Aksel Bertelsen

On the Problem of Testing Reality of a Complex Multivariate Normal Distribution, II.

Preprint
November
1987

8

Institute of Mathematical Statistics
University of Copenhagen

ISSN 0902-8846

Aksel Bertelsen

ON THE PROBLEM OF TESTING REALITY OF A COMPLEX MULTIVARIATE NORMAL DISTRIBUTION , II .

Preprint 1987 No.8

INSTITUTE OF MATHEMATICAL STATISTICS
UNIVERSITY OF COPENHAGEN

November 1987

Summary and introduction.

In Bertelsen (1987) zonal polynomials were used to obtain a series expansion for the density of the non-central distibution of the maximal invariant corresponding to testing that a $2m \times 2m$ covariance matrix with complex structure has real structure. However an explicit expression for the coefficients in the series expansion was only given for m less than five. In this paper we find an explicit expression for these coefficients.

AMS 1980 subject classifications. Primary 62 H 10

secondary 62 H 15

Key words and phrases. Complex multivariate normal distributions, testing for reality, zonal polynomials, non-central-distributions.

1. We first list the results and notation from Bertelsen (1987) that we need here. Let $A(m,\mathbb{R})$ be the set of $m \times m$ skew-symmetric matrices, and let $P_n(k)$ be the set of partitions of k in at most n parts, where $n = [m/2]$. To each $\bar{k} \in P_n(k)$ we defined a polynomial $C_{2\bar{k}}$ in the different elements of $B \in A(m,\mathbb{R})$. These polynomials were shown to have the following properties (1) - (4).

Let $O(m)$ be the group of orthogonal $m \times m$ matrices, and let α be the Haar-measure with unit mass on $O(m)$. Then

(1)
$$\int_{O(m)} \operatorname{tr}(B_1HB_2H')^{2k} d\alpha(H) = \sum_{\bar{k}\in P_n(k)} c(\bar{k}) C_{2\bar{k}}(B_1) C_{2\bar{k}}(B_2) \quad \text{and}$$
$$\int_{O(m)} \operatorname{tr}(B_1HB_2H')^{2k+1} d\alpha(H) = 0$$

where $B_1, B_2 \in A(m,\mathbb{R})$.

Let $T_+(m)$ be the group of upper-triangular $m \times m$ matrices with positive diagonal elements, and assume that T is distributed on $T_+(m)$ such that $T'T$ has a Wishart (I,m) distribution. It then follows from Bertelsen (1987) that

(2)
$$E_T C_{2\bar{k}}(TBT') = d(\bar{k}) \cdot C_{2\bar{k}}(B)$$

where $d(\bar{k}) = (m/2)_{\bar{k}2}$, and $\bar{k}2 = (k_1,k_1,k_2,k_2,\ldots,k_n,k_n)$ when $\bar{k} = (k_1,\ldots,k_n)$.

The symbol $(a)_{\bar{k}}$ is defined as $\prod_{i=1}^{n} (a - (i-1)/2)_{k_i}$ where $(x)_{k_i} = x(x+1)\cdots(x+k_i-1)$.

Further we have that

$$(3) \qquad C_{2\bar{k}}(HBH') = C_{2\bar{k}}(B) \qquad \text{for all} \quad H \in O(m).$$

Finally it was shown that $C_{2\bar{k}}$ could be expressed by the Schur-functions, $S_{\bar{k}}$, (the complex zonal polynomials) by

$$(4) \qquad C_{2\bar{k}}(\Lambda) = S_{\bar{k}}(\lambda_1^2,\ldots,\lambda_n^2)$$

where Λ has the form

$$(5) \qquad \Lambda = \mathrm{diag}\left[\begin{bmatrix} 0 & \lambda_1 \\ -\lambda_1 & 0 \end{bmatrix},\cdots,\begin{bmatrix} 0 & \lambda_n \\ -\lambda_n & 0 \end{bmatrix}\right] \quad \text{if } m = 2n$$

$$\Lambda = \mathrm{diag}\left[\begin{bmatrix} 0 & \lambda_1 \\ -\lambda_1 & 0 \end{bmatrix},\cdots,\begin{bmatrix} 0 & \lambda_n \\ -\lambda_n & 0 \end{bmatrix},0\right] \quad \text{if } m = 2n+1$$

The problem now is to find an explicit expression for the coefficients $c(\bar{k})$ given by (1).

2 Consider a $m \times m$ matrix $X = (x_{i,j})$ of independent standard normal variables $x_{i,j}$, and define the generating function g by

(6) $$g(z,\Gamma,\Lambda) = E_X \exp(z\cdot \mathrm{tr}(\Gamma X \Lambda X'))$$

for z sufficiently small, and Γ,Λ has the form (5).

Let $$\det_{ij} X = \det \begin{bmatrix} x_{2i-1,2j-1} & x_{2i-1,2j} \\ x_{2i,2j-1} & x_{2i,2j} \end{bmatrix}$$.

for $i=1,\ldots,n; j=1,\ldots,n$.

It is then seen that $$\mathrm{tr}(\Gamma X \Lambda X') = 2 \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_i \gamma_j \det_{ij} X$$

and using this we get that

(7) $$g(z,\Gamma,\Lambda) = \prod_{i=1}^{n} \prod_{j=1}^{n} (1 - 4z^2\lambda_i^2\gamma_j^2)^{-1}$$

From Takemura (1984, page 93) we find that $g(z,\Gamma,\Lambda)$ is a sum of terms of the form

(8) $$2^{2k} z^{2k} \sum_{\bar{k}\in P_n(k)} S_{\bar{k}}(\lambda_1^2,\ldots,\lambda_n^2)\cdot S_{\bar{k}}(\gamma_1^2,\ldots,\gamma_n^2)$$

We get another expression for g by expanding (6) in a sum of terms of the form

(9) $$(k!)^{-1}\, z^k\, E_X(\mathrm{tr}((\Gamma X \Lambda X')^k))$$

Now $E_X = E_T E_H$, where H has the uniform distribution on O(m), and T is distributed on $T_+(m)$ such that T'T has the Wishart(I,m) distribution.Using (1) we get that

$$E_H(tr(B_1HB_2H')^{2k}) = \sum_{\bar{k}\in P_n(k)} c(\bar{k})\cdot C_{2\bar{k}}(B_1)C_{2\bar{k}}(B_2)$$

while
$$E_H(tr(B_1HB_2H')^{2k+1}) = 0$$

Then from (2) and (4) we get that $g(z,\Gamma,\Lambda)$ is a sum of terms of the form

$$(10) \qquad (2k)!^{-1}\, 2^{2k}\, z^{2k} \sum_{\bar{k}\in P_n(k)} c(\bar{k})(m/2)_{\bar{k}2}\cdot S_{\bar{k}}(\lambda_1^2,\ldots,\lambda_n^2)\cdot S_{\bar{k}}(\gamma_1^2,\ldots,\gamma_n^2)$$

Comparing (8) with (10) we finally find that

$$(11) \qquad \boxed{c(\bar{k}) = (2k)!\cdot(m/2)_{\bar{k}2}^{-1}}$$

REFERENCES

1. Bertelsen,A. (1987) "On the problem of testing reality of a complex multivariate normal distribution." Preprint nr.3,1987,University of Copenhagen.

2. Takemura, A. (1984) "Zonal Polynomials",Lecture notes - Monograph Series.

PREPRINTS 1986

COPIES OF PREPRINTS ARE OBTAINABLE FROM THE AUTHOR OR FROM THE INSTITUTE OF MATHEMATICAL STATISTICS, UNIVERSITETSPARKEN 5, 2100 COPENHAGEN Ø, DENMARK, TELEPHONE + 45 1 35 31 33.

No. 1 Jespersen, N.C.B.: On the Structure of Simple Transformation Models.

No. 2 Dalgaard, Peter and Johansen, Søren: The Asymptotic Properties of the Cornish-Bowden-Eisenthal Median Estimator.

No. 3 Jespersen, N.C.B.: Dichotomizing a Continuous Covariate in the Cox Regression Model.

No. 4 Asmussen, Søren: On Ruin Problems and Queues of Markov-Modulated M/G/1 Type.

No. 5 Asmussen, Søren: The Heavy Traffic Limit of a Class of Markovian Queueing Models.

No. 6 Jacobsen, Martin: Right Censoring and the Kaplan - Meier and Nelson - Aalen Estimators.

PREPRINTS 1987

COPIES OF PREPRINTS ARE OBTAINABLE FROM THE AUTHOR OR FROM THE INSTITUTE OF MATHEMATICAL STATISTICS, UNIVERSITETSPARKEN 5, 2100 COPENHAGEN Ø, DENMARK, TELEPHONE + 45 1 35 31 33.

No. 1 Jensen, Søren Tolver and Johansen, Søren: Estimation of Proportional Covariances.

No. 2 Rootzén, Holger: Extremes, Loads, and Strengths.

No. 3 Bertelsen, Aksel: On the Problem of Testing Reality of a Complex Multivariate Normal Distribution.

No. 4 Gill, Richard D. and Johansen, Søren: Product-Integrals and Counting Processes.

No. 5 Leadbetter, M.R. and Rootzén, Holger: Extremal Theory for Stochastic Processes.

No. 6 Tjur, Tue: Block Designs and Electrical Networks.

No. 7 Johansen, Søren: Statistical Analysis of Cointegration Vectors.

No. 8 Bertelsen, Aksel: On the Problem of Testing Reality of a Complex Multivariate Normal Distribution, II.