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Abstract

Certain polynomials of a skew-symmetric matrix are considered. These polynomials can be expressed in terms of the zonal polynomials on the Hermitian matrices, and they are used to obtain a series expansion for the density of the non-central distribution of the maximal invariant corresponding to the problem of testing for reality of the covariance matrix of a complex multivariate normal distribution.

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1. Introduction

In a paper by Andersson, Brøns and Jensen (1983) ten fundamental tests concerning the structure of covariance matrices in multivariate analysis are treated. Each of the ten problems is invariant under a group of linear transformations and the maximal invariant statistic is obtained in terms of eigenvalues of matrices with certain structures; the distribution of the maximal invariant under the null hypothesis was found in terms of a density with respect to a Lebesgue measure. A series expansion for the density of the distribution of the maximal invariant under the alternative hypothesis has been obtained for some of the ten problems by James (1964) and Constantine (1963) by use of zonal polynomials and hypergeometric functions; it concerns the tests for independence and the tests for identity of two sets of variates where the simultaneous covariance matrix has real or complex structure. In this paper one of the remaining non-central distribution problems are treated by using methods similar to those of James and Constantine. It concerns the test that a $2m \times 2m$ covariance matrix with complex structure has real structure; this test was considered for the first time by Khatri (1965). Andersson and Perlman (1984) study the noncentral distribution of the maximal invariant and we use their results as a starting point. The theory of group representations is used to define polynomials of a skew-symmetric matrix. The polynomials are shown to be eigenfunctions of a certain differential operator, and using this it is shown that they can be expressed in a simple way in terms of the complex zonal polynomials on the Hermitian matrices. Finally the polynomials are used to get a series expansion for the distribution of the maximal invariant.

<u>2. The statistical problem</u> Following Andersson et.al.(1983) let $x_1, \cdots, x_N$ $N \geq m$, be i.i.d..observations from a normal distribution on $\mathbb{R}^{2m}$ with mean vector 0 and unknown covariance matrix

$$\Sigma = \begin{bmatrix} \Phi & -\psi \\ \psi & \Phi \end{bmatrix} \tag{1}$$

where Σ is positive definite; Φ belongs to $H^+(m,\mathbb{R})$, the set of positive definite $m \times m$ matrices; ψ belongs to $A(m,\mathbb{R})$, the set of $m \times m$ skew-symmetric matrices. The set of positive definite matrices of the form (1) is called $H^+(m,\mathbb{C})$.

Let H_1 denote the hypothesis that $\Sigma \in H^+(m,\mathbb{C})$, i.e., that Σ has complex structure.

The group of $2m \times 2m$ non-singular matrices of the form

$$\begin{bmatrix} A & -B \\ B & A \end{bmatrix}$$

where A and B are $m \times m$ matrices is called $GL(m,\mathbb{C})$. This group acts on $H^+(m,\mathbb{C})$

$$GL(m,\mathbb{C}) \times H^+(m,\mathbb{C}) \to H^+(m,\mathbb{C}), \quad (M,S) \to MSM' \tag{2}$$

The emperical covariance matrix is $\bar{S} = N^{-1} \sum_{i=1}^{N} x_i \cdot x_i'$; let

$$\bar{S} = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} \\ \bar{S}_{21} & \bar{S}_{22} \end{bmatrix}$$

be a partion into $m \times m$ matrices, and put

$$S = \frac{1}{2}\begin{bmatrix} \bar{S}_{11} + \bar{S}_{22} & \bar{S}_{12} - \bar{S}_{21} \\ \bar{S}_{21} - \bar{S}_{12} & \bar{S}_{11} + \bar{S}_{22} \end{bmatrix}$$

In Andersson et. al. (1983) it is shown that $S \in H^+(m,\mathbb{C})$ a.s., that S is the maximum likelihood estimator of Σ (under H_1) and that the distribution of S has density

$$\left[\frac{\det S}{\det \Sigma}\right]^{\frac{N}{2}} \exp(-\tfrac{1}{2}\mathrm{tr}(\Sigma^{-1}S)) \qquad (3)$$

w.r.t. a unique measure ν, which is invariant under the action (2).

For $m > 1$ we shall consider the hypothesis H_0 that $\psi = 0$, i.e. that Σ has real structure. The statistical problem of testing H_0 is invariant under the restriction of the action (2) to the subgroup $G = \{\mathrm{diag}(L,L) \mid L \in GL(m,\mathbb{R})\}$.

Let π be the orbit projection

$$\pi : H^+(m,\mathbb{C}) \to H^+(m,\mathbb{C})/G \qquad (4)$$

A representation of π is found by using the following lemma.

<u>Lemma 1.</u> For every $R \in H^+(m,\mathbb{R})$ and $F \in A(m,\mathbb{R})$ there exists $L \in GL(m,\mathbb{R})$ such that $LRL' = I_m$ and $LFL' = \Lambda$, where

$$\Lambda = \mathrm{diag}\left[\begin{bmatrix} 0 & \lambda_1 \\ -\lambda_1 & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & \lambda_n \\ -\lambda_n & 0 \end{bmatrix}\right] \quad \text{if } m = 2n$$

(5)

$$\Lambda = \mathrm{diag}\left[\begin{bmatrix} 0 & \lambda_1 \\ -\lambda_1 & 0 \end{bmatrix}, \cdots, \begin{bmatrix} 0 & \lambda_n \\ -\lambda_n & 0 \end{bmatrix}, 0\right] \text{ if } m = 2n + 1$$

Here $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$

Proof. Bourbaki (1959 page 123)

The lemma implies that for every $S \in H^+(m,\mathbb{C})$ there exists a $M \in G$ such that

$$MSM' = \begin{bmatrix} I_m & -\Lambda \\ \Lambda & I_m \end{bmatrix} \tag{6}$$

where Λ has the form (5)

It is seem that $1 > \lambda_1$, when $S \in H^+(m,\mathbb{C})$ and that $\mp\lambda_1, \cdots, \mp\lambda_{[m/2]}$ are uniquely determined as the eigenvalues of $\begin{bmatrix} 0 & -F \\ F & 0 \end{bmatrix}$ w.r.t. $\begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix}$, each with multiplicity two (when m is odd, 0 is always an eigenvalue with multiplicity two), i.e. the solutions to the equation

$$\det\left(\begin{bmatrix} 0 & -F \\ F & 0 \end{bmatrix} - \lambda \begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix}\right) = 0 \tag{7}$$

As a consequence of the lemma we have that for every $F \in A(m,\mathbb{R})$ there exists a $H \in O(m)$, the set of real orthogonal $m \times m$ matrices, such that $HFH' = \Lambda$ where Λ has the form (5).

The eigenvalues of F are $\mp\lambda_1 i, \cdots, \mp\lambda_{[m/2]} i$, and 0 when m is odd.

Let t be the linear map

$$t : H^+(m,\mathbb{C}) \to H^+(m,\mathbb{R}) \otimes I_2, \quad S = \begin{bmatrix} R & -F \\ F & R \end{bmatrix} \to \begin{bmatrix} R & 0 \\ o & R \end{bmatrix}$$

It now follows (Andersson et. al. 1983)) that (4) can be represented by

$$\pi : H^+(m,\mathbb{C}) \to \Lambda_{[m/2]} \tag{8}$$

where $\pi(S) = (\lambda_1,\cdots,\lambda_{[m/2]})$ is the ordered family of non-negative eigenvalues of $S - t(S)$ w.r.t. $t(S)$.

In the same paper it is shown that the likelihood ratio statistic for testing H_0 is

$$Q(\lambda_1,\cdots,\lambda_{[m/2]}) = \prod_{i=1}^{[m/2]} (1 - \lambda_i^2)^N \tag{9}$$

and that under H_0 the distribution of $\pi(S)$ has density (9) w.r.t. a quotient measure ν/β on $\Lambda_{[m/2]}$, where β is a Haar measure on G (G is unimodular), see Andersson (1982). ν/β has density

$$\prod_{1\le i<j\le[m/2]} (\lambda_i^2 - \lambda_j^2)^2 \prod_{i=1}^{[m/2]} \lambda_i^{2\epsilon}(1 - \lambda_i^2)^{-m} \tag{10}$$

where $\epsilon = m - 2[m/2]$, w.r.t. a Lebesgue measure on $\Lambda_{[m/2]}$.

The problem now is to find the distribution of $\pi(S)$ under the alternative

$\Sigma \in H^+(m,\mathbb{C})$

The density (3) is called p. It follows from Andersson(1982) that the distribution of $\pi(S)$ has density q, where

$$q(\pi(S)) = \int_G p(MSM')d\beta(M) \tag{11}$$

w.r.t. the quotient measure υ/β. We get that

$$q(\pi(S)) = Q(\pi(S)) \int_G \left[\frac{\det(t(MSM'))}{\det \Sigma}\right]^{N/2} \exp(-1/2\mathrm{tr}(\Sigma^{-1}MSM'))d\beta(M) \tag{12}$$

Using that the integral in (12) as a function of Σ and S is invariant under the action of G on $H^+(m,\mathbb{C})$ we can write (12) as

$$q(\pi(S)) = Q(\pi(S)) \cdot K(\pi(S),\pi(\Sigma)) \tag{13}$$

This function K is called the correction factor, following Andersson (1982).

There exists $M_1 \in G$, $M_2 \in G$ such that $M_1SM_1' = \begin{bmatrix} I_m & -\Lambda \\ \Lambda & I_m \end{bmatrix}$ and

$M_2\Sigma M_2' = \begin{bmatrix} I_m & -\Gamma \\ \Gamma & I_m \end{bmatrix}$ where Λ and Γ has the form (5). Let $\pi\begin{bmatrix} I_m & -\Gamma \\ \Gamma & I_m \end{bmatrix} =$ $(\gamma_1,\cdots,\gamma_{[m/2]})$. Then

$$K(\pi(S),\pi(\Sigma))=\det\begin{bmatrix} I_m & -\Gamma \\ \Gamma & I_m \end{bmatrix}^{-N/2} \int_{GL(m,\mathbb{R})} f(\Lambda,\Gamma,L)d\beta_1(L)$$

where

$$f(\Lambda,\Gamma,L) = \det\begin{bmatrix} L & 0 \\ 0 & L \end{bmatrix}^N \exp(-1/2\mathrm{tr}\begin{bmatrix} I_m & -\Gamma \\ \Gamma & I_m \end{bmatrix}\begin{bmatrix} L & 0 \\ 0 & L \end{bmatrix}\begin{bmatrix} I_m & -\Lambda \\ \Lambda & I_m \end{bmatrix}\begin{bmatrix} L' & 0 \\ 0 & L' \end{bmatrix}) \qquad (14)$$

and β_1 is a Haar measure on $GL(m,\mathbb{R})$.

Letting $M_1 = \begin{bmatrix} I_m + \Gamma^2 & 0 \\ 0 & I_m + \Gamma^2 \end{bmatrix}^{-1/2}$ it is seen that $\begin{bmatrix} I_m & -\Gamma \\ \Gamma & I_m \end{bmatrix}^{-1} = M_1 \begin{bmatrix} I_m & \Gamma \\ -\Gamma & I_m \end{bmatrix} M_1$

and we get

$$K(\pi(S),\pi(\Sigma)) = \prod_{i=1}^{[m/2]} (1-\gamma_i^2)^N \cdot I(\Lambda,\Gamma), \text{ where}$$

$$I(\Lambda,\Gamma) = \int_{GL(m,\mathbb{R})} (\det L)^{2N}\exp(-\mathrm{tr}(LL'))\exp(-\mathrm{tr}(\Gamma L\Lambda L'))d\beta_1(L) \qquad (15)$$

Let α be the normed Haar measure on $O(m)$ and $\prod_{i=1}^{m} t_{ii}^{-i} \otimes_{i\leq j} dt_{ij}$ a right Haar measure on $T_+(m)$, the group of upper triangular matrices with positive diagonal elements. Using Andersson (1978, page 45) we can write $I(\Lambda,\Gamma)$ as

$$\int_{T_+(m)}(\det T)^{2N}\exp(-\mathrm{tr}(TT'))\int_{O(m)}\exp(-\mathrm{tr}(\Gamma HT\Lambda T'H')d\alpha(H) \prod_{i=1}^{m} t_{ii}^{-i} \otimes_{i\leq j} dtj \qquad (16)$$

The problem is to evaluate these integrals.

3. <u>The polynomials</u> $C_{2\bar{k}}$. Let V(k) be the real vector space of homogeneous polynomials $\Phi(B)$ of degree k in the m(m − 1)/2 different elements of B $\in$ $A(m,\mathbb{R})$.

The action

$$GL(m,\mathbb{R}) \times A(m,\mathbb{R}) \to A(m,\mathbb{R}), \quad (L,B) \to LBL' \tag{17}$$

induces transformations T(L) of V(k)

$$\Phi \to T(L)\Phi, \quad (T(L)\Phi) = \Phi(L^{-1}BL^{-1\,\prime}) \tag{18}$$

These transformations define a representation of $GL(m,\mathbb{R})$ on V(k).

Let $P_n(k)$ be the set of ordered sequences $\bar{k} = (k_1,\cdots,k_n)$, where $k_i \in \mathbb{N} \cup \{0\}$, $k_1 \geq k_2 \geq\cdots\geq k_n$ and $\sum_{i=1}^{n} k_i = k$. The elements of $P_n(k)$ are ordered lexicographically, (see Constantine (1963, page 1272)). For $\bar{k} \in P_n(k)$ we let $2\bar{k} = (2k_1,\cdots,2k_n) \in P_n(2k)$ and $\bar{k}2 = (k_1,k_1,\cdots,k_n,k_n) \in P_{2n}(2k)$.

Thrall ((1942), page 380) and Hua (1963) have shown that the representation of $GL(m,\mathbb{R})$ given by (18) decomposes into the irreducible representations corresponding to the partitions $\bar{k}2$, each of which is contained exactly once, and $\bar{k}$ runs through $P_n(k)$. Let $V(\bar{k})$ be the invariant irreducible subspace of V(k) in which the irreducible representation of $GL(m,\mathbb{R})$ corresponding to the partition $\bar{k}2$ acts.

For each $\bar{k} \in P_n(k)$ it is seen that (18) with L restricted to be orthogonal defines a representation of $O(m)$ on $V(\bar{k})$; by this representation $V(\bar{k})$ decomposes into a direct sum of irreducible invariant subspaces $V(\bar{k},i)$, $i = 1,\cdots,n(k)$. It follows from Littlewood (1940) that if $\bar{k}$ is a partion in even parts, i.e. each k_i is even, then exactly one of the subspaces, say $V(\bar{k},1)$, has the following property: it is one dimensional and the corresponding representation of $O(m)$ is the identity representation; if $\bar{k}$ is not a partition in even parts none of the subspaces has this property. Using a method similar to that of Constantine (1963), page 1272-1273) it can be shown that a polynomial Φ, which generates $V(2\bar{k})$ has the form $\Phi(\Lambda) = d(\bar{k}) \cdot \lambda_1^{2k_1}\cdots\lambda_n^{2k_n}$ + terms of lower weight ($n = [m/2]$). Here is Λ of the form (5), and terms $\lambda_1^{l_1}\cdots\lambda_n^{l_n}$ are ordered corresponding to the ordering of the partitions $\bar{l} \in P_n(2k)$.

<u>Definition</u> $C_{2\bar{k}}$ is the polynomial which generates $V(2\bar{k},1)$ normed such that the coefficient to the term with highest weight is 1. (19)

It follows that

$$C_{2\bar{k}}(LBL') \in V(2\bar{k}) \text{ for each } L \in GL(m,\mathbb{R}) \tag{20}$$

$$C_{2\bar{k}}(HBH') = C_{2\bar{k}}(B) \text{ for each } H \in O(m) \tag{21}$$

and $C_{2\bar{k}}(\Lambda)$ is a homogeneous, symmetric polynomial of degree 2k in $\lambda_1,\ldots,\lambda_n$.

We shall later obtain an explicit expression for $C_{2\bar{k}}(\Lambda)$; writing $C_{2\bar{k}}(\Lambda)$ by means of the elementary symmetric functions of $\lambda_1,\cdots,\lambda_n$ (Muirhead(1982, page 247)) it is then possible to get an explicit expression for $C_{2\bar{k}}(B)$.

For $S \in H^+(m,\mathbb{R})$ there exists $T \in T_+(m)$ such that $S = T'T$ and since the eigenvalues of SB and TBT' are equal for $B \in A(m,\mathbb{R})$ we can define $C_{2\bar{k}}(SB)$ by $C_{2\bar{k}}(SB) = C_{2\bar{k}}(TBT')$.

As in Constantine (1963, page 1273) we find

$$C_{2\bar{k}}(S\Lambda) = \lambda_1^{2k_1}\cdots\lambda_n^{2k_n} \det S_{2,2}^{k_1-k_2} \det S_{4,4}^{k_2-k_3}\cdots\det S_{2m,2m}^{k_n} + \text{lower terms} \qquad (22)$$

where $S_{i,i}$ is the principal $i \times i$ minor of S.

<u>Lemma 2</u> For $T \in T_+(m)$

$$\int_{O(m)} C_{2\bar{k}}(THBH'T')d\alpha(H) = c_1(\bar{k},T)C_{2\bar{k}}(B) \qquad (23)$$

where α is the normed Haar measure on $O(m)$.

Proof: For each $T \in T_+(m)$ we have that $C_{2\bar{k}}(TBT') \in V(2\bar{k})$; assuming a basis has been chosen in each $V(2\bar{k},i)$, $i = 1,\ldots,n(2\bar{k})$, we can write $C_{2\bar{k}}(TBT')$ as a linear combination of terms each of which belongs to a

$V(2\bar{k},i)$. Now $\int_{O(m)} \Phi(HBH')d\alpha(H) = 0$ when $\Phi \in V(2\bar{k},i)$, $i > 1$, (Littlewood (1940)), and $\int_{O(m)} C_{2\bar{k}}(HBH')d\alpha(H) = C_{2\bar{k}}(B)$ for which the lemma follows. □

Theorem 1 Let $S \in H^{+}(m,\mathbb{R})$ then

$$\int_{H^{+}(m,\mathbb{R})} \exp(-tr(S))(\det S)^{a} C_{2\bar{k}}(SB)d\beta(S) = \Gamma_{m}(a,2\bar{k})\cdot C_{2\bar{k}}(B) \tag{24}$$

where β is a measure on $H^{+}(m,\mathbb{R}]$ invariant under the action

$$GL(m,\mathbb{R}) \times H^{+}(m,\mathbb{R}) \to H^{+}(m,\mathbb{R}), \quad (L,S) \to LSL' \tag{25}$$

and

$$\Gamma_{m}(a,\bar{\ell}) = \pi^{1/4m(m-1)} \prod_{i=1}^{m} \Gamma(a+\ell_{i} - 1/2(i-1))$$

(if $\bar{\ell} \in P_{n}(\ell)$ and $n < m$ we set $\ell_{n+1} = 0,\cdots,\ell_{m} = 0$).

Proof Let $f(B)$ be the left side of (24). Using the fact that β is invariant under (25) we get that $f(HBH') = f(B)$ for $H \in O(m)$. Then

$$f(B) = \int_{O(m)} f(HBH')d\alpha(H) = \int_{H^{+}(m,\mathbb{R})} \exp(-tr(S)(\det S)^{a} \int_{O(m)} C_{2\bar{k}}(SHBH')d\alpha(H)d\beta(S)$$

and (23) gives

$$f(B) = cC_{2\bar{k}}(B) \tag{26}$$

To evaluate c we put B = Λ in (26), and compare the coefficients of the term of highest weight on both sides of (26). From (22) it follows that

$$c = \int_{H+(m,\mathbb{R})} \exp(-\mathrm{tr}(S)(\det S)^{a} \det S_{2,2}^{k_1-k_2} \cdot\cdot \det S_{2n,2n}^{k_n} d\beta(S)$$

β has density $(\det S)^{-(m+1)/2}$ w.r.t.a Lebesque measure on $H^{+}(M,\mathbb{R})$ and $c = \Gamma_m(a,\bar{k}2)$ follows from Constantine (1963, page 1274). □

<u>4. Calculation of $C_{2\bar{k}}$</u> James (1968) used the fact that the zonal polynomials of the positive definite real symmetric matrices are eigenfunctions of the Laplace-Beltrami operator to obtain a recurrence relation between the coefficients of these polynomials. We use his method.

Let $A^{+}(m,\mathbb{R})$ be the subset of $A(m,\mathbb{R})$ consisting of those $B \in A(m,\mathbb{R})$ for which $\det(B) > 0$. Assume that m is even.

$$\Omega = \mathrm{tr}(B^{-1}dBB^{-1}dB) \qquad (27)$$

is a differential 2-form on $A^{+}(m,\mathbb{R})$, which is invariant under (17). Put $p = m(m-1)/2$ and let x be the $p \times 1$ vector $x = (x_1,\cdots,x_p)' = (b_{1,2},b_{1,3},\cdots,b_{m-1,m})'$ where $b_{i,j}$ is the elements of the matrix B. Then can write (27) on the form

$$\Omega = (dx)'G(x)dx \qquad (28)$$

where G(x) is a $p \times p$ non-singular symmetric matrix with elements $g_{ij}(x)$.

The elements of $G(x)^{-1}$ are called $g^{i,j}(x)$. The Laplace-Beltrami operator is then given by

$$\Delta = \det G(x)^{-1/2} \sum_{j=1}^{p} \frac{d}{dx_j}\left[\det G(x)^{1/2} \sum_{i=1}^{p} g^{i,j}(x)\frac{d}{dx_i}\right] \tag{29}$$

Using Helgason (1962, page 387) it is seen that Δ is invariant under (17). Writing

$$B = H\Lambda H' \tag{30}$$

where $H \in O(m)$ and Λ is of the form (5) we want to express (27) and (29) in terms of Λ (and H). Put $\lambda = (\lambda_1,\cdots,\lambda_{m/2})'$; the elements of the skew-symmetric matrix $d\theta = H'dH$ are called $d\theta_{i,j}$. Using (30) in (27) we get

$$\Omega = tr(\Lambda^{-1}d\Lambda\Lambda^{-1}d\Lambda) - 2tr(d\theta\Lambda^{-1}d\theta\Lambda) + 2tr(d\theta d\theta) \tag{31}$$

A direct calculation shows that for m = 4

$$= 2(d\lambda' d\theta')\left[\begin{array}{cc:cccc} \lambda_1^{-2} & 0 & & & 0 & \\ 0 & \lambda_2^{-2} & & & & \\ \hdashline & & -2 & 0 & 0 & \lambda_{12} \\ & 0 & 0 & -2 & \lambda_{12} & 0 \\ & & 0 & \lambda_{12} & -2 & 0 \\ & & \lambda_{12} & 0 & 0 & -2 \end{array}\right]\begin{bmatrix} d\lambda \\ d\theta \end{bmatrix} \tag{32}$$

$$= 2(d\lambda'\ d\theta')G(\lambda)\begin{bmatrix} d\lambda \\ d\theta \end{bmatrix},\ \text{where } \lambda_{12} = \left[\lambda_1^2 + \lambda_2^2\right]\left[\lambda_1\lambda_2\right]^{-1} \text{ and}$$

$d\theta=(d\theta_{1,3},d\theta_{1,4},d\theta_{2,3},d\theta_{2,4})$.

Using this one sees that for any even m $G(\lambda)$ has the form

$$G(\lambda) = \left[\begin{array}{cc:c} \lambda_1^{-2} & 0 & \\ 0 & \lambda_{m/2}^{-2} & 0 \\ \hdashline & 0 & A(\lambda) \end{array}\right] \tag{33}$$

and

$$\det G(\lambda) = \prod_{i=1}^{m/2} \lambda_i^{-2} \prod_{i<j} (\lambda_i^2 - \lambda_j^2)^4 (\lambda_i \lambda_j)^{-6} \tag{34}$$

Again for m = 4 calculation shows that the part of Δ concerned with λ has the form

$$\Delta_\lambda = \sum_{i=1}^{2} \lambda_i^2 \frac{d^2}{d\lambda_i^2} - \sum_{i=1}^{2} \lambda_i \frac{d}{d\lambda_i} + 4 \sum_{i=1}^{2} \sum_{j\neq i} \lambda_i^3 (\lambda_i^2 - \lambda_j^2)^{-1} \frac{d}{d\lambda_i} \tag{35}$$

For any even m we get (with n = m/2)

$$\Delta_\lambda = \sum_{i=1}^{n} \lambda_i^2 \frac{d^2}{d\lambda_i^2} - \sum_{i=1}^{n} \lambda_i \frac{d}{d\lambda_i} + 4 \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \lambda_i^3 (\lambda_i^2 - \lambda_j^2)^{-1} \frac{d}{d\lambda_i} \tag{36}$$

Now let $S_{\bar{k}} : \mathbb{C}^n \to \mathbb{C}$ be the Schur function corresponding to the partition

$\bar{k} \in P_n(k)$, see Littlewood (1940, page 191) or Garcia and Remmel (1981) for an explicit expression. These functions are the same as the zonal polynomials on the Hermitian matrices with the matrices restricted to be diagonal (Takemura (1984)) It then follows from Sugiura (1973) that the functions $S_{\bar{k}}$ are the eigenfunctions of the operator

$$\sum_{i=1}^{n} y_i \frac{d^2}{dy_i^2} + 2\sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} y_i y_j (y_i - y_j)^{-1} \frac{d}{dy_i} \tag{37}$$

For $\bar{k} \in P_n(k)$ define $f_{2\bar{k}} : \mathbb{R}^n \to \mathbb{R}$ by

$$f_{2\bar{k}}(\lambda_1, \cdots, \lambda_n) = S_{\bar{k}}(\lambda_1^2, \cdots, \lambda_n^2) \tag{38}$$

Using the chain rule it is seen that the functions $f_{2\bar{k}}$ are the eigenfunctions of Δ_λ.

We shall also consider $f_{2\bar{k}}$ as a function defined on $A(m,\mathbb{R})$ by requiring that $f_{2\bar{k}}(B) = f_{2\bar{k}}(HBH')$ for $H \in O(m)$ and $f_{2\bar{k}}(\Lambda) = f_{2\bar{k}}(\lambda)$

Theorem 2

$$C_{2\bar{k}} = f_{2\bar{k}} \tag{39}$$

Proof m even: By using the fact that Λ is invariant under (17) it can be shown that the function g, where

$$g(B) = \int_{O(m)} f(THBH'T')d\alpha(H)$$

is an eigenfunction of Λ with eigenvalue c if f is an eigenfunction of Λ with eigenvalue c. Hence any $f_{2\bar{k}}$ satisfies (23) (with $C_{2\bar{k}} = f_{2\bar{k}}$). $f_{2\bar{k}}$ then also satisfies (24) and it follows that $C_{2\bar{k}} = f_{2\bar{k}}$ since the properties (19), (20) and (24) determins the function $C_{2\bar{k}}$.

m odd: We consider $A(2m,\mathbb{R})$ $(GL(2m,\mathbb{R}))$ as a subset of $A(2m+1,\mathbb{R})$ $(GL(2m+1,\mathbb{R}))$ by identifying B with $\begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix}$ $\left[L \text{ with } \begin{bmatrix} L & 0 \\ 0 & 1 \end{bmatrix}\right]$, and the vector space $V(2k,2m)$ of homogeneous polynomials of degree 2k is in a natural way a subspace of $V(2k,2m+1)$. We then have

$$V(2k,2m) = \bigoplus_{\bar{k} \in P_n(k)} V(2\bar{k},2m+1) \cap V(2k,2m)$$

and since $V(2\bar{k},2m+1) \cap V(2k,2m) \neq 0$ it follows that this subspace is the same as $V(2\bar{k},2m)$, and we get $C_{2\bar{k}}^{2m+1}(\Lambda) = C_{2\bar{k}}^{2m}(\Lambda)$. □

5. The correction factor In the same way as James (1960, page 155-156) it can be shown that

Lemma 3

$$\int_{O(m)} \mathrm{tr}(B_1 H B_2 H')^{2k} d\alpha(H) = \sum_{\bar{k} \in P_n(k)} c(\bar{k}) C_{2\bar{k}}(B_1) C_{2\bar{k}}(B_2) \qquad (40)$$

and

$$\int_{O(m)} \mathrm{tr}(B_1 H B_2 H')^{2k+1} d\alpha(H) = 0$$

where $B_1, B_2 \in A(m, \mathbb{R})$.

Expanding $\exp(-\mathrm{tr}(\Gamma H T \Lambda T' H'))$ as a power serie and using (40) and (24) we get

Theorem 3 Let Γ and Λ be of the form (5).Then

$$\int_{T_+(m)} (\det T)^{2N} \exp(-\mathrm{tr}(TT')) \int_{O(m)} \exp(-\mathrm{tr}(\Gamma H T \Lambda T' H') d\alpha(H) \prod_{i=1}^{m} t_{ii}^{-i} \otimes_{i \leq j} dt_{ij}$$

$$= \sum_{k=0}^{\infty} (2k)!^{-1} \sum_{\bar{k} \in P_n(k)} c(\bar{k}) \Gamma_m(N, \bar{k}2) C_{2\bar{k}}(\Gamma) C_{2\bar{k}}(\Lambda) \qquad (41)$$

$n = [m/2]$ and $c(\bar{k})$ is given by

$$\int_{O(m)} \operatorname{tr}(\Gamma H \Lambda H)^{2k} d\alpha(H) = \sum_{\bar{k} \in P_n(k)} c(\bar{k}) C_{2\bar{k}}(\Gamma) C_{2\bar{k}}(\Lambda). \tag{42}$$

Using (16) we then have an expression for the correction factor K.

The coefficients $c(\bar{k})$ have only been evaluated in the cases m = 2, m = 3 and m = 4. The cases m = 2 and m = 3 are easy.

The case m = 4. For $H = (h_{ij}) \in O(4)$ we let

$$H_{ij} = \begin{bmatrix} h_{i,1} & h_{i,2} \\ h_{j,1} & h_{j,2} \end{bmatrix} \text{ and } G_{ij} = \begin{bmatrix} h_{i,3} & h_{i,4} \\ h_{j,3} & h_{j,4} \end{bmatrix}$$

for $1 \leq i < j \leq 4$. Then

$$H = \begin{bmatrix} H_{12} & G_{12} \\ H_{34} & G_{34} \end{bmatrix}$$

and the left side of (42) becomes

$$\int_{O(4)} 2^{2k} (\gamma_1\lambda_1 \det H_{12} + \gamma_1\lambda_2 \det G_{12} + \gamma_2\lambda_1 \det H_{34} + \gamma_2\lambda_2 \det G_{34})^{2k} d\alpha(H)$$

The relation

$$\sum_{1 \leq i < j \leq 4} \det H_{ij}^2 = 1 \tag{43}$$

and invariance of α gives

$$\int_{O(4)} \det H_{12}^2 d\alpha(H) = 1/6 \tag{44}$$

Again the invariance of α implies that

$$\int_{O(4)} f(PH)d\alpha(H) = \int_{O(4)} f(H)d\alpha(H) \text{ for } P \in O(4) \tag{45}$$

From (45) with $P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & u_1 & 0 & u_2 \\ & & P_2 & \end{bmatrix}$ and $f(H) = (\det H_{12})^{2k}$ we get

$$(u_1^2+u_2^2)^k \int_{O(4)} (\det H_{12})^{2k} d\alpha(H) = \int_{O(4)} (u_1 \det H_{12} + u_2 \det H_{14})^{2k} d\alpha(H)$$

and comparing coefficients to $u_1^{2k-2} \cdot u_2^2$ gives

$$k \int_{O(4)} (\det H_{12})^{2k} d\alpha(H) = \begin{bmatrix} 2k \\ 2 \end{bmatrix} \int_{O(4)} (\det H_{12})^{2k-2} (\det H_{14})^2 d\alpha(H) \tag{46}$$

Comparing coefficients to $\lambda_1^{2k} \gamma_1^{2k-2} \gamma_2^2$ and $(\gamma_1 \lambda_1)^{2k}$ in (42) gives

$$\int_{O(4)} (\det H_{12})^{2k} d\alpha(H) = \begin{bmatrix} 2k \\ 2 \end{bmatrix} \int_{O(4)} (\det H_{12})^{2k-2} (\det H_{34})^2 d\alpha(H) \tag{47}$$

From (43) we get

$$\int_{O(4)} (\det H_{12})^{2k} d\alpha(H) = \int_{O(4)} (\det H_{12})^{2k} \sum_{1\le i<j\le 4} (\det H_{ij})^2 d\alpha(H).$$

From this relation together with (44),(46),(19) and using induction we get

$$\int_{O(4)} (\det H_{12})^{2k} d\alpha(H) = \binom{2k+2}{2}^{-1} \tag{48}$$

Now

$$\det H_{12}^2 = \det G_{34}^2, \ \det H_{34}^2 = \det G_{12}^2 \text{ and}$$

$$\det(H_{12}) \cdot \det(H_{34}) \cdot \det(G_{12}) \cdot \det(G_{34}) \ge 0 \tag{49}$$

and it follows that

$$\begin{aligned} c(k,0) &= 2^{2k}\binom{2k+2}{2}^{-1} \\ c(i,k-i) &= 2^{2k}(2i+k+1)^{-1}(2k+1)^{-1}\left[\binom{2k}{2i}-\binom{2k}{2i+2}\right] \end{aligned} \tag{50}$$

where $i = k - [k/2], \cdots, k-1$.

These coefficients and theorem 3 have been checked by simulation of the distribution of $\pi(S)$ for $m = 4$.

This paper is based on the author's dissertation at the Institute of Mathematical Statistics, University of Copenhagen, under the supervision of Steen Andersson.

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