S. A. Andersson M. D. Perlman

Group-Invariant Analogues of Hadamard's Inequality

Preprint
November
1987
9

Institute of Mathematical Statistics
University of Copenhagen

ISSN 0902-8846

S.A. Andersson and M.D. Perlman

GROUP-INVARIANT ANALOGUES OF HADAMARD'S INEQUALITY

Preprint 1987 No. 9

INSTITUTE OF MATHEMATICAL STATISTICS
UNIVERSITY OF COPENHAGEN

November 1987

Group-Invariant Analogues of Hadamard's Inequality*

Steen A. Andersson

Institute of Mathematical Statistics
University of Copenhagen
DK-2100 Copenhagen Ø
Denmark

Michael D. Perlman

Department of Statistics, GN-22
University of Washington
Seattle, Washington 98195
U.S.A.

Revised September, 1987

ABSTRACT

A wide class of inequalities for the determinant and other real-valued functions of an $n \times n$ complex Hermitian (or real symmetric) matrix $H \equiv (h_{jk})$ may be obtained by generalizing Marshall and Olkin's [9] proof of Hadamard's inequality

(1) $$\det H \leq \prod_{j=1}^{n} h_{jj}$$

for positive definite (pd) H. We shall see that each subgroup G of the group $\mathbf{U}_n$ of $n \times n$ unitary matrices not only determines an analogue of (1) for $\det H$, but also provides inequalities for a large family of unitarily invariant functions of H (not necessarily pd).

* This research was supported in part by National Science Foundation Grant MCS 83-01807 at the University of Washington.

1. INTRODUCTION

To prove the classical Hadamard inequality (1), Marshall and Olkin observed that

(2) $$\frac{1}{2^n}\sum_{\alpha=1}^{2^n} D_\alpha H D_\alpha = \mathrm{Diag}\,(h_{11}, \ldots, h_{nn})$$

where D_α ranges over all $n \times n$ diagonal matrices $\mathrm{Diag}\,(\pm 1, \ldots, \pm 1)$, then invoked the concavity and unitary invariance of $\log\det H$ for pd H. They also remarked that further inequalities may be obtained from (2) by replacing $\log\det H$ by any concave (or convex) unitarily invariant function $\phi(H) \equiv f(\boldsymbol{\lambda}(H))$.[1]

Of particular interest are the convex functions (see [8], p. 478)

$$\phi_{(m)}(H) \equiv \sum_{j=1}^{m} \lambda_j(H), \qquad 1 \le m \le n-1,$$

defined for all Hermitian H. (Note that $\phi_{(n)}(H) \equiv \mathrm{tr}\, H$ is linear.) When applied to (2), the convexity of $\phi_{(m)}$ yields *Schur's inequalities*

(3) $$\sum_{j=1}^{m} \lambda_j(H) \ge \sum_{j=1}^{m} h_{(jj)}, \qquad 1 \le m \le n-1,$$

where $h_{(11)} \ge \cdots \ge h_{(nn)}$ denote the ordered values of $h_{11}, \ldots, h_{nn}$, while

(3′) $$\sum_{j=1}^{n} \lambda_j(H) = \sum_{j=1}^{n} h_{(jj)}$$

by the linearity of $\phi_{(n)}$. The relations (3), (3′) together are equivalent to the vector relation

[1] A function of $\phi(H)$ is unitarily invariant if $\phi(UHU^*) = \phi(H)$ for all $U \in \mathbf{U}_n$. Such a function depends on H only through its *ordered* eigenvalues $\lambda_1(H) \ge \cdots \ge \lambda_n(H)$ (necessarily real), i.e., $\phi(H) = f(\boldsymbol{\lambda}(H))$ for some function f, where $\boldsymbol{\lambda}(H) = (\lambda_1(H), \ldots, \lambda_n(H))$. Note that H is pd iff $\lambda_n(H) > 0$. We shall be concerned mainly with functions ϕ whose domain is either $\mathbb{H}_n \equiv$ the set of all $n \times n$ complex Hermitian matrices or $\mathbb{H}_n^+ \equiv \{H \in \mathbb{H}_n \mid H \,\mathrm{pd}\}$.

$$\text{(4)} \qquad \boldsymbol{\lambda}(H) \succ \mathbf{h}(H) \equiv (h_{11}, \ldots, h_{nn}),$$

i.e., $\boldsymbol{\lambda}(H)$ *majorizes* $\mathbf{h}(H)$.[2] The relation (4) in turn implies that

$$\text{(5)} \qquad f(\boldsymbol{\lambda}(H)) \begin{Bmatrix} \geq \\ \leq \end{Bmatrix} f(\mathbf{h}(H)) \text{ if } f \text{ is } \begin{Bmatrix} \text{Schur-convex} \\ \text{Schur-concave} \end{Bmatrix},$$

where by definition a real-valued function f is *Schur-convex (Schur-concave)* on its domain if it preserves (reverses) the majorization preordering ([8], Chapter 3). The inequality (1) is the special case of (5) for the Schur-concave[3] function $f(x_1, \ldots, x_n) = \prod_1^n x_j$ defined for $x_j \geq 0$, $1 \leq j \leq n$.

From (5), one obtains upper or lower bounds for functions of $\boldsymbol{\lambda}(H)$ in terms of the diagonal elements of H. The main purpose of this paper is to demonstrate that many different bounds are available in terms of other simple linear functions of the elements of H. Each bound is determined by the projection of H onto a group-invariant subspace $\mathbb{H}_G \subset \mathbb{H}_n$ (see Section 2). Which of these bounds are most informative (i.e., sharpest) will depend upon which group-invariant symmetry properties are most nearly satisfied by H (see Remark 2 to follow). Several examples are presented in Section 3.

---

[2] If $\mathbf{x} \equiv (x_1, \ldots, x_n)$ and $\mathbf{y} \equiv (y_1, \ldots, y_n)$ are real vectors, then $\mathbf{x}$ *weakly majorizes* $\mathbf{y}$ (written $\mathbf{x} \succ_w \mathbf{y}$) if $x_{(1)} + \cdots + x_{(k)} \geq y_{(1)} + \cdots + y_{(k)}$ for $1 \leq k \leq n$, where $x_{(1)} \geq \cdots \geq x_{(n)}$ and $y_{(1)} \geq \cdots \geq y_{(n)}$ denote the components of $\mathbf{x}$ and $\mathbf{y}$ in decreasing order. We say $\mathbf{x}$ *majorizes* $\mathbf{y}$ (written $\mathbf{x} \succ \mathbf{y}$) if $x \succ_w y$ and $x_1 + \cdots + x_n = y_1 + \cdots + y_n$. See Marshall and Olkin [8] for a comprehensive account of the majorization preordering and its applications.

[3] A function $f(x_1, \ldots, x_n)$ is Schur-convex (Schur-concave) if it is convex (concave) and a permutation-invariant, but not conversely. If f is Schur-convex (Schur-concave), so is $\psi(f)$ for any increasing function ψ. Thus, $\prod x_j \equiv \exp\left(\sum \log x_j\right)$ is Schur-concave.

2. THE GROUP-INVARIANT FORMULATION

Our generalization of Marshall and Olkin's argument is based upon consideration of the matrix

(6)
$$\frac{1}{\#(G)} \sum_{g \in G} gHg^* \equiv H_G ,$$

where G denotes a finite *subgroup* of $\mathbf{U}_n$, $\#(G)$ is the order of G, and $H \in \mathbb{H}_n$. Clearly H_G is a *linear* function of H (also see Footnote 4), $H_G \in \mathbb{H}_n$, $\mathrm{tr}\, H_G = \mathrm{tr}\, H$, and H_G is pd whenever H is pd. Because H_G is a convex combination (in fact, the barycenter) of the G-orbit $\{gHg^* \mid g \in G\}$ of H, applying $\phi_{(m)}(1 \le m \le n)$ to (6) immediately yields the following generalization of (4):

(7)
$$\boldsymbol{\lambda}(H) \succ \boldsymbol{\lambda}(H_G) ,$$

which in turn extends (5):

(8)
$$f(\boldsymbol{\lambda}(H)) \begin{Bmatrix} \ge \\ \le \end{Bmatrix} f(\boldsymbol{\lambda}(H_G)) \text{ if } \boldsymbol{\lambda} \textit{ is } \begin{Bmatrix} \text{Schur-convex} \\ \text{Schur-concave} \end{Bmatrix} ;$$

in particular, by setting $f(x_1, \ldots, x_n) = \prod_1^n x_j$ we obtain

(9)
$$\det H \le \det H_G \qquad (H \text{ pd}).$$

Although (7) and (8) are valid for an arbitrary finite *subset* $G \subset \mathbf{U}_n$, their main interest occurs when G is a *subgroup* of $\mathbf{U}_n$, for in this case H_G possesses symmetry properties which (i) facilitate its calculation, and (ii) make $f(\boldsymbol{\lambda}(H_G))$ an interesting bound. If we let

$$\mathbb{H}_G \equiv \{A \in \mathbb{H}_n \mid gAg^* = A \;\forall g \in G\}$$

denote the (real) linear subspace of all G-invariant Hermitian matrices, then the group property of G implies that

$$g_1 H_G g_1^* = \frac{1}{\#(G)} \sum_{g \in G} (g_1 g) H (g_1 g)^* = H_G$$

for each $g_1 \in G$, i.e., H_G is G-invariant, so

$$H_G \in \mathbb{H}_G .^4$$

For example, if $G = \{D_\alpha \mid 1 \le \alpha \le 2^n\}$ (see (2)) then $\mathbb{H}_G$ is the set of all $n \times n$ diagonal matrices with real elements and $H_G = \mathrm{Diag}(h_{11}, \ldots, h_{nn})$, so (2), (4), (5), and (1) are special cases of (6), (7), (8), and (9), respectively. By computing H_G for other subgroups $G \subseteq \mathbf{U}_n$, other interesting bounds for $\boldsymbol{\lambda}(H)$ and $f(\boldsymbol{\lambda}(H))$ can be obtained.

We remark that when G is a subgroup, equality holds in (7) iff H is G-invariant, i.e.,

$$H \in \mathbb{H}_G \iff \boldsymbol{\lambda}(H) = \boldsymbol{\lambda}(H_G) .$$

Trivially, $H \in \mathbb{H}_G \implies H = H_G \implies \boldsymbol{\lambda}(H) = \boldsymbol{\lambda}(H_G)$. To see the converse, assume that $\boldsymbol{\lambda}(H) = \boldsymbol{\lambda}(H_G)$, so that

$$\mathrm{tr}\, H^2 = \mathrm{tr}\,(H_G)^2 \le \frac{1}{\#(G)} \sum_{g \in G} \mathrm{tr}\,(gHg^*)^2$$

by the convexity of $\mathrm{tr}\, A^2$ for $A \in \mathbb{H}_n$. However, $\mathrm{tr}\,(gHg^*)^2 = \mathrm{tr}\, H^2$, hence equality holds, so by the *strict* convexity of $\mathrm{tr}\, A^2$ it follows that the matrices gHg^*, $g \in G$, are all identical. Since

---

[4] In fact, H_G is the orthogonal projection of H onto $\mathbb{H}_G$ with respect to the inner product $\langle H_1, H_2 \rangle = \mathrm{tr}\, H_1 H_2$ on $\mathbb{H}_n$. To verify this fact, simply note that $H_G \in \mathbb{H}_G$ and $\langle H - H_G, \mathbb{H}_G \rangle = 0$. When H is pd, such projections occur as maximum likelihood estimators of covariance matrices in normal statistical models determined by group invariance (e.g., [2]).

the $n \times n$ identity matrix $I_n \in G$, $gHg^* = H$ for every $g \in G$, hence $H \in \mathbb{H}_G$.

Before presenting examples in the next section, we point out that (7) and (8) may be extended by considering nested subgroups of $\mathbf{U}_n$. It is easy to verify that if $G' \subset G$ then $\mathbb{H}_G \subseteq \mathbb{H}_{G'}$ (in fact, $\mathbb{H}_G = (\mathbb{H}_{G'})_G$) and

$$(6') \qquad (H_{G'})_G = H_G ,$$

so that

$$(7') \qquad \boldsymbol{\lambda}(H_{G'}) \succ \boldsymbol{\lambda}(H_G) ,$$

$$(8') \qquad f(\boldsymbol{\lambda}(H_{G'})) \begin{Bmatrix} \geq \\ \leq \end{Bmatrix} f(\boldsymbol{\lambda}(H_G)) \text{ if } f \text{ is } \begin{Bmatrix} \text{Schur-convex} \\ \text{Schur-concave} \end{Bmatrix};$$

in particular,

$$(9') \qquad \det H_{G'} \leq \det H_G \qquad (H \text{ pd}).$$

The inequalities (7) and (8) are the special cases of (7′) and (8′) obtained by setting $G' = G_0 \equiv \{I_n\}$.

Remark 1. For positive definite $H \in \mathbb{H}_n$, certain reversals of (7) and (8) may be obtained by replacing H by H^{-1} in (6)-(8). Thus we find that

$$\boldsymbol{\lambda}^{-1}(H) \equiv (\lambda_n^{-1}(H), \ldots, \lambda_1^{-1}(H)) = \boldsymbol{\lambda}(H^{-1}) \succ \boldsymbol{\lambda}((H^{-1})_G) ,$$

hence

$$f(\boldsymbol{\lambda}^{-1}(H)) \begin{Bmatrix} \geq \\ \leq \end{Bmatrix} f(\boldsymbol{\lambda}((H^{-1})_G)) \text{ if } f \text{ is } \begin{Bmatrix} \text{Schur-convex} \\ \text{Schur-concave} \end{Bmatrix}.$$

In particular,

$$\det H^{-1} \le \det (H^{-1})_G \qquad (H \text{ pd}),$$

hence

$$\det H \ge \frac{1}{\det (H^{-1})_G} \qquad (H \text{ pd}).$$

Such reversed inequalities are not of much practical interest, however, since the calculation of H^{-1} and $(H^{-1})_G$ usually is no simpler than that of $\boldsymbol{\lambda}(H)$ or $\det H$. □

3. EXAMPLES AND REMARKS

The general inequalities (7) and (8) are elementary and straightforward—their interest depends upon whether, for specific G, the projection (≡ orbit barycenter) H_G and its eigenvalues $\boldsymbol{\lambda}(H_G)$ are readily obtainable and provide interesting bounds for $f(\boldsymbol{\lambda}(H))$. Because H_G is a linear function of H and is G-invariant, usually it is easy to determine H_G and $\boldsymbol{\lambda}(H_G)$. This is illustrated by the following six examples.

Example 1 (Block-diagonal matrices). Choose positive integers q, $n_1, \ldots, n_q$ such that $\sum n_j = n$ and let G_1 be the subgroup of $\mathbf{U}_n$ consisting of all block-diagonal matrices of the form $\mathrm{Diag}(\pm I_{n_1}, \ldots, \pm I_{n_q})$ $(\#(G_1) = 2^q)$. It is easy to verify that $\mathbb{H}_{G_1}$ consists of all block-diagonal matrices $A = \mathrm{Diag}(A_1, \ldots, A_q)$ with each A_j an $n_j \times n_j$ Hermitian matrix, and that

$$H_{G_1} = \text{Diag}\,(H_{11}, \ldots, H_{qq})$$

where $H_{jj} : n_j \times n_j$ is the j^{th} diagonal block of H. Thus (7) implies that

$$\boldsymbol{\lambda}(H) \succ \tilde{\boldsymbol{\lambda}}(H_{G_1}) \equiv (\boldsymbol{\lambda}(H_{11}), \ldots, \boldsymbol{\lambda}(H_{qq})), \tag{10}$$

where $\tilde{\boldsymbol{\lambda}}(A)$ denotes the vector of eigenvalues of A in arbitrary order.[5] (See [8], p. 225, for an alternate proof.) Applying (9) yields Fischer's inequality

$$\det H \le \prod_{j=1}^{q} \det H_{jj} \qquad (H \text{ pd}).$$

If we take $q = 2$, $n_1 = n - 1$, $n_2 = 1$, then (10) implies (compare to (3))

$$\begin{aligned} \sum_{j=1}^{m} \lambda_j(H) &\ge \sum_{j=1}^{m} \lambda_j(H_{11}), && 1 \le m \le n-1, \\ \sum_{j=1}^{m} \lambda_{n-j+1}(H) &\le \sum_{j=1}^{m} \lambda_{n-j+1}(H_{11}), && 1 \le m \le m-1, \end{aligned} \tag{11}$$

where H_{11} is an $(n-1)\times(n-1)$ principal submatrix of H. (See Remark 3 for further discussion of (11)). □

Example 2 (Completely symmetric matrices). Take G_2 to be the subgroup of $\mathbf{U}_n$ consisting of all $n \times n$ permutation matrices ($\#(G_2) = n!$). The reader may verify that $\mathbb{H}_{G_2}$ consists of all $n \times n$ matrices $A \equiv (a_{jk})$ such that $a_{jj} = a$ (real), $1 \le j \le n$, and $a_{jk} = b$ (real), $1 \le j \ne k \le n$,[6] so H_{G_2} must be of the form

---

[5] Under the majorization preordering, $\boldsymbol{\lambda}(A)$ and $\tilde{\boldsymbol{\lambda}}(A)$ are equivalent, i.e., $\boldsymbol{\lambda} \succ \tilde{\boldsymbol{\lambda}}$ and $\tilde{\boldsymbol{\lambda}} \succ \boldsymbol{\lambda}$.

[6] Recall that if A is Hermitian, each a_{jj} is real and $a_{jk} = \bar{a}_{kj}$ for $j \ne k$. For $A \in \mathbb{H}_{G_2}$, additionally $a_{jj} = a_{kk}$ and $a_{jk} = a_{kj}$ for $j \ne k$: this follows from the requirement that $A = P_{jk} A P_{jk}^*$, where P_{jk}

(12)
$$H_{G_2} = \begin{bmatrix} a & b & . & . & . & b \\ b & a & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & a & b \\ b & . & . & . & b & a \end{bmatrix}$$

To calculate a and b in terms of the elements of H, first note from (12) that

$$\operatorname{tr} H_{G_2} = na$$

$$e^* H_{G_2} e = na + n(n-1)b ,$$

where $e = (1, \ldots, 1)^*$. From (6) and the facts that $g^* g = I_n$ $(g \in \mathbf{U}_n)$ and $e^* g = e^* (g \in G_2)$, however,

$$\operatorname{tr} H_{G_2} = \operatorname{tr} H = \sum_{j=1}^{n} h_{jj}$$

$$e^* H_{G_2} e = e^* H e = \sum_{j=1}^{n} h_{jj} + \sum\sum_{j \neq k} h_{jk} ,$$

so that

$$a = \frac{1}{n} \sum_{j=1}^{n} h_{jj} \equiv h_0 ,$$

$$b = \operatorname{Re} \left[\frac{2}{n(n-1)} \sum\sum_{j<k} h_{jk} \right] \equiv h_+ ,$$

---

is the permutation matrix that transposes the j^{th} and k^{th} coordinates. Thus all diagonal elements of A are equal (and real), while all off-diagonal elements are real. To show that all off-diagonal elements of A must be equal, replace P_{jk} by other appropriate permutation matrices.

respectively the averages of the diagonal and off-diagonal elements of H.

Since the eigenvalues of the matrix (12) are $a-b$ (with multiplicity $n-1$) and $a+(n-1)b$, (7) yields

(13) $$\boldsymbol{\lambda}(H) \succ \boldsymbol{\lambda}(H_{G_2})$$

$$\equiv ((h_0-h_+)+n(h_+ \vee 0),\ h_0-h_+, \ldots, h_0-h_+,\ (h_0-h_+)+n(h_+ \wedge 0))\,.$$

By applying (8) for suitable choices of f we obtain the following inequalities (some of which may be new):

(14) $$\det H \le nh(h_0-h_+)^{n-1} \qquad (H \text{ pd})$$

(15) $$\sum_{j=1}^{m} \lambda_j(H) \ge m(h_0-h_+)+n(h_+ \vee 0)$$

(16) $$\sum_{j=1}^{m} \lambda_{n-j+1}(H) \le m(h_0-h_+)+n(h_+ \wedge 0)$$

(17) $$\prod_{j=1}^{m} \lambda_{n-j+1}(H) \le (h_0-h_+)^m + n(h_+ \wedge 0)(h_0-h_+)^{m-1} \qquad (H \text{ pd}),$$

where $1 \le m \le n-1$ and $nh = h_0+(n-1)h_+$ (h is the average of *all* the elements of H).

If $H = R \equiv (r_{jk})$ is a *correlation matrix* (i.e., $r_{jj}=1,\ 1 \le j \le n$), then these inequalities assume simpler forms:

(18) $$\det R \le [1+(n-1)r_+]\,(1-r_+)^{n-1} \qquad (R \text{ pd})$$

(19) $$\sum_{j=1}^{m} \lambda_j(R) \ge m(1-r_+)+n(r_+ \vee 0)$$

(20) $$\sum_{j=1}^{m} \lambda_{n-j+1}(R) \le m(1-r_+) + n(r_+ \wedge 0)$$

(21) $$\prod_{j=1}^{m} \lambda_{n-j+1}(R) \le (1-r_+)^m + n(r_+ \wedge 0)(1-r_+)^{m-1} \qquad (R \text{ pd}),$$

where r_+ denotes the average of the off-diagonal elements of R. (Inequality (18) is attributed to L. J. Gleser in [1] (p. 328).) □

Remark 2. If G and G' are not comparable (i.e., $G \not\subset G'$, $G' \not\subset G$), then in general $\mathbb{H}_G$ and $\mathbb{H}_{G'}$ are not comparable and the lower bounds $\boldsymbol{\lambda}(H_G)$ and $\boldsymbol{\lambda}(H_{G'})$ are not comparable with respect to the partial ordering of majorization. For example, if we set $q = n$, $n_1 = \cdots = n_n = 1$ in Example 1, then $\boldsymbol{\lambda}(H_{G_1}) \equiv (h_{(11)}, \ldots, h_{(nn)})$ and $\boldsymbol{\lambda}(H_{G_2})$ are not comparable in general. If restrictions are imposed on H, however, then comparability may result. Thus, if H is restricted to be a correlation matrix R, then $\boldsymbol{\lambda}(R_{G_2}) \succ \boldsymbol{\lambda}(R_{G_1}) \equiv (1, \ldots, 1)$ and the inequalities (18) and (19) are sharper than (1) and (3). On the other hand, if the restriction $h_+ = 0$ is imposed on H, then $\boldsymbol{\lambda}(H_{G_1}) \succ \boldsymbol{\lambda}(H_{G_2}) = (h_0, \ldots, h_0)$ and (1) and (3) are sharper than (18) and (19). This suggests that one should take into account any symmetry properties or other restrictions *approximately* satisfied by a particular H in order to determine those subgroups G that will provide sharp bounds. □

Remark 3. The primary appeal of this group-invariant approach is that it is *suggestive,* providing (possibly different) bounds for $\phi(H) \equiv f(\boldsymbol{\lambda}(H))$ for every subgroup $G \subseteq \mathbf{U}_n$ (but see

Remark 5). Once these bounds are determined, however, alternate derivations and/or sharper bounds may become apparent. For example, from the extremal representations

$$\left\{ \begin{aligned} \lambda_1(H) &= \sup_{\|u\|=1} u^* H u \\ \lambda_n(H) &= \inf_{\|u\|=1} u^* H u \end{aligned} \right. \tag{22}$$

for a Hermitian matrix H one obtains

$$\lambda_1(H) \geq h_0 + (n-1)h_+ \geq \lambda_n(H) \tag{23}$$

$$\lambda_1(H) \geq \tfrac{1}{2}(h_{jj} + h_{kk}) - \mathrm{Re}(h_{jk}) \geq \lambda_n(H) \qquad (j \neq k) \tag{24}$$

by taking $u = e$ and $u = (e_j - e_k)/\sqrt{2}$, respectively, where e_j is the unit vector $(0, \ldots, 0, 1, \ldots, 0)$ with the 1 in the j^{th} position. Averaging (24) over all $j \neq k$ yields

$$\lambda_1(H) \geq h_0 - h_+ \geq \lambda_n(H)\,; \tag{25}$$

combining (23) and (25) gives (15) and (16) for the case $m = 1$.

Sharper bounds are also immediate. From (24), for example,

$$\begin{aligned} \lambda_1(H) &\geq \max_{j \neq k} [\tfrac{1}{2}(h_{jj} + h_{kk}) - \mathrm{Re}\,(h_{jk})] \\ &\geq \min_{j \neq k} [\tfrac{1}{2}(h_{jj} + h_{kk}) - \mathrm{Re}\,(h_{jk})] \\ &\geq \lambda_n(H)\,, \end{aligned} \tag{26}$$

which is sharper than (25), but which requires more information about H. When $H = R$ (a correlation matrix), (26) becomes

$$\lambda_1(R) \geq 1 - \min_{j \neq k} \mathrm{Re}\,(r_{jk}) \geq 1 - \max_{j \neq k} \mathrm{Re}\,(r_{jk}) \geq \lambda_n(R)\,. \tag{27}$$

Combining (26) or (27) with (23) provides bounds sharper than (15), (16) or (19), (20) for the case $m = 1$.

As another example, the interlacing inequalities ([8], p.227).

$$\lambda_1(H) \geq \lambda_1(H_{11}) \geq \lambda_2(H) \geq \cdots \geq \lambda_{n-1}(H) \geq \lambda_{n-1}(H_{11}) \geq \lambda_n(H)$$

clearly are stronger than (11). These inequalities are obtained from the Courant-Fischer min-max representation of $\lambda_j(H)$, which shows that $\lambda_j(H)$ is neither convex nor concave in H if $2 \leq j \leq n-1$. □

Example 3 (Hermitian circulants). Take G_3 to be the subgroup of all cyclic permutation matrices ($\#(G_3) = n$), i.e., $G = \{I, P, \ldots, P^{n-1}\}$ where $I = I_n$ and

$$P = \begin{bmatrix} 0 & 1 & 0 & . & . & 0 & 0 \\ 0 & 0 & 1 & . & . & 0 & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & 0 & 1 \\ 1 & 0 & 0 & . & . & 0 & 0 \end{bmatrix}. \tag{28}$$

To determine $\mathbb{H}_{G_3}$ it is convenient to use the fact that $\mathbb{H}_{G_3} = \mathbb{H}_{\{P\}}$, which holds since G_3 is generated by $\{P\}$. It is readily verified that $\mathbb{H}_{G_3}$ consists of all $n \times n$ matrices $A \equiv (a_{jk})$, $0 \leq j$, $k \leq n-1$, such that

$$a_{j,\{j+\alpha\}} = b_\alpha , \qquad 0 \leq \alpha \leq n-1 , \ \ 0 \leq j \leq n-1 ,$$

where $\{j + \alpha\} = (j + \alpha)(\text{mod } n)$, b_0 is real, and $b_\alpha = \bar{b}_{n-\alpha}$ for $1 \leq \alpha \leq n-1$. (Note that this last condition implies that $b_{n/2}$ is real if n is even.) Thus, H_{G_3} takes the form

$$H_{G_3} = h_0 I + h_1 P + h_2 P^2 + \cdots + \bar{h}_2 P^{n-2} + \bar{h}_1 P^{n-1}$$

$$= \begin{bmatrix} h_0 & h_1 & h_2 & . & . & . & \bar{h}_2 & \bar{h}_1 \\ \bar{h}_1 & h_0 & h_1 & . & . & . & . & \bar{h}_2 \\ \bar{h}_2 & \bar{h}_1 & h_0 & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & h_0 & h_1 & h_2 \\ h_2 & . & . & . & . & \bar{h}_1 & h_0 & h_1 \\ h_1 & h_2 & . & . & . & \bar{h}_2 & \bar{h}_1 & h_0 \end{bmatrix}$$

where, from (6),

(30) $$h_\alpha \equiv \frac{1}{n} \sum_{j=1}^{n} h_{j,\{j+\alpha\}} \ , \qquad 0 \le \alpha \le n-1 \ .$$

(Note that h_0 is real, while $h_\alpha = \bar{h}_{n-\alpha}$ for $1 \le \alpha \le n-1$.) The *unordered* eigenvalues of H_{G_3} are given by

$$\tilde{\lambda}_j(H_{G_3}) = \sum_{\alpha=0}^{n-1} h_\alpha \omega^{j\alpha}$$

$$= \begin{cases} h_0 + 2\sum_{\alpha=1}^{[\frac{n}{2}]} \left[(\mathrm{Re}\, h_\alpha) \cos\frac{j\alpha\pi}{n} - (\mathrm{Im}\, h_\alpha) \sin\frac{j\alpha\pi}{n} \right], & n \text{ odd} \\ h_0 + 2\sum_{\alpha=1}^{\frac{n}{2}-1} \left[(\mathrm{Re}\, h_\alpha) \cos\frac{j\alpha\pi}{n} - (\mathrm{Im}\, h_\alpha) \sin\frac{j\alpha\pi}{n} \right] + (-1)^j h_{n/2}, & n \text{ even,} \end{cases}$$

$1 \le j \le n$, where $\omega = e^{2\pi i/n}$ ([10], pp. 65-66). Thus, (7)-(9) yield

$$\boldsymbol{\lambda}(H) \succ \boldsymbol{\lambda}(H_{G_3})$$

(31) $$\det H \le \prod_{j=1}^{n} \left[\sum_{\alpha=0}^{n-1} h_\alpha \omega^{j\alpha} \right] \qquad (H \text{ pd})$$

$$\lambda_1(H) \ge \max_{1\le j\le n} \left[\sum_{\alpha=0}^{n-1} h_\alpha \omega^{j\alpha} \right] \ge \min_{1\le j\le n} \left[\sum_{\alpha=0}^{n-1} h_\alpha \omega^{j\alpha} \right] \ge \lambda_n(H)$$

and other inequalities. (The *ordered* eigenvalues $\lambda_j(H_{G_3})$ depend on the relative values of h_0, $\mathrm{Re}\, h_\alpha$, and $\mathrm{Im}\, h_\alpha$, so cannot be expressed concisely.) □

Example 4 (Symmetric circulants). For $n \ge 3$ let $G_4 = \{I, P, \ldots, P^{n-1}, Q, QP, \ldots, QP^{n-1}\}$[7] ($\#(G_4) = 2n$), where P is given by (28) and

$$Q = \begin{bmatrix} 0 & . & . & . & 0 & 1 \\ . & . & . & . & 1 & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 1 & . & . & . & . \\ 1 & 0 & . & . & . & 0 \end{bmatrix}.$$

Since $G_3 \subset G_4 \Longrightarrow \mathbb{H}_{G_4} = (\mathbb{H}_{G_3})_{G_4}$, the space $\mathbb{H}_{G_4}$ consists of all matrices of the form (29) with each h_α real. By (6′), therefore,

[7] For $0 \le \alpha \le n-1$, P^α and QP^α are, respectively, the rotations and reflections that leave a regular n-gon invariant in n-space, i.e., $G_4 \cong$ the dihedral group.

$$H_{G_4} = \begin{bmatrix} h_0 & \hat{h}_1 & \hat{h}_2 & . & . & \hat{h}_2 & \hat{h}_1 \\ \hat{h}_1 & h_0 & \hat{h}_1 & . & . & . & \hat{h}_2 \\ \hat{h}_2 & \hat{h}_1 & h_0 & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & h_0 & \hat{h}_1 & \hat{h}_2 \\ \hat{h}_2 & . & . & . & \hat{h}_1 & h_0 & \hat{h}_1 \\ \hat{h}_1 & \hat{h}_2 & . & . & \hat{h}_2 & \hat{h}_1 & h_0 \end{bmatrix}$$

where

$$\hat{h}_\alpha = \tfrac{1}{2}(h_\alpha + h_{n-\alpha}) = \mathrm{Re}\, h_\alpha \ , \qquad 1 \le \alpha \le n-1 \, .$$

(Note that $H_{G_4} = H_{G_3}$ if H is real symmetric, but not in general.) The *unordered* eigenvalues of H_{G_4} are given by

$$\tilde{\lambda}_j(H_{G_4}) = \sum_{\alpha=0}^{n-1} \hat{h}_\alpha \omega^{j\alpha}$$

$$= \begin{cases} h_0 + 2 \displaystyle\sum_{\alpha=1}^{[\frac{n}{2}]} (\mathrm{Re}\, h_\alpha) \cos \frac{j\alpha\pi}{n} \, , & n \text{ odd} \\ h_0 + 2 \displaystyle\sum_{\alpha=1}^{\frac{n}{2}-1} (\mathrm{Re}\, h_\alpha) \cos \frac{j\alpha\pi}{n} + (-1)^j h_{n/2} \, , & n \text{ even,} \end{cases}$$

$1 \le j \le n$. (Note that $\tilde{\lambda}_j = \tilde{\lambda}_{n-j}$, $1 \le j \le n-1$, so that $\left[\frac{n-1}{2}\right]$ of these eigenvalues occur with multiplicity two.) Thus we obtain

$$\boldsymbol{\lambda}(H) \succ \boldsymbol{\lambda}(H_{G_4})$$

(32) $$\det H \le \prod_{j=1}^{n} \left[\sum_{\alpha=0}^{n-1} \hat{h}_\alpha \omega^{j\alpha} \right] \qquad (H \text{ pd})$$

$$\lambda_1(H) \ge \max_{1 \le j \le n} \left[\sum_{\alpha=0}^{n-1} \hat{h}_\alpha \omega^{j\alpha} \right] \ge \min_{1 \le j \le n} \left[\sum_{\alpha=0}^{n-1} \hat{h}_\alpha \omega^{j\alpha} \right] \ge \lambda_n(H),$$

plus other inequalities from (8). □

Remark 4. To illustrate (7′) and (8′), $\{I_n\} \equiv G_0 \subset G_3 \subset G_4 \subset G_2 \subset \mathbf{U}_n$ implies that

$$\boldsymbol{\lambda}(H) \equiv \boldsymbol{\lambda}(H_{G_0}) \succ \boldsymbol{\lambda}(H_{G_3}) \succ \boldsymbol{\lambda}(H_{G_4}) \succ \boldsymbol{\lambda}(H_{G_2}) \succ \boldsymbol{\lambda}(H_{\mathbf{U}_n}) \equiv \boldsymbol{\lambda} \left[(\frac{1}{n} \operatorname{tr} H) I_n \right]$$

$$\det H \le \det H_{G_3} \le \det H_{G_4} \le \det H_{G_2} \le (\frac{1}{n} \operatorname{tr} H)^n \qquad (H \text{ pd})$$

$$\lambda_1(H) \ge \lambda_1(H_{G_3}) \ge \lambda_1(H_{G_4}) \ge \lambda_1(H_{G_2}) \ge \frac{1}{n} \operatorname{tr} H,$$

and so on. Thus the inequalities in (14)-(17), (32), and (31) are increasingly sharp. (The group $\mathbf{U}_n$ is not finite, but see (60).) □

Before presenting our two final examples, we recall several facts about the eigenvalues of a structured Hermitian matrix. First, if A and B are $p \times p$ Hermitian matrices, then $A+B$, $A-B$, and

(33) $$\begin{bmatrix} A & B \\ B & A \end{bmatrix} : 2p \times 2p$$

are also Hermitian and their eigenvalues (necessarily real) are related by

(34)
$$\tilde{\boldsymbol{\lambda}} \begin{bmatrix} A & B \\ B & A \end{bmatrix} = (\boldsymbol{\lambda}(A+B),\ \boldsymbol{\lambda}(A-B)),$$ [8]

which implies that

(35)
$$\det \begin{bmatrix} A & B \\ B & A \end{bmatrix} = \det(A+B)\det(A-B).$$

Next, if A is Hermitian ($A^* = A$) and B is anti-Hermitian ($B^* = -B$), then $A+iB$, $A-iB$, and

(36)
$$\begin{bmatrix} A & -B \\ B & A \end{bmatrix} :\ 2p \times 2p$$

are Hermitian and their eigenvalues (necessarily real) satisfy

(37)
$$\tilde{\boldsymbol{\lambda}} \begin{bmatrix} A & -B \\ B & A \end{bmatrix} = (\boldsymbol{\lambda}(A+iB),\ \boldsymbol{\lambda}(A-iB)),$$ [9]

so that

---

[8] This follows from the relation

$$\begin{bmatrix} A & B \\ B & A \end{bmatrix} = U \begin{bmatrix} A+B & 0 \\ 0 & A-B \end{bmatrix} U^*,$$

where

$$U \equiv \frac{1}{\sqrt{2}} \begin{bmatrix} I_p & I_p \\ I_p & -I_p \end{bmatrix}$$

is a unitary matrix.

[9] This follows from the fact that

$$\begin{bmatrix} A & -B \\ B & A \end{bmatrix} = U \begin{bmatrix} A+iB & 0 \\ 0 & A-iB \end{bmatrix} U^*,$$

where

$$U \equiv \frac{1}{\sqrt{2}} \begin{bmatrix} iI_p & -I_p \\ I_p & -iI_p \end{bmatrix}$$

is unitary.

(38) $$\det \begin{bmatrix} A & -B \\ B & A \end{bmatrix} = \det (A + iB) \det (A - iB) .$$

If A and B are restricted to be real matrices in (36)-(38) so that A is symmetric ($A' = A$) and B is anti-symmetric ($B' = -B$), then $A + iB$ and $A - iB$ $(= (A + iB)')$ have the same eigenvalues, so (37) and (38) become

(39) $$\tilde{\boldsymbol{\lambda}} \begin{bmatrix} A & -B \\ B & A \end{bmatrix} = (\boldsymbol{\lambda}(A + iB), \ \boldsymbol{\lambda}(A + iB))$$
$$\det \begin{bmatrix} A & -B \\ B & A \end{bmatrix} = [\det (A + iB)]^2 .$$

Finally, if $E : p \times p$ is an arbitrary complex matrix, then

(40) $$\begin{bmatrix} 0 & E \\ E^* & 0 \end{bmatrix} : \ 2p \times 2p$$

is also Hermitian, and

(41) $$\tilde{\boldsymbol{\lambda}} \begin{bmatrix} 0 & E \\ E^* & 0 \end{bmatrix} = (\boldsymbol{\sigma}(E), \ -\boldsymbol{\sigma}(E)),$$ [10]

where $\boldsymbol{\sigma}(E) = (\sigma_1(E), \ldots, \sigma_p(E))$ and $\sigma_1(E) \geq \cdots \geq \sigma_p(E) \geq 0$ are the ordered *singular values* of E, i.e.,

---

[10] From the *singular value decomposition* $E = \Gamma D_\sigma \Psi^*$ ([8], p.498), where Γ and Ψ are $p \times p$ unitary matrices and $D_\sigma = \mathrm{Diag}\,(\sigma_1(E), \ldots, \sigma_p(E))$, one has

$$\begin{bmatrix} 0 & E^* \\ E & 0 \end{bmatrix} = \begin{bmatrix} \Gamma & 0 \\ 0 & \Psi \end{bmatrix} \begin{bmatrix} 0 & D_\sigma \\ D_\sigma & 0 \end{bmatrix} \begin{bmatrix} \Gamma^* & 0 \\ 0 & \Psi^* \end{bmatrix} .$$

Since

$$\tilde{\boldsymbol{\lambda}} \begin{bmatrix} 0 & D_\sigma \\ D_\sigma & 0 \end{bmatrix} = (\boldsymbol{\sigma}(E), -\boldsymbol{\sigma}(E)),$$

(41) follows.

(42) $$\sigma_j(E) = [\lambda_j(EE^*)]^{1/2} .$$

For later use, we record here the simple equivalence

(43) $$(\mathbf{x}, -\mathbf{x}) \succ (\mathbf{y}, -\mathbf{y}) \Longleftrightarrow |\mathbf{x}| \succ_w |\mathbf{y}| ,$$

where $\mathbf{x} = (x_1, \ldots, x_n)$, $|\mathbf{x}| = (|x_1|, \ldots, |x_n|)$, and $\succ_w$ denotes weak majorization (cf. Footnote 2).

Example 5 (2×2 block symmetry). Set $n = 2p$ and partition the complex $2p \times 2p$ Hermitian matrix H as

(44) $$H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}, \quad H_{jk} : p \times p .$$

(Note that $H_{11}^* = H_{11}$, $H_{22}^* = H_{22}$, $H_{12}^* = H_{21}$.) Let G_5 be the subgroup of $\mathbf{U}_{2p}$ consisting of the two matrices

$$\begin{bmatrix} I_p & 0 \\ 0 & I_p \end{bmatrix}, \quad \begin{bmatrix} 0 & I_p \\ I_p & 0 \end{bmatrix}.$$

It is readily seen that $\mathbb{H}_{G_5}$ consists of all complex matrices of the form (33) with A and B Hermitian, and that

(45) $$H_{G_5} = \begin{bmatrix} \tfrac{1}{2}(H_{11}+H_{22}) & \tfrac{1}{2}(H_{12}+H_{21}) \\ \tfrac{1}{2}(H_{12}+H_{21}) & \tfrac{1}{2}(H_{11}+H_{22}) \end{bmatrix} \equiv \begin{bmatrix} H_0 & H_1 \\ H_1 & H_0 \end{bmatrix}$$

with H_0 and H_1 Hermitian. From (7), (8), (34), and (35) we obtain the comparisons

$$\boldsymbol{\lambda}\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \succ (\boldsymbol{\lambda}(H_0+H_1)\,,\ \boldsymbol{\lambda}(H_0-H_1))$$

$$\det\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \le \det(H_0+H_1)\det(H_0-H_1) \qquad (H\ \text{pd})$$

$$\text{(46)} \qquad \lambda_1\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \ge \max(\lambda_1(H_0+H_1)\,,\ \lambda_1(H_0-H_1))$$

$$\ge \min(\lambda_p(H_0+H_1)\,,\ \lambda_p(H_0-H_1))$$

$$\ge \lambda_{2p}\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}, \qquad \text{etc.}$$

(Note that H pd $\Longrightarrow$ H_{G_5} pd $\Longrightarrow$ $H_0 \pm H_1$ pd by (34), or by direct calculation.) □

Example 6 (Complex structure). It is interesting to compare the results in Example 5 to those obtained by considering the subgroup G_6 consisting of the four matrices

$$\begin{bmatrix} I_p & 0 \\ 0 & I_p \end{bmatrix}, \quad \begin{bmatrix} 0 & -I_p \\ I_p & 0 \end{bmatrix}, \quad \begin{bmatrix} -I_p & 0 \\ 0 & -I_p \end{bmatrix}, \quad \begin{bmatrix} 0 & I_p \\ -I_p & 0 \end{bmatrix}.$$

Since $gHg^* = (-g)H(-g)^*$, only the first two matrices need be considered in determining $\mathbb{H}_{G_6}$ and H_{G_6}. Thus, it is readily seen that $\mathbb{H}_{G_6}$ consists of all complex matrices of the form (36) with $A^* = A$ and $B^* = -B$, and that

$$\text{(47)} \qquad H_{G_6} = \begin{bmatrix} \tfrac{1}{2}(H_{11}+H_{22}) & \tfrac{1}{2}(H_{12}-H_{21}) \\ \tfrac{1}{2}(H_{21}-H_{12}) & \tfrac{1}{2}(H_{11}+H_{22}) \end{bmatrix} \equiv \begin{bmatrix} H_0 & -\tilde{H}_1 \\ \tilde{H}_1 & H_0 \end{bmatrix},$$

a matrix of *complex structure* (see [2], p. 133). Note that $H_0^* = H_0$ and $\tilde{H}_1^* = -\tilde{H}_1$.) From (7)-(9), (37), and (38) we therefore obtain

$$
\begin{aligned}
& \boldsymbol{\lambda} \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \succ (\boldsymbol{\lambda}(H_0 + i\tilde{H}_1),\ \boldsymbol{\lambda}(H_0 - i\tilde{H}_1)) \\
& \det \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \leq \det(H_0 + i\tilde{H}_1)\det(H_0 - i\tilde{H}_1) \qquad (H \text{ pd}) \\
(48) \qquad & \lambda_1 \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \geq \max(\lambda_1(H_0 + i\tilde{H}_1),\ \lambda_1(H_0 - i\tilde{H}_1)) \\
& \qquad \geq \min(\lambda_p(H_0 + i\tilde{H}_1),\ \lambda_p(H_0 - i\tilde{H}_1)) \\
& \qquad \geq \lambda_{2p} \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}, \qquad \text{etc.}
\end{aligned}
$$

(Note that H pd $\Longrightarrow$ $H_0 \pm i\tilde{H}_1$ pd.) If H is restricted to be real symmetric, then these inequalities take the form (see (39))

$$
\begin{aligned}
& \boldsymbol{\lambda} \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \succ (\boldsymbol{\lambda}(H_0 + i\tilde{H}_1),\ \boldsymbol{\lambda}(H_0 + i\tilde{H}_1)) \\
(49) \quad & \det \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \leq [\det(H_0 + i\tilde{H}_1)]^2 \qquad (H \text{ pd}) \\
& \lambda_1 \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \geq \lambda_1(H_0 + i\tilde{H}_1) \geq \lambda_p(H_0 + i\tilde{H}_1) \geq \lambda_{2p} \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}. \qquad \square
\end{aligned}
$$

Remark 2 (continued). From (10) (with $q = 2$, $n_1 = n_2 = p$), (46), and (48) we see that

$$
(50) \qquad \boldsymbol{\lambda} \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \succ \left\{ \begin{matrix} (\boldsymbol{\lambda}(H_{11}),\ \boldsymbol{\lambda}(H_{22})) \\ (\boldsymbol{\lambda}(H_0 + H_1),\ \boldsymbol{\lambda}(H_0 - H_1)) \\ (\boldsymbol{\lambda}(H_0 + i\tilde{H}_1)),\ \boldsymbol{\lambda}(H_0 - i\tilde{H}_1)) \end{matrix} \right\} \succ (\boldsymbol{\lambda}(H_0),\ \boldsymbol{\lambda}(H_0)).
$$

The three intermediate bounds are non-comparable, reflecting the fact that the three groups G_1 (with $q = 2$, $n_1 = n_2 = p$), G_5, and G_6 are not comparable with respect to inclusion. $\square$

By applying the majorization relations in (46) and (48) to the matrix in (40) we can obtain the following comparisons, due to Fan and Hoffman [4], among the singular values of an arbitrary complex matrix E and its Hermitian and anti-Hermitian parts, $\frac{1}{2}(E+E^*)$ and $\frac{1}{2}(E-E^*)$:

$$(51) \qquad \begin{aligned} \sigma(E) &\succ_w \sigma(\tfrac{1}{2}(E+E^*)) \\ \sigma(E) &\succ_w \sigma(\tfrac{1}{2}(E-E^*)) . \end{aligned}$$

To see this, simply write

$$H \equiv \begin{bmatrix} 0 & E \\ E^* & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2}(E+E^*) \\ \frac{1}{2}(E+E^*) & 0 \end{bmatrix} + \begin{bmatrix} 0 & \frac{1}{2}(E-E^*) \\ \frac{1}{2}(E^*-E) & 0 \end{bmatrix}$$

$$= H_{G_5} + H_{G_6},$$

then apply (7) and (41) to obtain

$$(52) \qquad \begin{aligned} (\sigma(E), -\sigma(E)) &\succ (\sigma(\tfrac{1}{2}(E+E^*)), -\sigma(\tfrac{1}{2}(E+E^*))) \\ (\sigma(E), -\sigma(E)) &\succ (\sigma(\tfrac{1}{2}(E-E^*)), -\sigma(\tfrac{1}{2}(E-E^*))) . \end{aligned}$$

By (43), (52) and (51) are equivalent.[11]

---

[11] Marshall and Olkin derive (51) from the apparently stronger relations $\sigma_i(\frac{1}{2}(E \pm E^*)) \leq \sigma_i(E)$ ([8], p. 240, eqn. (5)). These relations are false, however. Fan and Hoffman ([4], p. 115) give the counterexample

$$E = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} .$$

Marshall and Olkin's alternate derivation of (51) ([8], p. 244) is essentially equivalent to that presented here.

A modification of Marshall and Olkin's identity (2) (see (54) below) yields the following comparison between the diagonal elements e_{jj} of E and its singular values, essentially due to von Neumann (cf. [8], pp. 228-9):

$$\sigma(E) \succ_w |\mathbf{e}| \equiv (|e_{11}|, \ldots, |e_{pp}|) \tag{53}$$

(compare to (4)). To see this, let $G \subset \mathbf{U}_{2p}$ denote the subgroup consisting of all matrices of the form

$$\begin{bmatrix} D_\alpha & 0 \\ 0 & D_\alpha \end{bmatrix}, \quad 1 \le \alpha \le 2^p ,$$

where D_α is defined as in (2) with n replaced by p. Then

$$\begin{bmatrix} 0 & E \\ E^* & 0 \end{bmatrix}_G = \begin{bmatrix} 0 & D_{\mathbf{e}} \\ D_{\bar{\mathbf{e}}} & 0 \end{bmatrix}, \tag{54}$$

where $\mathbf{e} = (e_{11}, \ldots, e_{pp})$, $\bar{\mathbf{e}} = (\bar{e}_{11}, \ldots, \bar{e}_{pp})$. Hence by (41),

$$(\sigma(E), -\sigma(E)) \succ (|\mathbf{e}|, -|\mathbf{e}|),$$

which is equivalent to (53).

As a final application, bounds for the eigenvalues of a complex Hermitian $p \times p$ matrix $C \equiv \mathrm{Re}\, C + i\, \mathrm{Im}\, C$ can be obtained in terms of its real part $\mathrm{Re}\, C$. From (39) and (46),

$$(\boldsymbol{\lambda}(C), \boldsymbol{\lambda}(C)) = \tilde{\boldsymbol{\lambda}} \begin{bmatrix} \mathrm{Re}\, C & -\mathrm{Im}\, C \\ \mathrm{Im}\, C & \mathrm{Re}\, C \end{bmatrix} \succ (\boldsymbol{\lambda}(\mathrm{Re}\, C), \boldsymbol{\lambda}(\mathrm{Re}\, C)),$$

hence

$$
\begin{aligned}
&\boldsymbol{\lambda}(C) \succ \boldsymbol{\lambda}(\mathrm{Re}\, C) && \\
(55) \qquad & \det C \le \det(\mathrm{Re}\, C) && (C \text{ pd}) \\
&\lambda_1(C) \ge \lambda_1(\mathrm{Re}\, C) \ge \lambda_p(\mathrm{Re}\, C) \ge \lambda_p(C), && \text{etc.}
\end{aligned}
$$

In turn, these yield alternate bounds (compare (51) and (53)) for the (squared) singular values of an arbitrary complex $p \times p$ matrix E:

$$
\begin{aligned}
&\sigma^2(E) \succ \begin{cases} \boldsymbol{\lambda}(\mathbb{R}\mathbb{R}' + \mathbb{I}\mathbb{I}') \\ \boldsymbol{\lambda}(\mathbb{R}'\mathbb{R} + \mathbb{I}'\mathbb{I}) \end{cases} \\
&|\det E|^2 \le \min\{\det(\mathbb{R}\mathbb{R}' + \mathbb{I}\mathbb{I}'),\ \det(\mathbb{R}'\mathbb{R} + \mathbb{I}'\mathbb{I})\}, \\
(56) \qquad &\sigma_1^2(E) \ge \max\{\lambda_1(\mathbb{R}\mathbb{R}' + \mathbb{I}\mathbb{I}'),\ \lambda_1(\mathbb{R}'\mathbb{R} + \mathbb{I}'\mathbb{I})\} \\
&\qquad \ge \min\{\lambda_p(\mathbb{R}\mathbb{R}' + \mathbb{I}'\mathbb{I}),\ \lambda_p(\mathbb{R}'\mathbb{R} + \mathbb{I}'\mathbb{I})\} \\
&\qquad \ge \sigma_p^2(E), \qquad \text{etc.,}
\end{aligned}
$$

where $\mathbb{R} = \mathrm{Re}\, E$, $\mathbb{I} = \mathrm{Im}\, E$. (Apply (55) with $C = EE^*$ and $C = E^*E$.)

The reader is invited to develop further examples by computing $\mathbb{H}_G$ and H_G for other subgroups $G \subseteq \mathbf{U}_n$. If G_6 in Example 6 is replaced by a subgroup G' isomorphic to the quaternion group ($\#(G') = 8$), for example, then H_G has the *quaternion structure* displayed in Andersson ([2], p. 133).

Once $\mathbb{H}_G$ and the projection $H \to H_G$ are known, additional patterned examples and eigenvalue bounds for $H \in \mathbb{H}_n$ may be generated via the relations

$$(57) \qquad \mathbb{H}_{G_U} = U^* \mathbb{H}_G U$$

$$(58) \qquad H_{G_U} = U^*(UHU^*)_G U$$

$$(59) \qquad \boldsymbol{\lambda}(H) \succ \boldsymbol{\lambda}(H_{G_U})) = \boldsymbol{\lambda}((UHU^*)_G),$$

where U is a fixed unitary matrix in $\mathbf{U}_n$ and G_U is the conjugate subgroup in $\mathbf{U}_n$ given by

$$G_U \equiv U^* G U = \{U^* g U \mid g \in G\} \subseteq \mathbf{U}_n .$$

Lastly, (7)-(9) remain valid for any closed (therefore compact) subgroup G of $\mathbf{U}_n$, provided that H_G is defined by

(60) $$H_G = \int_G g H g^* \, d\nu_G(g) ,$$

where ν_G is the (unique) Haar ($\equiv G$-invariant) probability measure on G. If G is finite, (60) reduces to (6).

Remark 5. It should be noted that distinct subgroups G, G' do not necessarily give rise to distinct classes $\mathbb{H}_G$, $\mathbb{H}_{G'}$. For example, let $G = G_2 =$ the group of all $n \times n$ permutation matrices (see Example 2) and $G' =$ the group of all even $n \times n$ permutation matrices. Then $G \cong \mathbf{S}_n$ (the symmetric group) and $G' \cong \mathbf{A}_n$ (the alternating group) so G' is a subgroup of G of index 2, but $\mathbb{H}_G = \mathbb{H}_{G'}$ and $H_G = H_{G'}$ provided $n \geq 4$. □

4. EXTENSIONS

The bounds $\boldsymbol{\lambda}(H) \succ \boldsymbol{\lambda}(H_G)$ and $\det H \leq \det H_G$ (H pd) involve H_G whose elements are *linear* in the elements of H (cf (6)-(9)). Many strengthenings of Hadamard's inequality are available in terms of matrices whose elements are *nonlinear* functions of those of H, e.g. [6]. As one example, consider the k^{th} *multiplicative compound matrix* $H^{(k)}$ ([8], p. 502). This is the matrix of dimension $\binom{n}{k} \times \binom{n}{k}$ whose elements consist of all $k \times k$ minors of H $(1 \leq k \leq n,$

$H^{(1)} \equiv H, H^{(n)} = \det H$). By applying (1) to $H^{(k)}$ one obtains

$$\det H^{(k)} \leq \prod_{1 \leq i_1 < \cdots < i_k \leq n} H(i_1, \ldots, i_k) \equiv C_k \qquad (H \text{ pd}) \tag{61}$$

where $H(i_1, \ldots, i_k)$ denotes the principal minor formed from the rows and columns of H with indices $i_1, \ldots, i_k$. Since

$$\det H^{(k)} = (\det H)^{\binom{n-1}{k-1}},$$

(61) yields the bound

$$\det H \leq C_k^{1/\binom{n-1}{k-1}} \equiv B_k \qquad (H \text{ pd}).$$

By applying this inequality with H replaced by all $k \times k$ principal submatrices of H and n, k replaced by $k, k-1$ we obtain

$$C_k \leq (C_{k-1})^{(n-k+1)/(k-1)},$$

so we obtain a sequence of successively sharper bounds for $\det H$ known as *Szasz's inequalities:*

$$\det H = B_n \leq B_{n-1} \leq \cdots \leq B_2 \leq B_1 \equiv \prod_{j=1}^{n} h_{jj} \qquad (H \text{ pd}).$$

It should be remarked that we have spoken somewhat loosely by referring to G as a "finite subgroup" of the unitary group $\mathbf{U}_n$. Actually, G should be thought of as a *matrix representation* of a finite group; different representations of the same group may have different representation spaces and will in general provide different bounds for the eigenvalues and determinants of matrices in those spaces. Thus, Example 2 of Section 3 deals with the usual representation of the symmetric group $\mathbf{S}_n$ in terms of $n \times n$ matrices. There are many other representations of $\mathbf{S}_n$, (cf. [5]), however — for example, representations in terms of $\binom{n}{k} \times \binom{n}{k}$ matrices — and these representations will yield new bounds for the eigenvalues and determinants of Hermitian matrices of these dimensions.

By sacrificing symmetry, (6)-(9) may be generalized as follows. For $H \in \mathbb{H}_n$ and P a probability measure on $\mathbf{U}_n$, consider

$$\int_{\mathbf{U}_n} gHg^* dP(g) \equiv H_P . \tag{62}$$

Clearly, $H_P \in \mathbb{H}_n$ and H_P is pd whenever H is pd. The argument that led to (7) now shows that

$$\boldsymbol{\lambda}(H) \succ \boldsymbol{\lambda}(H_P) ;^{12} \tag{63}$$

in particular,

$$\det H \le \det H_P \qquad (H \text{ pd}).$$

If $P = \nu_G$ (see (60)) where G is a finite or compact subgroup of $\mathbf{U}_n$, then $H_P = H_G$ and (63) reduces to (7). Here $H_P \in \mathbb{H}_G$, a strong symmetry property. If support $(P) \subseteq G$ but $P \neq \nu_G$, then H_P still lies in the convex hull of the G-orbit $\{gHg^* \mid g \in G\}$ but is not necessarily its barycenter, hence H_P need not belong to $\mathbb{H}_G$. In this case $\boldsymbol{\lambda}(H_P)$ may not be easy to determine and/or may not provide interesting bounds for $\boldsymbol{\lambda}(H)$.

---

[12] In fact, the converse is also true: for $H_1, H_2 \in \mathbb{H}_n$, $\boldsymbol{\lambda}(H_1) \succ \boldsymbol{\lambda}(H_2)$ if and only if there exists a probability measure P on $\mathbf{U}_n$ such that $H_2 = (H_1)_P$ — cf. Ando [3], Theorem 7.1, or Karlin and Rinott [7], Theorem 8.1.

Similarly, the relations (6′)-(9′) may be generalized. Suppose that P' is also a probability measure on $\mathbf{U}_n$ and let $P * P'$ denote the convolution of P and P'. (That is, $P * P'$ is the probability distribution of the random element $gg' \in \mathbf{U}_n$, where $g \sim P$, $g' \sim P'$, with g and g' independent.) It is readily verified that

(64) $$(H_{P'})_P = H_{P * P'},$$

hence (63) yields

(65) $$\boldsymbol{\lambda}(H_{P'}) \succ \boldsymbol{\lambda}(H_{P * P'});$$

in particular,

$$\det H_{P'} \leq \det H_{P * P'} \qquad (H \text{ pd}).$$

If $P = \nu_G$ and support $(P') \subseteq G$ with G as above, then $P * P' = P$ and $H_P = H_G$, so (64) and (65) become

(66) $$(H_{P'})_G = H_G,$$

(67) $$\boldsymbol{\lambda}(H_{P'}) \succ \boldsymbol{\lambda}(H_G).$$

If $G' \subseteq G$ are two finite or compact subgroups of $\mathbf{U}_n$ and $P = \nu_G$, $P' = \nu_{G'}$, then also $H_{P'} = H_{G'}$ and (66) and (67) reduce to (6′) and (7′), respectively.

Finally, we discuss the possibility of extending (6) (or (60)) and (7)-(9) to matrix groups G that are not necessarily contained in $\mathbf{U}_n$. If G is a closed subgroup of $GL(n)$ (the group of all nonsingular $n \times n$ complex matrices), it is well-known that a Haar ($\equiv G$-invariant) probability measure ν_G exists on G if and only if G is compact. A related fact is that $\mathbb{H}_G^+ \neq \emptyset$ if and only if G is compact: if G is compact, then $\int gg^* d\nu_G(g) \in \mathbb{H}_G^+$, while if $H_0 \in \mathbb{H}_G^+$, then $G = B^{-1}G_0B$, where $B \in GL(n)$ satisfies $H_0^{-1} = B^*B$ and $G_0 \equiv BGB^{-1}$ is a closed subgroup of $\mathbf{U}_n$, hence G_0 and therefore G are compact. For these reasons, in order that H_G may be defined as in (60), it must be assumed that G is finite or compact.

In this case, for any $H \in \mathbb{H}_n$ it is immediate that

$$H_G \in \mathbb{H}_G = B^{-1}\mathbb{H}_{G_0}B^{-1*} ;$$

in fact,

$$H_G = B^{-1}(BHB^*)_{G_0}B^{-1*} .$$

Since $G_0 \subseteq \mathbf{U}_n$, it follows from (7) or (63) that

$$\boldsymbol{\lambda}(BHB^*) \succ \boldsymbol{\lambda}((BHB^*)_{G_0}) = \boldsymbol{\lambda}(BH_GB^*) , \tag{68}$$

but this does not provide a majorization bound for $\boldsymbol{\lambda}(H)$ if $B \notin \mathbf{U}_n$ (i.e., if $I_n \notin \mathbb{H}_G^+$). However, we are able to deduce from (68) that

$$\det BHB^* \leq \det (BHB^*)_{G_0} = \det BH_GB^*$$

for H pd, whence

$$\det H \leq \det H_G = \frac{\det (BHB^*)_{G_0}}{|\det B|^2} \qquad (H \text{ pd}). \tag{69}$$

Thus, every subgroup $G_0 \subseteq \mathbf{U}_n$ and $B \in GL(n)$ determines a Hadamard-type upper bound for $\det H$.

We conclude by presenting several examples of (69) with $G_0 = \{D_\alpha \mid 1 \le \alpha \le 2^n\}$ (see (2)) and various choices of $B \equiv (b_{ij})_{i,j=1,\ldots,n}$. If $b_{ij} = 1$ for $i \ge j$ and $b_{ij} = 0$ otherwise, (69) becomes

$$\det H \le \prod_{k=1}^{n} \left[\sum_{i=1}^{k} \sum_{j=1}^{k} h_{ij} \right] \qquad (H \text{ pd}). \tag{70}$$

If $b_{ii} = 1$ for $1 \le i \le n$, $b_{i+1,i} = \varepsilon_i$ for $1 \le i \le n-1$, and $b_{ij} = 0$ otherwise, where each $\varepsilon_i = \pm 1$, (69) becomes

$$\det H \le h_{11} \prod_{i=1}^{n-1} [h_{ii} + h_{i+1,i+1} + 2\varepsilon_i \mathrm{Re}\,(h_{i,i+1})] \qquad (H \text{ pd}).$$

By taking the minimum over all choices of $\varepsilon_1, \ldots, \varepsilon_{n-1}$ we obtain

$$\det H \le h_{11} \prod_{i=1}^{n-1} [h_{ii} + h_{i+1,i+1} - 2\,|\mathrm{Re}\,(h_{i,i+1})|\,] \qquad (H \text{ pd}). \tag{71}$$

This bound for $\det H$ will be sharper than the classical Hadamard bound $\prod h_{ii}$ if (but not only if) $|\mathrm{Re}\,(h_{i,i+1})| \ge \frac{1}{2} h_{ii}$ for $1 \le i \le n-1$. A further strengthening of (71) may be obtained by minimizing the right-hand side over all permutations $\pi \equiv (\pi(1), \ldots, \pi(n))$ of $(1, \ldots, n)$:

$$\det H \le \min_{\pi} h_{\pi(1)\pi(1)} \prod_{i=1}^{n-1} [h_{\pi(i)\pi(i)} + h_{\pi(i+1)\pi(i+1)} - 2\,|\mathrm{Re}\,(h_{\pi(i)\pi(i+1)})|\,] \qquad (H \text{ pd}). \tag{72}$$

Acknowledgement

We wish to thank the referee, whose careful reading and suggestions substantially improved the paper.

References

[1] M. A. Aitken, W. C. Nelson, and K. H. Reinfurt. Tests for correlation matrices. *Biometrika* **55** (1968), 327-334.

[2] S. A. Andersson. Invariant normal models. *Ann. Statist.* **3** (1975), 132-154.

[3] T. Ando. *Majorization, Doubly Stochastic Matrices, and Comparison of Eigenvalues.* Division of Applied Mathematics, Research Institute of Applied Electricity, Hokkaido University, Sapporo, Japan, 1982.

[4] K. Fan and A. Hoffman. Some metric inequalities in the space of matrices. *Proc. Amer. Math. Soc.* **6** (1955), 111-116.

[5] G. D. James. *The Representation Theory of the Symmetric Groups.* Lecture Notes in Mathematics, v. 682, Springer-Verlag, Berlin, 1978.

[6] C. R. Johnson and W. W. Barrett. Spanning-tree extensions of the Hadamard-Fischer inequalities. *Linear Algebra Applic.* **66** (1985), 177-193.

[7] S. Karlin and Y. Rinott. Entropy inequalities for classes of probability distributions. II. The multivariate case. *Advances Appl. Probab.* **13** (1981), 325-351.

[8] A. W. Marshall and I. Olkin. *Inequalities: Theory of Majorization and Its Applications.* Academic Press, New York, 1979.

[9] A. W. Marshall and I. Olkin. A convexity proof of Hadamard's inequality. *Amer. Math. Monthly* **89** (1982), 687-688.

[10] H. Minc and M. Marcus. *A Survey of Matrix Theory and Matrix Inequalities.* Prindle, Weber, and Schmidt, Boston, 1964.

PREPRINTS 1986

COPIES OF PREPRINTS ARE OBTAINABLE FROM THE AUTHOR OR FROM THE INSTITUTE OF MATHEMATICAL STATISTICS, UNIVERSITETSPARKEN 5, 2100 COPENHAGEN Ø, DENMARK, TELEPHONE + 45 1 35 31 33.

No. 1 Jespersen, N.C.B.: On the Structure of Simple Transformation Models.

No. 2 Dalgaard, Peter and Johansen, Søren: The Asymptotic Properties of the Cornish-Bowden-Eisenthal Median Estimator.

No. 3 Jespersen, N.C.B.: Dichotomizing a Continuous Covariate in the Cox Regression Model.

No. 4 Asmussen, Søren: On Ruin Problems and Queues of Markov-Modulated M/G/1 Type.

No. 5 Asmussen, Søren: The Heavy Traffic Limit of a Class of Markovian Queueing Models.

No. 6 Jacobsen, Martin: Right Censoring and the Kaplan - Meier and Nelson - Aalen Estimators.

PREPRINTS 1987

COPIES OF PREPRINTS ARE OBTAINABLE FROM THE AUTHOR OR FROM THE INSTITUTE OF MATHEMATICAL STATISTICS, UNIVERSITETSPARKEN 5, 2100 COPENHAGEN Ø, DENMARK, TELEPHONE + 45 1 35 31 33.

No. 1 Jensen, Søren Tolver and Johansen, Søren: Estimation of Proportional Covariances.

No. 2 Rootzén, Holger: Extremes, Loads, and Strengths.

No. 3 Bertelsen, Aksel: On the Problem of Testing Reality of a Complex Multivariate Normal Distribution.

No. 4 Gill, Richard D. and Johansen, Søren: Product-Integrals and Counting Processes.

No. 5 Leadbetter, M.R. and Rootzén, Holger: Extremal Theory for Stochastic Processes.

No. 6 Tjur, Tue: Block Designs and Electrical Networks.

No. 7 Johansen, Søren: Statistical Analysis of Cointegration Vectors.

No. 8 Bertelsen, Aksel: On the Problem of Testing Reality of a Complex Multivariate Normal Distribution, II.

No. 9 Andersson, S.A. and Perlman, M.D.: Group-Invariant Analogues of Hadamard's Inequality.