamma' CW,$$

$$V = (\alpha'\Lambda^{-1}\alpha)^{-1}\alpha'\Lambda^{-1}W,$$

such that the variance of V *is given by* $Var(V) = (\alpha'\Lambda^{-1}\alpha)^{-1}$.

The expression for the variance is found as follows. In Johansen (1988b) it was shown that (3.22) holds with $\Lambda^{-1}\alpha(\alpha'\Lambda^{-1}\alpha)^{-1}$ replaced by

$$\Sigma_{00}^{-1}\Sigma_{0k}\beta(\beta'\Sigma_{k0}\Sigma_{00}^{-1}\Sigma_{0k}\beta)^{-1}\beta'\Sigma_{kk}\beta.$$

Now apply (3.12) to replace $\beta'\Sigma_{k0}$ by an expression in α and we find

$$\Sigma_{00}^{-1}\alpha(\alpha'\Sigma_{00}^{-1}\alpha)^{-1}$$

which by (3.14) equals the expression in (3.22) from which the variance follows easily.

Note that the limiting distribution for fixed U is Gaussian with mean zero and variance

$$\gamma\int_0^1 UU'du\gamma' \otimes (\alpha'\Lambda^{-1}\alpha)^{-1}.$$

Thus the limiting distribution of that part of $\hat{\beta}$ which is orthogonal to β is a mixture of Gaussian distributions.

Next we shall find an asymptotic representation for $\hat{\alpha}$ as well as the asymptotic distribution. The estimate of α has to be normalized in a way similar to that of β.

PROPOSITION 3.9: *Under the hypothesis* H_2: $\Pi = \alpha\beta'$ *the estimator* $\hat{\alpha}$ *has the representation*

$$(3.23)\qquad T^{1/2}(\hat{\alpha}\hat{b}' - \alpha) = T^{1/2}(-S_{0k}\beta(\beta' S_{kk}\beta)^{-1} - \alpha) + o_P(1)$$

which converges weakly to a Gaussian distribution of dimension p×r *with mean zero and variance matrix* $\Lambda\otimes(\beta'\Sigma_{kk}\beta)^{-1}$.

PROOF: The definition of $\hat{\alpha}$ is

$$\hat{\alpha} = -S_{0k}\hat{\beta}(\hat{\beta}' S_{kk}\hat{\beta})^{-1}$$

and by (3.20) we can replace $\hat{\beta}$ by $\beta\hat{b}$, which shows

$$\hat{\alpha} = - S_{0k}\beta(\beta' S_{kk}\beta)^{-1}\hat{b}'^{-1} + O_P(T^{-1})$$

and hence the representation (3.23)

$$T^{1/2}(\hat{\alpha}\hat{b}' - \alpha) = -T^{1/2}(S_{0k}\beta(\beta' S_{kk}\beta)^{-1} + \alpha) + O_P(T^{-1/2}).$$

From (3.21) we find

$$T^{1/2}(\hat{\alpha}\hat{b}' - \alpha) = - T^{1/2}U_T(\beta' S_{kk}\beta)^{-1} + o_P(1).$$

which proves Proposition 3.9.

Note that the asymptotic distribution of $\hat{\alpha}$ is not influenced by the asymptotic variance of $\hat{\beta}\hat{b}^{-1}$, since $\hat{\beta}\hat{b}^{-1}$ converges so fast to β, see Stock (1987) for a similar result for regression estimates of cointegration vectors.

COROLLARY 3.10: *Under the hypothesis* H_1 : $\Pi = \alpha\beta'$ *the asymptotic distribution of* $T^{1/2}(\hat{\Pi} - \Pi)$ *is Gaussian in* p×r *dimensions with mean* 0 *and a variance matrix given by*

$$\Lambda\otimes\beta(\beta'\Sigma_{kk}\beta)^{-1}\beta',$$

which is consistently estimated by

$$(S_{00} - \hat{\alpha}\hat{\alpha}')\otimes(\hat{\beta}\hat{\beta}').$$

We have found the estimate $\hat{\alpha} = -S_{0k}\hat{\beta}$ and its asymptotic properties but it is convenient to have a representation of $\hat{\alpha}$ as the solution to an eigenvalue problem, since this will give a function that $\hat{\alpha}$ maximizes and the asymptotic properties of $\hat{\alpha}$ and in particular the test statistics can be found by the usual methods. The idea is to maximize with respect to Λ, β, as well as the matrices Γ_j, and find the likelihood profile with respect to α. We introduce δ such that δ is px(p-r), (α,δ) has full rank and $\delta'\alpha = 0$ i.e. $sp(\delta) = sp(\alpha)^{\perp}$or $\delta = \alpha_{\perp}$.

We then find as in section 2, see (2.9), that

$$L_{max}^{-2/T}(\alpha,\beta) = |S_{00} + \alpha\beta'S_{k0} + S_{0k}\beta\alpha' +\alpha\beta'S_{kk}\beta\alpha'|,$$

and hence for fixed α and δ

$$|\alpha'\alpha|\,|\delta'\delta|\min_{\beta} L_{max}^{-2/T}(\alpha,\beta) =$$

$$\min_{\beta}\begin{vmatrix} \alpha'S_{00}\alpha + \beta'S_{k0}\alpha + \alpha'S_{0k}\beta + \beta'S_{kk}\beta & \alpha'S_{00}\delta + \beta'S_{k0}\delta \\ \delta'S_{0k}\beta + \delta'S_{00}\alpha & \delta'S_{00}\delta \end{vmatrix}.$$

In the last expression we have replaced $\beta\alpha'\alpha$ by β since this gives the same minimum. To simplify the notation we introduce $S_{\alpha\alpha} = \alpha'S_{00}\alpha$, $S_{\alpha k} = \alpha'S_{0k}$, and $S_{\alpha\alpha.\delta} = S_{\alpha\alpha} - S_{\alpha\delta}S_{\delta\delta}^{-1}S_{\delta\alpha}$ etc., then

$$|\alpha'\alpha|\,|\delta'\delta|\min_{\beta} L_{max}^{-2/T}(\alpha,\beta) =$$

$$|S_{\delta\delta}|\min_{\beta} |S_{\alpha\alpha.\delta} + \beta'S_{k\alpha.\delta} + S_{\alpha k.\delta}\beta + \beta'S_{kk.\delta}\beta|.$$

This expression is minimized by

$$\hat{\beta}(\alpha) = -S_{kk.\delta}^{-1}S_{k\alpha.\delta},$$

giving a minimum value of

$$|S_{\delta\delta}||S_{\alpha\alpha.\delta} - S_{\alpha k.\delta}S_{kk.\delta}^{-1}S_{k\alpha.\delta}| = |S_{\delta\delta}||S_{\alpha\alpha.\delta k}|.$$

Now we use the identity

$$|\alpha'\alpha||\delta'\delta|\begin{vmatrix} S_{00} & S_{0k} \\ S_{k0} & S_{kk} \end{vmatrix} = \begin{vmatrix} S_{\alpha\alpha} & S_{\alpha\delta} & S_{\alpha k} \\ S_{\delta\alpha} & S_{\delta\delta} & S_{\delta k} \\ S_{k\alpha} & S_{k\delta} & S_{kk} \end{vmatrix} =$$

$$|S_{kk}||S_{\delta\delta.k}||S_{\alpha\alpha.\delta k}|,$$

which gives the representation

$$L_{max}^{-2/T}(\alpha) = |S_{00.k}||\delta' S_{00}\delta|/|\delta'(S_{00} - S_{0k}S_{kk}^{-1}S_{k0})\delta|. \tag{3.24}$$

Thus $\hat{\delta}$ and thereby $\hat{\alpha}$ can be determined by solving the eigenvalue problem

$$|\lambda S_{00} - S_{0k}S_{kk}^{-1}S_{k0}| = 0, \tag{3.25}$$

and choosing $\hat{\delta} = (\hat{u}_{r+1}, \ldots, \hat{u}_p)$, corresponding to the p-r smallest eigenvalues.

We can now apply Lemma 3.7 to see that the solutions of (3.25) are the same as the solutions of (2.18), and that we can choose $\hat{u}_i = S_{00}^{-1}S_{0k}\hat{v}_i$. From $(\hat{u}_1, \ldots, \hat{u}_r)' S_{00}\hat{\delta} = 0$ it follows that we can choose $\hat{\alpha} = -S_{00}(\hat{u}_1, \ldots, \hat{u}_r) = -S_{0k}\hat{\beta}$.

Thus we get the same solution as in section 2 but now as the solution of an optimization problem, a representation which will be convenient in the following.

We shall now find the properties of $\hat{\delta}$. We let $\hat{\delta} = \alpha\hat{a} + \delta\hat{d}$, then d $= (\delta'\delta)^{-1}\delta'\hat{\delta}$, and we have

PROPOSITION 3.11 *For $T \to \infty$ we find the representation*

$$T^{1/2}(\hat{\delta}\hat{d}^{-1} - \delta) = \alpha(\beta'\Sigma_{k0}\alpha)^{-1}(T^{1/2}\beta' S_{k0}\delta) + o_P(1) \tag{3.26}$$

which converges weakly to a Gaussian distribution with mean zero and variance matrix $\alpha(\alpha'(\Sigma_{00} - \Lambda)\alpha)^{-1}\alpha' \otimes \delta'\Lambda\delta$.

PROOF: The normalization $\hat{\delta}'S_{00}\hat{\delta} = I$ implies that $\hat{\delta}, \hat{a}$ and $\hat{d}$ are bounded in probability. Now consider for $i = r+1, \ldots, p$ the equations

$$(3.27) \qquad \hat{\lambda}_i \alpha' S_{00} \alpha \hat{a}_i + \hat{\lambda}_i \alpha' S_{00} \delta \hat{d}_i = \alpha' S_{0k} S_{kk}^{-1} S_{k0} \alpha \hat{a}_i + \alpha' S_{0k} S_{kk}^{-1} S_{k0} \delta \hat{d}_i .$$

By Lemma 3.5 we have $\hat{\lambda}_i \in O_P(T^{-1})$ and Lemma 3.6 shows that the coefficient to $\hat{d}_i$ is of the order $O_P(T^{-1/2})$, which implies that $\hat{a}_i \in O_P(T^{-1/2})$. Thus $\hat{\delta}'S_{00}\hat{\delta} = I$ implies that $\hat{d}'\delta' S_{00} \delta \hat{d} \overset{P}{\to} I$ and hence that $|\hat{d}|$ is bounded away from zero and finally that $\hat{d}^{-1}$ is bounded in probability. From (3.27) we then find that

$$\hat{\delta}\hat{d}^{-1} - \delta = \alpha\hat{a}\hat{d}^{-1} = \alpha(\alpha' S_{0k} S_{kk}^{-1} S_{k0} \alpha)^{-1} (\alpha' S_{0k} S_{kk}^{-1} S_{k0} \delta) + o_P(1)$$

which by Lemma 3.5 has the required representation and limit distribution.

3.4. *The asymptotic distribution of the likelihood ratio test statistics*

We shall now find the limiting distributions of the likelihood ratio test statistics for the various hypotheses discussed in Section 2.

The tests of H_2 in H_1 and H_3^* in H_2 were discussed in Johansen (1988b). The following results were obtained:

THEOREM 3.12: *Under the hypothesis $H_2 : \Pi = \alpha\beta'$ the statistic $-2\ln(Q;H_2|H_1)$ has a limit distribution which can be expressed in terms of a p-r dimensional Brownian motion B with i.i.d components as*

$$(A.28) \qquad tr\{\int_0^1 dBB' (\int_0^1 BB'du)^{-1} \int_0^1 BdB'\}.$$

This distribution was tabulated by simulation, and approximated by a χ^2 *distribution of the form* $(.85 - .58/f)\chi^2(f)$ *with* $f = 2(p-r)^2$. *Further it was found that under the hypothesis* $H_3^* : \beta = H\varphi$ *the statistic* $-2ln(Q;H_3^*|H_2)$ *is asymptotically distributed as* χ^2 *with* $f = (p-s)r$ *degrees of freedom.*

The reason for getting a strange limit distribution for the first statistic is that it involves the p-r smallest eigenvalues of (2.18) which are associated with the non-stationary part of the process.

We shall now as a preliminary result consider the test of a simple hypothesis concerning α or δ.

PROPOSITION 3.13: *The test statistic* $-2ln(Q;\delta|H_2)$ *of a simple hypothesis concerning* δ *has the representation*

(3.29) $-2ln(Q;\delta|H_2) = Ttr\{(\delta'\Lambda\delta)^{-1}\delta'S_{0k}\beta(\beta'\Sigma_{kk}\beta)^{-1}\beta'S_{k0}\delta\} + o_P(1)$.

and is asymptotically distibuted as χ^2 *with* $r(p-r)$ *degrees of freedom.*

PROOF: From the expression (3.24) for the likelihood profile in α we get the expression

$$-2\ln(Q) = T\ln\{|\hat{\delta}'S_{00.k}\hat{\delta}|/|\hat{\delta}'S_{00}\hat{\delta}|\} - T\ln\{|\delta'S_{00.k}\delta|/|\delta'S_{00}\delta|\}$$

which by a Taylors expansion equals

$$Ttr\{(\hat{\delta}'S_{00}\hat{\delta})^{-1}(\hat{\delta}-\delta)'S_{00}(\hat{\delta}-\delta) - (\hat{\delta}'S_{00.k}\hat{\delta})^{-1}(\hat{\delta}-\delta)'S_{00.k}(\hat{\delta}-\delta)\} + O_P(T(\hat{\delta}-\delta)^3).$$

Now replace $\hat{\delta}$ by $\hat{\delta}^* = \hat{\delta}\hat{d}^{-1}$, see (3.26), which is also a maximizing point for the likelihood function, and which converges in probability to δ. We then find that

$$\hat{\delta}^{*\prime}S_{00.k}\hat{\delta}^{*} = \hat{\delta}^{*\prime}(S_{00} - S_{0k}S_{kk}^{-1}S_{k0})\hat{\delta}^{*} \xrightarrow{P} \delta'\Sigma_{00}\delta = \delta'\Lambda\delta,$$

and

$$\hat{\delta}^{*\prime}S_{00}\hat{\delta}^{*} \xrightarrow{P} \delta'\Sigma_{00}\delta = \delta'\Lambda\delta,$$

since $\beta'\Sigma_{k0}\delta = \beta'\Sigma_{kk}\beta\alpha'\delta = 0$. We thus get the representation

$$-2\ln(Q) = tr\{(\delta'\Lambda\delta)^{-1}T^{1/2}(\hat{\delta}^{*}-\delta)'\Sigma_{0k}\beta(\beta'\Sigma_{kk}\beta)^{-1}\beta'\Sigma_{k0}T^{1/2}(\hat{\delta}^{*}-\delta)\} + o_P(1).$$

From the representation of $\hat{\delta}^{*}-\delta$ in (3.26) we then get

$$-2\ln(Q) = tr\{(\delta'\Lambda\delta)^{-1}T^{1/2}\delta'S_{0k}\beta(\beta'\Sigma_{kk}\beta)^{-1}\beta'S_{k0}\delta T^{1/2}\} + o_P(1)$$

which by Lemma 3.5 is asymptotically distributed as χ^2 with (p-r)r degrees of freedom.

Consider now finally the composite hypothesis

$$\tilde{H}_3\colon\ \alpha = A\psi.$$

where A is pxm of full rank. The restriction on α can be formulated as $sp(\alpha) \subset sp(A)$ which corresponds to the requirement that $sp(\delta)$ should contain the vectors in $sp(A)^{\perp}$ the orthogonal complement of A. Thus let B span this space, i.e. (A,B) has full rank, and $A'B = 0$, then we can formulate the hypothesis as

$$\tilde{H}_3\colon\ \delta = (A\tau, B)$$

where $\tau(m\times(m-r))$ has to be estimated. We shall then solve the maximization problem, see (3.24)

$$\max_{\tau} \left| \begin{matrix} \tau'S_{aa.k}\tau & \tau'S_{ab.k} \\ S_{ba.k}\tau & S_{bb.k} \end{matrix} \right| \Big/ \left| \begin{matrix} \tau'S_{aa}\tau & \tau'S_{ab} \\ S_{ba}\tau & S_{bb} \end{matrix} \right| =$$

$$|S_{bb.k}|\,|S_{bb}|^{-1}\max_{\tau} |\tau'(S_{aa.k} - S_{ab.k}S_{bb.k}^{-1}S_{ba.k})\tau| / |\tau'S_{aa.b}\tau| =.$$

$$|S_{bb.k}|\,|S_{bb}|^{-1}\max_{\tau} |\tau'(S_{aa.b} - S_{ak.b}S_{kk.b}^{-1}S_{ka.b})\tau| / |\tau'S_{aa.b}\tau|.$$

Hence we shall solve the equation

$$(3.30) \qquad |\lambda S_{aa.b} - S_{ak.b}S_{kk.b}^{-1}S_{ka.b}| = 0,$$

giving the eigenvalues $\tilde{\lambda}_1 > \ldots \tilde{\lambda}_m > 0$ and eigenvectors $\tilde{u}_1,\ldots,\tilde{u}_m$. We then choose $\tilde{\tau} = (\tilde{u}_{r+1},\ldots,\tilde{u}_m)$ corresponding to the m−r smallest eigenvalues. It follows from Lemma 3.7, that (3.30) and (2.38) have the same positive solutions. The eigenvectors are related by

$$\tilde{u}_i = S_{aa.b}^{-1}S_{ak.b}\tilde{v}_i, \quad i = 1,\ldots,m.$$

From the relation

$$(\tilde{u}_1,\ldots,\tilde{u}_r)'A'S_{00.b}A(\tilde{u}_{r+1},\ldots,\tilde{u}_m) = 0,$$

it follows that we can take

$$\tilde{\alpha} = -A(A'A)^{-1}A'S_{00.b}A(\tilde{u}_1,\ldots,\tilde{u}_r) = -A(A'A)^{-1}S_{ak.b}\tilde{\beta},$$

see (2.40). Thus we get the same solution as in section 2.

THEOREM 3.14: *The likelihood ratio test of the hypothesis*

$$\tilde{H}_3\colon \alpha = A\psi \text{ or } B'\alpha = 0$$

where $A(p\times m)$ *and* $B(p\times(m-r))$ *are of full rank, and* $A'B = 0$ *is denoted by* $-2\ln(Q;\tilde{H}_3|H_2)$, *see (2.44), and is asymptotically distributed as* $\chi^2(r(p-m))$.

PROOF: Let us formulate the hypotheses in terms of δ and use the result of Proposition 3.13 to get

$$(3.31) \qquad -2\ln(Q;\tilde{H}_3|H_2) = \mathrm{tr}\{(\delta'\Lambda\delta)^{-1}T^{1/2}\delta'S_{0k}\beta(\beta'\Sigma_{kk}\beta)^{-1}\beta'S_{k0}\delta T^{1/2}\}$$
$$- \mathrm{tr}\{(\tau'\Lambda_{aa.b}\tau)^{-1}T^{1/2}\tau'S_{ak.b}\beta(\beta'\Sigma_{kk.b}\beta)^{-1}\beta'S_{ka.b}\tau T^{1/2}\} + o_P(1),$$

where δ has the form $(A\tau,B)$. Now $\alpha'B = 0$ implies that $\beta'\Sigma_{k0}B = -\beta'\Sigma_{kk}\beta\alpha'B = 0$ and hence that $\beta'\Sigma_{kk.b}\beta = \beta'(\Sigma_{kk} - \Sigma_{k0}B(B'\Sigma_{kk}B)^{-1}B'\Sigma_{k0})\beta = \beta'\Sigma_{kk}\beta$. From the form of δ we get that the first term contains the factors

$$(3.32)\qquad \begin{bmatrix} \tau'S_{ak}\beta \\ S_{bk}\beta \end{bmatrix}' \begin{bmatrix} \tau'\Lambda_{aa}\tau & \tau'\Lambda_{ab} \\ \Lambda_{ba}\tau & \Lambda_{bb} \end{bmatrix}^{-1} \begin{bmatrix} \tau'S_{ak}\beta \\ S_{bk}\beta \end{bmatrix} =$$

$$\beta'S_{ka.b}\tau(\tau'\Lambda_{aa.b}\tau)^{-1}\tau S_{ak.b}\beta + \beta'S_{kb}\Lambda_{bb}^{-1}S_{bk}\beta.$$

The first term of (3.32) cancels the second term in (3.31) and we get the representation

$$(3.33)\qquad -2\ln(Q;\tilde{H}_3|H_2) = T\mathrm{tr}\{(\beta'\Sigma_{kk}\beta)^{-1}\beta'S_{kb}\Lambda_{bb}^{-1}S_{bk}\beta\} + o_P(1),$$

which by Lemma 3.5 is asymptotically distributed as χ^2 with $r(p-m)$ degrees of freedom since $B'\alpha = 0$.

5. *Wald tests for hypotheses about α and β*

We shall consider Wald tests which are very easy to calculate once the eigenvectors and eigenvalues have been calculated under the hypothesis H_2. Let us first consider a test for the hypothesis concerning α and let us express it as

$$\tilde{H}_3 \;:\; B'\alpha = 0.$$

A Wald test can be constructed by suitably normalizing the statistic $B'\hat{\alpha}$.

THEOREM 3.15: *Under the hypothesis $\tilde{H}_3$: $B'\alpha = 0$ where B is $p\times(p-m)$ of full rank the asymptotic distribution of*

$$(3.34)\qquad T\mathrm{tr}\{(B'(S_{00} - \hat{\alpha}\hat{\alpha}')B)^{-1}(B'\hat{\alpha}\hat{\alpha}'B)\}$$

is χ^2 with $(p-m)r$ degrees of freedom.

PROOF: In view of the results of Proposition 3.9 we can consider

$$T\mathrm{tr}\{\Lambda_{bb}^{-1}B'\hat{\alpha}\hat{b}'(\beta'\Sigma_{kk}\beta)\hat{b}\hat{\alpha}'B\}$$

which is asymptotically distributed as χ^2 with r(p-m) degrees of freedom. To apply the test we need consistent estimates for the variance matrices, and we thus insert the consistent estimates

$$\hat{\beta}^{*\prime}S_{kk}\hat{\beta}^{*} = \hat{b}'^{-1}\hat{\beta}'S_{kk}\hat{\beta}\hat{b}^{-1} = \hat{b}'^{-1}\hat{b}^{-1}$$

and

$$B'(S_{00} - \hat{\alpha}\hat{\alpha}')B,$$

and the result follows. If we apply the result (3.23) we get the representation

$$B'(\hat{\alpha}\hat{b}' - \alpha) = - S_{bk}\beta(\beta'S_{kk}\beta)^{-1} + O_P(T^{-1/2})$$

and we see that the Wald test is just the quadratic approximation derived to the likelihood ratio test in (3.33)

One can derive a different expression for this statistic by using the alternative derivation of $\hat{\alpha} = - S_{00}\hat{u}$, where $\hat{u}$ consists of the r eigenvectors of (3.25) corresponding to the r largest eigenvalues. Since $\hat{u}\hat{u}' + \hat{\delta}\hat{\delta}' = S_{00}^{-1}$ we find the expression

$$T\mathrm{tr}\{(B'S_{00}\hat{\delta}\hat{\delta}'S_{00}B)^{-1}(B'S_{00}\hat{u}\hat{u}'S_{00}B)\} \tag{3.35}$$

Let us next consider the hypothesis H_3^* but expressed as

$$H_3^* : K'\beta = 0.$$

where K is p×(p-s) of full rank. This suggests a Wald test on the statistic $K'\hat{\beta}$ and the problem is again how to normalize it.

THEOREM 3.16: *Under the hypothesis* H_3^* : $K'\beta = 0$, *where* K *is* $p\times(p-s)$ *of full rank, the asymptotic distribution of*

$$(3.36) \qquad T\,tr\{(K'\hat{\Pi}'(S_{00} - \hat{\alpha}\hat{\alpha}')^{-1}\hat{\Pi}K)(K'\hat{\gamma}\hat{\gamma}'K)^{-1}\}$$

is χ^2 *with* $(p-s)r$ *degrees of freedom.*

PROOF: The idea of this test is to note that the limiting distribution of $K'\hat{\beta}\hat{b}^{-1}$ is a mixture of Gaussian distributions, see Proposition 3.8, hence the asymptotic distribution of

$$(3.37) \qquad (K'\gamma(\int_0^1 UU'du)^{-1}\gamma'K)^{-1/2}(TK'\hat{\beta}\hat{b}^{-1})(\alpha'\Lambda^{-1}\alpha)^{1/2}$$

will be, for fixed U, a Gaussian random matrix of dimension $(p-s)\times r$ with mean zero and covariance matrix $I\otimes I$. Since this result is the same for all fixed U it also holds unconditionally, from which we derive that

$$(3.38) \qquad tr\{(K'\gamma(\int_0^1 UU'du)^{-1}\gamma'K)^{-1}(TK'\hat{\beta}\hat{b}^{-1})(\alpha'\Lambda^{-1}\alpha)(\hat{b}'^{-1}\hat{\beta}'KT)\}$$

is asymptotically χ^2 with $(p-s)r$ degrees of freedom. We shall now insert consistent estimates for the variance matrices. The first one follows from Proposition 3.9 and the consistency of $\hat{\Lambda} = S_{00} - \hat{\alpha}\hat{\alpha}'$:

$$\hat{b}\hat{\alpha}'(S_{00} - \hat{\alpha}\hat{\alpha}')^{-1}\hat{\alpha}\hat{b}' \xrightarrow{P} \alpha'\Lambda^{-1}\alpha.$$

For the second we first apply Lemma 3.4 and the definition of U in Proposition 3.8 :

$$K'\gamma(\gamma'S_{kk}/T\gamma)^{-1}\gamma'K \xrightarrow{w} K'\gamma(\int_0^1 UU'du)^{-1}\gamma'K.$$

We shall now show how to estimate the left hand side from the data. We first decompose $\hat{\gamma} = \gamma\hat{c} + \beta\hat{e}$, and find from the equation

$$\hat{\gamma}_i'S_{k0}S_{00}^{-1}S_{0k}\hat{\gamma}_i = \hat{\lambda}_i \in O_P(T^{-1}), \; i = r+1,\dots,p,$$

that $\hat{\gamma}_i$ and hence $\hat{c}_i$ and $\hat{e}_i \in O_P(T^{-1/2})$. Then the normalization

$$I = \hat{\gamma}'S_{kk}\hat{\gamma} = \hat{e}'\beta'S_{kk}\beta\hat{e} + \hat{c}'\beta'S_{kk}\gamma\hat{c} + \hat{c}'\gamma's_{kk}\beta\hat{e} + \hat{c}'\gamma'S_{kk}\gamma\hat{c}$$

shows that

$$\hat{c}'\gamma'S_{kk}\gamma\hat{c} \xrightarrow{P} I. \tag{3.39}$$

Note that $K'\hat{\gamma} = K'\gamma\hat{c}$, such that

$$TK'\gamma(\gamma'S_{kk}\gamma)^{-1}\gamma'K = TK'\hat{\gamma}(\hat{c}'\gamma'S_{kk}\gamma\hat{c})^{-1}\hat{\gamma}'K.$$

By (3.39) this has the same limit distribution as $TK'\hat{\gamma}\hat{\gamma}'K$. Combining these results we obtain Theorem 3.16.

Note that the test looks like a Wald test on $\hat{\Pi}K$ but one has not normalized it by the asymptotic variance of $\hat{\Pi}K$ since, as is seen from Corollary 3.10, this will be zero when $K'\beta = 0$.

An alternative form of the test statistic which is very easy to calculate is

$$T\mathrm{tr}\{(K'\hat{\beta}(\hat{D}^{-1} - I)^{-1}\hat{\beta}'K)(K'\hat{\gamma}\hat{\gamma}'K)^{-1}\} \tag{3.40}$$

where $\hat{D} = \mathrm{diag}(\hat{\lambda}_1,\ldots,\hat{\lambda}_r)$. This is seen from the identity

$$\beta'\Sigma_{kk}\beta(\beta'\Sigma_{k0}\Sigma_{00}^{-1}\Sigma_{0k}\beta)^{-1}\beta'\Sigma_{kk}\beta - \beta'\Sigma_{kk}\beta = (\alpha'\Lambda^{-1}\alpha)^{-1}$$

by inserting the estimates.

We shall apply it to the special case when $r = 1$ and $K' = (k_1,\ldots,k_p)$, hence $s = p-1$, and we have only one cointegration relation where we want to test some linear constraint on the coefficients. We shall formulate the result as a Corollary.

COROLLARY 3.17: *If only 1 cointegration vector β is present ($r = 1$), and if we want to test the hypothesis*

$$K'\beta = 0$$

then the test statistic

(3.41) $\quad T^{1/2}K'\hat{\beta}/\{(\hat{\lambda}_1^{-1} - 1)(K'\hat{\gamma}\hat{\gamma}'K)\}^{1/2}$

is asymptotically normalized Gaussian. Here $\hat{\lambda}_1$ *is the maximal eigenvalue and* $\hat{\beta}$ *the corresponding eigenvector of the equation*

$$|\lambda S_{kk} - S_{k0}S_{00}^{-1}S_{0k}| = 0.$$

The remaining eigenvectors form $\hat{\gamma}$.

The normalization $\hat{\gamma}'S_{kk}\hat{\gamma} = I$ implies that $\hat{\gamma}'\hat{\gamma}$ is of the order of T^{-1} which shows that $\hat{\beta}$ is really normalized by T.

Thus if there is only one cointegration vector $\hat{\beta}$ one can think of the matrix $(\hat{\lambda}^{-1}-1)\hat{\gamma}\hat{\gamma}'$ as giving an estimate of the asymptotic "variance" of $\hat{\beta}$.

This result should be interpreted with care since $K'\hat{\beta}$ is not asymptotically Gaussian and may not have an asymptotic variance but one can normalize a linear combination of the components of $\hat{\beta}$ in such a way that it becomes asymptotically Gaussian.

A comparison with the proof of Theorem 4 (1987) shows that the representation of the asympototic distribution of $-2\ln(Q;H_2^*|H_1)$ given there involves the same Brownian motions, such that the χ^2 test suggested here is asymptotically the same as the likelihood ratio test. Only the introduction of $\hat{\beta}$ makes the result more transparent.

As a final remark one should add, that we can of course test further restriction on β or α, i.e. in the hypothesis H_2^* : $\beta = H\varphi$ we can test the hypothesis H_2^{**}: $\varphi = H_1\eta$ and derive the corresponding likelihood ratio test or Wald test. These will be asymptotically distributed as χ^2 with the appropriate degrees of freedom corresponding to the loss of parameters in going from H_2^* to H_2^{**}. We shall not formulate these results in detail here.

University of Copenhagen

REFERENCES

Ahn, S. K., and G. C. Reinsel (1987): "Estimation for Partially Non-stationary Multivariate Autoregressive Models," University of Wisconsin.

Anderson, T. W. (1984): *An introduction to multivariate statistical analysis*. New York: Wiley.

Box, G. E. P., and G. C. Tiao (1981): "A Canonical Analysis of Multiple Time Series with Applications," *Biometrika*, 64, 355-365.

Davidson, J. (1986): "Cointegration in Linear Dynamic Systems," Discussion paper, LSE.

Engle, R. F. and C. W. J. Granger (1987): " Co-integration and Error Correction: Representation, Estimation and Testing," *Econometrica*, 55, 251-276.

Granger, C. J. (1981): " Some Properties of Time Series Data and their Use in Econometric Model Specification,"*Journal of Econometrics*, 16, 251-276.

Granger, C. W. J., and A. A. Weiss (1983): "Time Series Analysis of Error Correction Models," in *Studies in economic time series and multivariate statistics*, ed. by S. Karlin, T. Amemiya and L. A. Goodman. New York: Academic Press.

Johansen, S.(1988a): "The Mathematical Structure of Error Correction Models," *Contemporary Mathematics* , American Mathematical Society (in press).

Johansen, S. (1988b): "Statistical Analysis of Cointegration Vectors," *Journal of Economic Dynamics and Control*, (in press).

Peña, D. and G. E. P. Box (1987): "Identifying a Simplifying Structure in Time Series," *Journal of the American Statistical Association*, 82, 836-843.

Phillips, P. C. B. (1987): "Multiple Regression with Integrated Time Series," Cowles Foundation discussion paper No. 852.

Phillips, P. C. B., and S. N. Durlauf (1986): "Multiple Time Series Regression with Integrated Processes," *Review of Economic Studies*, 53, 473-495.

Phillips, P. C. B., and S. Ouliaris (1986): " Testing for Cointegration," Cowles Foundation discussion paper No 809.

———(1987): "Asymptotic Properties of Residual based Tests for Cointegration," Cowles Foundation discussion paper No.847.

Phillips, P. C. B., and J. Y. Park (1986a): " Asymptotic Equivalence of OLS and GLS in Regression with Integrated Regressors," Cowles Foundation discussion paper No.802.

———(1986b): "Statistical Inference in Regressions with Integrated Processes : Part 1," Cowles Foundation discussion paper No. 811.

———(1987): "Statistical Inference in Regressions with Integrated Processes : Part 2," Cowles Foundation discussion paper No. 819.

Sims, A., J. H. Stock, and M. W. Watson (1986): " Inference in Linear Time Series Models with some Unit Roots," Preprint.

Stock, J.H. (1987): "Asymptotic Properties of Least Squares Estimates of Cointegration Vectors," *Econometrica*, 55, 1035-1056.

Stock, J. H., and M. W. Watson (1987): " Testing for Common Trends," Working paper in Econometrics, the Hoover Institution, Stanford.

Velu, R. P., G. C. Reinsel, and D. W. Wichern (1986): "Reduced Rank Models for multiple time series," *Biometrika*, 73, 105-118.

Velu, R. P., D. W. Wichern, and G. C. Reinsel (1987): "A Note on Non-stationary and Canonical Analysis of Multiple Time Series Models," *Journal of Time Series Analysis*, 8, 479-487.

M2 (REAL DEMAND FOR MONEY)
QUARTERLY DATA FROM 1974:1 TO 1985:4

1974:1	1138.137457	1119.057313	1087.323038	1116.470688
1975:1	1099.228093	1133.893587	1184.096733	1332.307015
1976:1	1312.702203	1318.434458	1339.031798	1324.606002
1977:1	1287.936212	1315.538012	1291.784214	1278.019706
1978:1	1224.091112	1227.749034	1227.297212	1221.063880
1979:1	1191.728770	1220.231183	1165.126856	1187.932151
1980:1	1135.894028	1131.120360	1101.585742	1157.760605
1981:1	1118.404654	1121.932420	1090.708796	1119.920756
1982:1	1086.286620	1089.560548	1075.939195	1086.321037
1983:1	1102.863178	1182.681762	1229.925540	1295.813977
1984:1	1315.669097	1375.594124	1373.428687	1494.216170
1985:1	1461.192515	1499.328375	1567.417736	1655.323700

Y^r (REAL NATIONAL INCOME)
QUARTERLY DATA FROM 1973:1 TO 1985:4

1973:1	224.7162	224.3259	221.4889	227.2918
1974:1	216.5592	211.3524	216.4148	211.8868
1975:1	218.9246	215.8073	222.8430	250.1341
1976:1	246.9495	240.6962	235.4124	239.1914
1977:1	238.8918	241.9483	242.4076	243.8743
1978:1	238.5137	241.1501	247.4554	242.6897
1979:1	247.5116	248.8452	238.6271	234.0031
1980:1	236.4621	240.7253	222.4843	230.0312
1981:1	232.3390	231.1393	232.8106	230.7839
1982:1	236.3070	241.5006	253.2976	247.5879
1983:1	250.3042	246.7591	244.2551	258.6220
1984:1	247.0589	255.7699	258.2597	244.0017
1985:1	256.1064	253.5935	255.0615	264.7311

I^b (BOND INTEREST RATE)
QUARTERLY DATA FROM 1973:1 TO 1985:4

1973:1	1.1208	1.1223	1.1343	1.1408
1974:1	1.1547	1.1779	1.1705	1.1522
1975:1	1.1342	1.1334	1.1284	1.1288
1976:1	1.1413	1.1531	1.1605	1.1618
1977:1	1.1665	1.1630	1.1692	1.1728
1978:1	1.1717	1.1757	1.1711	1.1820
1979:1	1.1700	1.1689	1.1782	1.1804
1980:1	1.1910	1.1967	1.1923	1.1829
1981:1	1.1845	1.1928	1.2032	1.1923
1982:1	1.2032	1.2107	1.2088	1.1970
1983:1	1.1611	1.1383	1.1417	1.1338
1984:1	1.1341	1.1417	1.1448	1.1412
1985:1	1.1317	1.1196	1.1070	1.1037

I^s (DEPOSITE INTEREST RATE)
QUARTERLY DATA FROM 1973: 1 TO 1985: 4

1973:1	1.0695	1.0705	1.0760	1.0780
1974:1	1.0940	1.0955	1.0955	1.0955
1975:1	1.0885	1.0790	1.0760	1.0740
1976:1	1.0720	1.0780	1.0800	1.1030
1977:1	1.0970	1.0880	1.0950	1.0970
1978:1	1.0990	1.0880	1.0810	1.0770
1979:1	1.0750	1.0770	1.0860	1.1010
1980:1	1.1090	1.1210	1.1210	1.1070
1981:1	1.1050	1.1090	1.1110	1.1090
1982:1	1.1070	1.1110	1.1110	1.1100
1983:1	1.1060	1.0870	1.0830	1.0850
1984:1	1.0850	1.0830	1.0850	1.0920
1985:1	1.0903	1.0876	1.0800	1.0756

M1 (DEMAND FOR NARROW MONEY)
QUARTERLY DATA FROM 1958:1 TO 1984:4

1958:1	1122	1174	1150	1246
1959:1	1279	1347	1395	1429
1960:1	1362	1430	1508	1496
1961:1	1534	1498	1578	1644
1962:1	1576	1588	1646	1722
1963:1	1798	1803	1827	1987
1964:1	1849	1901	1946	2046
1965:1	1996	2023	1999	2087
1966:1	1935	2074	2030	2213
1967:1	2042	2105	2016	2103
1968:1	2151	2316	2378	2671
1969:1	2552	2730	2737	3140
1970:1	3455	3627	3628	3959
1971:1	3218	3243	3415	3975
1972:1	3950	4275	4408	4974
1973:1	4644	5170	5004	6114
1974:1	5376	5903	6145	7283
1975:1	7403	7801	7391	9450
1976:1	8582	8652	8504	9286
1977:1	9496	9996	9670	9872
1978:1	10058	11247	11071	11496
1979:1	11442	13269	13003	14087
1980:1	13354	14588	14141	14979
1981:1	15093	15763	16045	17186
1982:1	16353	18664	18958	19917
1983:1	19401	21362	21403	21427
1984:1	20606	22026	22238	22426

Y (NOMINAL INCOME)
QUARTERLY DATA FROM 1958:1 TO 1984:4

1958:1	3082.7	3152.8	3369.4	3348.9
1959:1	3186.1	3415.1	3715.7	3762.0
1960:1	3711.3	3838.2	4128.4	4146.3
1961:1	4201.1	4295.0	4550.0	4579.6
1962:1	4501.7	4567.0	4857.1	4930.6
1963:1	4710.5	5111.3	5333.3	5386.0
1964:1	5613.2	5743.1	5893.1	6304.5
1965:1	6136.9	6258.5	6625.1	6807.3
1966:1	6266.5	6836.9	7143.8	7529.4
1967:1	7249.8	7484.6	7610.4	7764.6
1968:1	8006.2	8380.7	8789.6	8971.7
1969:1	9227.4	9533.8	9978.4	10273.0
1970:1	10167.8	10647.5	11013.8	11762.9
1971:1	10821.1	11838.8	12063.1	12937.8
1972:1	12668.8	13339.4	13823.0	15077.5
1973:1	15122.9	15703.9	16774.6	19144.6
1974:1	18748.2	20729.5	20720.8	23975.4
1975:1	22581.7	24704.9	24349.4	26324.6
1976:1	24624.2	26960.7	27850.1	30687.2
1977:1	27579.1	29656.8	30356.6	32988.7
1978:1	30199.1	32770.8	32481.6	36287.6
1979:1	34729.0	38014.1	37678.0	41367.9
1980:1	38549.2	42385.7	44836.8	48607.3
1981:1	44331.6	49379.4	49993.1	54197.1
1982:1	48986.4	54811.1	55242.3	60971.6
1983:1	54374.5	61662.5	63252.4	67983.3
1984:1	60899.4	68137.1	68628.9	

P (THE PRICE LEVEL)
QUARTERLY DATA FROM 1958:1 TO 1984:4

1958:1	67	68	68	68
1959:1	68	68	69	70
1960:1	71	71	72	72
1961:1	72	72	72	74
1962:1	74	76	76	77
1963:1	78	79	80	81
1964:1	86	88	89	89
1965:1	90	92	92	93
1966:1	94	95	97	98
1967:1	99	100	101	104
1968:1	108	110	110	111
1969:1	111	112	112	112
1970:1	114	114	115	116
1971:1	119	122	125	126
1972:1	127	131	133	135
1973:1	139	144	152	156
1974:1	163	169	179	182
1975:1	193	200	209	215
1976:1	225	228	237	242
1977:1	252	261	268	271
1978:1	275	280	284	286
1979:1	294	300	305	311
1980:1	322	336	345	354
1981:1	365	377	384	389
1982:1	402	412	415	424
1983:1	432	448	454	467
1984:1	469	478	485	488

I^m (THE MARGINAL INTEREST RATE)
QUARTERLY DATA FROM 1958:1 TO 1984:4

1958:1	11.55	18.90	10.65	14.12
1959:1	7.25	6.75	6.75	6.75
1960:1	6.75	17.50	27.57	17.74
1961:1	11.64	13.03	12.45	15.91
1962:1	19.16	25.75	27.43	27.17
1963:1	23.48	18.68	20.82	18.61
1964:1	14.65	18.37	19.48	21.80
1965:1	20.12	14.50	7.00	7.00
1966:1	7.00	12.23	24.61	34.24
1967:1	7.00	12.96	9.62	12.85
1968:1	9.60	7.00	7.00	7.00
1969:1	7.00	7.00	7.00	7.46
1970:1	8.58	16.34	26.00	20.37
1971:1	8.25	12.52	9.73	8.50
1972:1	7.75	7.75	7.75	7.75
1973:1	7.75	7.75	10.03	26.00
1974:1	18.35	15.93	9.41	14.77
1975:1	13.87	19.18	17.58	27.92
1976:1	19.55	19.48	17.28	18.23
1977:1	15.18	16.47	18.00	19.79
1978:1	18.62	9.58	9.10	9.73
1979:1	8.39	8.37	8.45	11.84
1980:1	12.31	14.92	15.99	16.55
1981:1	17.74	12.95	13.25	15.07
1982:1	13.99	13.81	13.68	15.15
1983:1	14.33	15.00	15.55	16.78
1984:1	17.50	16.70	16.38	15.55

APPENDIX B Model (2.1) for the Danish data

$$\Delta X_t = \Gamma_1 \Delta X_{t-1} - \Pi X_{t-2} + \text{const} + \Sigma_i k_i Q_{it} + \epsilon_t$$

$$\Gamma_1 = \begin{bmatrix} \underset{(.25)}{-.46} & \underset{(.22)}{.02} & \underset{(.60)}{-1.18} & \underset{(.82)}{.02} \\ \underset{(.28)}{-.01} & \underset{(.25)}{-.45} & \underset{(.67)}{.23} & \underset{(.93)}{-1.21} \\ \underset{(.07)}{-.05} & \underset{(.07)}{.08} & \underset{(.18)}{-.29} & \underset{(.25)}{-.08} \\ \underset{(.05)}{.05} & \underset{(.04)}{.02} & \underset{(.11)}{.39} & \underset{(.16)}{-.05} \end{bmatrix}$$

$$-\Pi = \begin{bmatrix} \underset{(.12)}{-.25} & \underset{(.17)}{.14} & \underset{(.55)}{-1.98} & \underset{(.64)}{1.77} \\ \underset{(.13)}{.09} & \underset{(.19)}{-.31} & \underset{(.63}{.16} & \underset{(.72)}{.23} \\ \underset{(.04)}{.00} & \underset{(.05)}{.00} & \underset{(.17)}{-.02} & \underset{(.20)}{.17} \\ \underset{(.02)}{-.00} & \underset{(.04)}{.01} & \underset{(.11)}{.14} & \underset{(.13)}{-.33} \end{bmatrix}$$

Estimated correlations and variances of regression residuals.

$$C_\epsilon = \begin{bmatrix} .028^2 & & & \\ .69 & .031^2 & & \\ -.39 & -.03 & .009^2 & \\ -.21 & .11 & .19 & .006^2 \end{bmatrix}$$

Residual autocorrelations, r_i $(i=1,..,8)$ and Box-Pierce Q-statistic, χ^2-distributed with 18 degrees of freedom.

r_i	1	2	3	4	5	6	7	8	Q(18)
$\Delta m1$	-.03	-.09	.12	-.08	.11	.02	.10	.06	7.8
Δy	-.10	-.17	.10	-.21	.13	-.02	.07	-.06	10.7
Δi^m	.14	.06	.10	-.18	.03	-.02	-.20	.00	10.9
$\Delta^2 p$	-.02	01	.00	-.10	.00	-.24	.04	-.07	7.5

$$\Delta X_t = \Gamma_1 \Delta X_{t-1} + \Pi X_{t-2} + \text{const} + \Sigma_i k_i Q_{it} + \epsilon_t$$

$$\Gamma_1 = \begin{bmatrix} \underset{(.11)}{-.21} & \underset{(.14)}{-.22} & \underset{(.13)}{-.10} & \underset{(.45)}{-.01} \\ \underset{(.07)}{.04} & \underset{(.09)}{-.69} & \underset{(.08)}{-.06} & \underset{(.27)}{-.16} \\ \underset{(.08)}{-.16} & \underset{(.11)}{-.02} & \underset{(.10)}{-.23} & \underset{(.33)}{.42} \\ \underset{(.03)}{.01} & \underset{(.03)}{.02} & \underset{(.03)}{.02} & \underset{(.11)}{-.82} \end{bmatrix}$$

$$\Pi = \begin{bmatrix} \underset{(.06)}{-.12} & \underset{(.06)}{.11} & \underset{(.11)}{-.23} & \underset{(.54)}{-.42} \\ \underset{(.04)}{.02} & \underset{(.04)}{-.04} & \underset{(.07)}{-.14} & \underset{(.33)}{-.21} \\ \underset{(.04)}{.09} & \underset{(.04)}{-.10} & \underset{(.08)}{-.48} & \underset{(.40)}{.67} \\ \underset{(.01)}{.00} & \underset{(.01)}{.01} & \underset{(.03)}{-.00} & \underset{(.13)}{-.48} \end{bmatrix}$$

Estimated correlations and variances of regression residuals.

$$C_\epsilon = \begin{bmatrix} .050^2 & & & \\ .25 & .031^2 & & \\ .19 & .03 & .037^2 & \\ -.30 & -.31 & -.02 & .012^2 \end{bmatrix}$$

Residual autocorrelations, r_i (i=1,..,8) and Box-Pierce Q-statistic, χ^2-distributed with 30 degrees of freedom.

r_i	1	2	3	4	5	6	7	8	Q(30)
$\Delta m1$	.01	-.01	-.17	.02	-.02	.01	-.05	.26	31.3
Δy	.01	-.23	-.15	.22	-.16	.03	-.02	.14	59.2
Δi^m	.01	-.01	-.05	-.05	-.04	.05	.10	-.08	26.0
$\Delta^2 p$	.04	-.02	-.14	.10	.08	-.04	.02	.02	24.6

APPENDIX C.

Product moment matrices from the regression of ΔX_t on ΔX_{t-1} and X_{t-2}, for Denmark $(m2, y, i^b, i^d)$ and Finland $(m1, y, i^m, \Delta p)$. The matrices are given in the form

$$\begin{bmatrix} S_{00} & S_{20}S_{00}^{-1}S_{02} \\ S_{02} & S_{22} \end{bmatrix} \times 10^3$$

DK

$$\begin{bmatrix} .874 & & & \\ .569 & .865 & & \\ -.067 & -.004 & .059 & \\ -.051 & -.028 & .013 & .029 \end{bmatrix} \quad \begin{bmatrix} .192 & & & \\ .093 & .237 & & \\ -.049 & .042 & .067 & \\ -.041 & -.014 & .021 & .018 \end{bmatrix}$$

$$\begin{bmatrix} -.190 & -.119 & .102 & .028 \\ -.360 & -.422 & .022 & .040 \\ -.143 & -.078 & -.034 & -.004 \\ .007 & -.007 & -.024 & -.018 \end{bmatrix} \quad \begin{bmatrix} 8.798 & & & \\ 3.340 & 2.401 & & \\ -1.284 & -.205 & .370 & \\ -.561 & -.145 & .168 & .125 \end{bmatrix}$$

SF

$$\begin{bmatrix} 2.488 & & & \\ .408 & .918 & & \\ .477 & .143 & 1.631 & \\ -.135 & -.082 & -.012 & .140 \end{bmatrix} \quad \begin{bmatrix} 4.155 & & & \\ 3.934 & 4.297 & & \\ .527 & .549 & .586 & \\ .222 & .223 & .046 & .027 \end{bmatrix}$$

$$\begin{bmatrix} -1.831 & -1.723 & .183 & .065 \\ -.893 & -1.923 & -.015 & .014 \\ -.615 & -.276 & -.863 & -.042 \\ -.068 & -.073 & .013 & -.042 \end{bmatrix} \quad \begin{bmatrix} 93.538 & & & \\ 93.885 & 101.750 & & \\ .584 & -.334 & 2.097 & \\ 1.974 & 2.194 & .076 & .134 \end{bmatrix}$$

TABLE I

The estimate of the unrestricted matrix Π in model H_1 for the Danish and Finnish data.

$$
\text{DK} \qquad
\begin{array}{c}
m2 \\ y \\ i^b \\ i^d
\end{array}
\begin{bmatrix}
.248 & -.133 & 1.978 & -1.767 \\
-.095 & .308 & -.156 & -.231 \\
.004 & -.002 & .024 & -.174 \\
.001 & -.010 & -.144 & .333
\end{bmatrix}
\qquad
\text{SF} \qquad
\begin{array}{c}
m1 \\ y \\ i^m \\ \Delta p
\end{array}
\begin{bmatrix}
.119 & -.110 & .227 & .418 \\
-.024 & .036 & .137 & .214 \\
-.095 & .104 & .479 & -.670 \\
-.004 & -.007 & .003 & .484
\end{bmatrix}
$$

TABLE II

The eigenvalues $\hat{\lambda}$ and eigenvectors $\hat{V}$ as well as $-S_{02}\hat{V}$ for the Danish and Finnish data.

DK					SF				
	eigenvalues $\hat{\lambda}$					eigenvalues $\hat{\lambda}$			
	(.5004	.1940	.1269	.0138)		(.3093	.2260	.0731	.0295)
	eigenvectors $\hat{V}$					eigenvectors $\hat{V}$			
m2	-19.39	-14.77	10.27	- 12.39	m1	-2.93	4.58	-11.13	1.38
y	18.61	25.04	-26.05	1.29	y	2.86	-6.06	10.24	2.22
i^b	-130.40	-17.44	29.81	.54	i^m	20.79	-9.23	-2.99	-1.90
i^d	105.05	-73.35	-83.90	-24.94	Δp	20.58	104.17	20.28	-21.62
	$-S_{02}\hat{V}\times 10^3$					$-S_{02}\hat{V}\times 10^3$			
m2	-16.33	4.22	-2.59	-1.64	m1	11.37	4.43	-11.68	1.86
y	-3.96	6.98	-8.00	-1.05	y	7.69	1.25	1.16	4.55
i^b	-.33	-1.38	-1.48	.66	i^m	18.25	-10.23	-.66	-1.58
i^d	1.22	-1.99	-.65	-.16	Δp	1.89	3.79	1.31	-1.11

TABLE III

Test statistics for the hypothesis H_2 for various values of r versus the general alternative H_1 for the Danish and Finnish data. The quantiles from the asymptotic distribution are taken from Johansen [9]

H_2	-2ln(Q;DK)	-2ln(Q;SF)	95% quantile	90% quantile
$r \leq 3$	.64	3.11	4.2	2.9
$r \leq 2$	6.88	11.01	12.0	10.3
$r \leq 1$	16.80	37.65	23.8	21.2
$r = 0$	48.72	76.14	38.6	35.6

TABLE IV

Eigenvalues λ^* and eigenvectors φ^* as well as $-S_{0k}H\varphi^*$ for the Danish and Finnish data under the restriction that money demand and national income have equal coefficients with opposite sign.

DK

eigenvalues λ^*

$$(.4999 \quad .1759 \quad .0462)$$

eigenvectors φ^*

$$\begin{array}{l} m2-y \\ i^b \\ i^d \end{array} \begin{bmatrix} -19.77 & -8.96 & 18.09 \\ -131.45 & .05 & 16.57 \\ 106.00 & -107.58 & -6.46 \end{bmatrix}$$

$-S_{0k}H\varphi^* \times 10^3$

$$\begin{array}{l} m2 \\ y \\ i^b \\ i^d \end{array} \begin{bmatrix} -16.15 & 2.27 & -.66 \\ -3.59 & 2.02 & -4.24 \\ -.33 & -1.85 & -1.04 \\ 1.20 & -2.07 & .17 \end{bmatrix}$$

SF

eigenvalues λ^*

$$(.3093 \quad .1994 \quad .0704)$$

eigenvectors φ^*

$$\begin{array}{l} m1-y \\ i^m \\ \Delta p \end{array} \begin{bmatrix} -2.95 & 6.02 & -10.30 \\ 20.92 & -8.67 & -3.61 \\ 18.56 & 86.87 & 17.05 \end{bmatrix}$$

$-S_{0k}H\varphi^* \times 10^3$

$$\begin{array}{l} m1 \\ y \\ i^m \\ \Delta p \end{array} \begin{bmatrix} 11.36 & 6.26 & -10.70 \\ 7.72 & 2.71 & 2.31 \\ 18.40 & -9.78 & -1.30 \\ 1.82 & 2.99 & 1.09 \end{bmatrix}$$

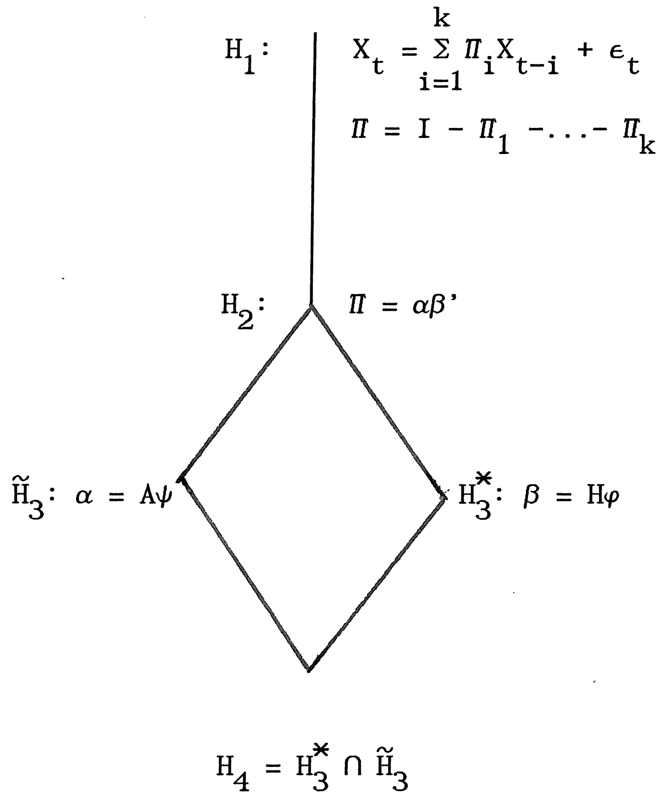
TABLE V

The estimate of the restricted matrix Π in model H_2 for the Danish and Finnish data. For the Danish data r = 1 and the coefficients to m2 and y are equal with opposite sign. The same holds for the coefficients to i^b and i^d. For the Finnish data r = 3 and the coefficients of m2 and y are equal with opposite sign.

DK SF

$$\begin{array}{c} m2 \\ y \\ i^b \\ i^d \end{array} \begin{bmatrix} .275 & -.275 & 2.076 & -2.076 \\ .068 & -.068 & .513 & -.513 \\ .001 & -.001 & -.008 & .008 \\ -.028 & .028 & -.211 & .211 \end{bmatrix} \quad \begin{array}{c} m1 \\ y \\ i^m \\ \Delta p \end{array} \begin{bmatrix} .114 & -.114 & -.222 & -.572 \\ -.030 & .030 & -.130 & -.417 \\ -.100 & .100 & -.474 & .531 \\ .001 & -.001 & -.008 & -.312 \end{bmatrix}$$

Figure 1 - Illustration of the relation between various hypotheses concerning the cointegration vectors



PREPRINTS 1987

COPIES OF PREPRINTS ARE OBTAINABLE FROM THE AUTHOR OR FROM THE INSTITUTE OF MATHEMATICAL STATISTICS, UNIVERSITETSPARKEN 5, 2100 COPENHAGEN Ø, DENMARK, TELEPHONE + 45 1 35 31 33.

No. 1 Jensen, Søren Tolver and Johansen, Søren: Estimation of Proportional Covariances.

No. 2 Rootzén, Holger: Extremes, Loads, and Strengths.

No. 3 Bertelsen, Aksel: On the Problem of Testing Reality of a Complex Multivariate Normal Distribution.

No. 4 Gill, Richard D. and Johansen, Søren: Product-Integrals and Counting Processes.

No. 5 Leadbetter, M.R. and Rootzén, Holger: Extremal Theory for Stochastic Processes.

No. 6 Tjur, Tue: Block Designs and Electrical Networks.

No. 7 Johansen, Søren: Statistical Analysis of Cointegration Vectors.

No. 8 Bertelsen, Aksel: On the Problem of Testing Reality of a Complex Multivariate Normal Distribution, II.

No. 9 Andersson, S.A. and Perlman, M.D.: Group-Invariant Analogues of Hadamard's Inequality.

No. 10 Hald, Anders: Two Generalizations of the Problem of Points by Bernoulli, de Moivre and Montmort.

No. 11 Andersson, Steen Arne: The Lattice Structure of Orthogonal Linear Models and Orthogonal Variance Component Models.

PREPRINTS 1988

COPIES OF PREPRINTS ARE OBTAINABLE FROM THE AUTHOR OR FROM THE INSTITUTE OF MATHEMATICAL STATISTICS, UNIVERSITETSPARKEN 5, 2100 COPENHAGEN Ø, DENMARK, TELEPHONE + 45 1 35 31 33.

No. 1 Jacobsen, Martin: Discrete Exponential Families: Deciding when the Maximum Likelihood Estimator Exists and is Unique.

No. 2 Johansen, Søren and Juselius, Katarina: Hypothesis Testing for Cointegration Vectors - with an Application to the Demand for Money in Denmark and Finland.