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Preprint
January
1987

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Preprint 1987 No. 2

INSTITUTE OF MATHEMATICAL STATISTICS
UNIVERSITY OF COPENHAGEN

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A metal bar or a glass fibre breaks when the load exceeds its inherent strength. Stochastic extreme value theory provides models both for predicting maximum loads and for the strength of the material. The talk this note is based on was about the "Weibull theory" where the observed statistical variation of strengths of brittle materials is explained via extreme value theory and about extremes of load processes of the "filtered Poisson process" type. For a general discussion of the connections between structural engineering and extreme values we refer to the paper by Bolotin in these proceedings, and to [10] for a survey of the last few years developments in stochastic extreme value theory. For reasons of space we here concentrate on the first part, and refer to [11] for the second part of the talk.

STRENGTH OF BRITTLE MATERIALS

The starting point is the empirical fact that strengths of pieces of material manufactured under similar conditions show a practically important stochastic variation from piece to piece. This is connected with extreme values through the so-called *weakest link principle*, which says that for some materials, such as glass fibres or iron bars, the strength of a piece is determined by the strength of its weakest part. Weibull's argument ([12], cf. also [3,7]), which were quite informal, involved a "microscopic", unobservable, model for strengths, which then motivates an observable "macroscopic" model.

Here mathematically formalized versions of the two models will be introduced. We will show that the two are in fact equivalent, and discuss ways of testing the assumptions on observed strength data. A longstanding point of debate is which distribution is appropriate for material strengths. Of course the Weibull distribution is the main contender, but e.g. in [2] strengths of glass fibres is described by the product of two different Weibull distribution functions, to take into account surface and interior defects, and many other distributions, such as normal, lognormal, and gamma have been suggested [3,7]. An important feature of the statistical procedure proposed here is that it makes it possible to investigate homogeneity and weakest link behaviour without involving any assumptions about distributional forms.

We only consider the simplest case, of a specimen subjected to uniaxial tension. In the microscopic theory it is noted that observed strengths are much lower than the strength of the bonds between molecules, and the discrepancy is explained by the presence of small *flaws* or *microcracks* ([5]). The flaws are assumed to be "randomly" distributed in a homogeneous material,

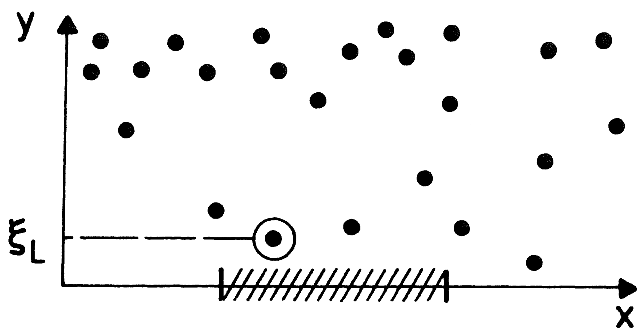

Figure 1 Plot of strengths x_i of microcracks against their location y_i.

with the material breaking when the local stress at any of the flaws exceeds its inherent strength. In Figure 1 this is illustrated by plotting the strength y_i of the i-th microcrack against its location x_i, measured along the specimen. The microscopic model for the strength of homogeneous brittle materials is that $\{(x_i,y_i)\}_{i=1}^{\infty}$ are the points of a Poisson process, N, in the first quadrant of the plane, with intensity measure $dx \times m(dy)$, and with m an arbitrary locally finite measure on $[0,\infty)$. The strength ξ_L of a piece of the specimen corresponding to the interval L is then

(1) $$\xi_L = \min\{y_i; x_i \in L\},$$

cf. Figure 1. Thus the survival function (s.f.) $S_L(x)$ of the piece L of length ℓ is given by

(2) $$S_L(x) = P(\xi_L > x)$$

$$= P(N(L \times [0,x]) = 0)$$
$$= \exp\{-\ell\, m([0,x])\},$$

since $E(N(L \times [0,x])) = \ell\, m([0,x])$.

The mathematical formalization of the macroscopic model is as follows. We again consider a piece of material L with length ℓ and strength ξ_L and assume that it can be subdivided, at least hypothetically, into smaller pieces $L_1,\dots,L_n$ of arbitrary lengths $\ell_1,\dots,\ell_n$, and with definite (random) strengths $\xi_{L_1},\dots,\xi_{L_n}$, respectively. We say that the material is stochastically[1]

(i) *brittle* if $\xi_L = \min(\xi_{L_1},\dots,\xi_{L_n})$,

(ii) *homogeneous* if the marginal distribution of $\xi_{L_1},\dots,\xi_{L_n}$ depends only on $\ell_1,\dots,\ell_n$,

(iii) *disconnected* if $\xi_{L_1},\dots,\xi_{L_n}$ are independent for all disjoint divisions $L_1,\dots,L_n$ of L.

Of these properties, (ii) and (iii) are of purely statistical character, while (i) depends on the mechanism involved in a failure. The properties all have definite physical meanings. It follows at once from (ii) that the s.f. $S_L(x) = P(\xi_L > x)$ only depends on the length ℓ of L, i.e. $S_L(x) = S_\ell(x)$, and (i) and (ii) are then seen to imply that

(3) $$S_\ell(x) = S(x)^\ell, \quad x, \ell > 0,$$

with $S(x) = S_1(x)$.

Both models of course involve idealizations of reality. E.g. it may not be meaningful to assume that very short pieces have a definite (measurable) strength as is done in the macroscopic model, and microcracks have a physical extension which is not taken into account in the microscopic model. Nevertheless, on the scale of interest the models might still be quite accurate. It is then interesting to note that *the microscopic and macroscopic models are mathematically equivalent*. Thus, e.g. if one believes that a material shows the behaviour specified by (i)-(iii) then necessarily the physical mechanism behind failures must be the one given by the microscopic model.

We briefly outline a proof of this result. One half is immediate: clearly a material which satisfies (1) with $\{(x_i,y_i)\}$ the points of a Poisson process with intensity $dx \times m(dy)$ also satisfies the assumptions (i)-(iii) of the macroscopic model. For the converse, suppose the material satisfies (i)-(iii) with s.f. $S_\ell(x)$ given by (3). It is straightforward to see that this determines all joint distributions of strengths (of intervals). Now, by the first part of the proof, the microscopic model specified by

$$m([0,x]) = -\log S(x), \quad x > 0,$$

satisfies (i)-(iii), and according to (2) also (3) holds. Hence, as was to be shown, this microscopic model leads to precisely the same distributions as the macroscopic model we started out with. The further, more difficult problem of how to recover the Poisson process of microcracks from (hypothetical) measurements of strengths of all pieces will be treated elsewhere.

In practical situations one often needs not only the assumptions (i)-(iii) but also a parametric model for S(x). One way to obtain this is to add an *ad hoc* notion which is that the material is

(iv) *Size-stable* if each S_ℓ is a location-scale transformation of S, i.e. if there are $\alpha_\ell > 0$, β_ℓ such that
$S_\ell(x) = S(\alpha_\ell^{-1}(x - \beta_\ell)), \quad \ell > 0.$

It follows from (3) and (iv) that S(x) is min-stable, and hence is one of the three extreme value s.f.'s for minima (see [9], p.271 - 273). A final assumption is that strengths are

(v) *Non-negative* if $\xi_L \geqq 0$ for all L and if values arbitrarily close to zero are possible.

This further assumption makes the type III extreme value (or Weibull) s.f. the only possibility, so that then

$$(4) \qquad S_\ell(x) = \begin{cases} 0 & x < 0 \\ \exp\{-\ell(x/\sigma)^\alpha\} & x \geqq 0, \end{cases}$$

where $\alpha, \sigma > 0$ are material parameters.

An alternative argument to (iv),(v) is to assume that S(x) decreases as a power at $x = 0$, i.e. $S(x) \sim 1 - (x/\sigma)^\alpha$ as $x \to 0$. Together with (3) and the standard criterion for the domain of attraction for minima for the Weibull distribution this again leads to (4). Further, in the engineering literature (4) is often advocated for directly, for its mathematical simplicity and flexibility.

We now turn to the problem of empirically testing the models. As far as I know, direct tests of the microscopic model is beyond present capabilities, and we will hence discuss how the assumptions of the macroscopic model can be checked using measurements of strengths of specimens of varying sizes. The independence assumption (iii) can be checked by applying any of the standard independence tests to a series of strength measurements. It seems less obvious how to test (i) and (ii) separately. Instead we investigate them together by embedding (3) into the larger model

(5) $$S_\ell(x) = S(x)^{\ell^\beta},$$

where $\beta > 0$ and $S(x)$ are free "parameters", and then test for $\beta = 1$. Writing $z = \log \ell$ and $h(x) = S'(x)/S(x)$ (assuming that $S(x)$ is differentiable) (5) takes the form $S_\ell(x) = \exp\{-e^{\beta z}\int_0^x h(t)dt\}$ and is recognized to be of the Cox-model type. We can hence use standard methods for the Cox model to estimatet β, test for $\beta = 1$, combine observations of strengths of pieces of different lengths to one estimate of the "underlying s.f." $S(x)$, and compare it with the Weibull s.f. resulting from (iv),(v).([8] is a general reference on the Cox model).

A. Deis in his masters thesis [4] makes a detailed study of these testing problems, and applies them to a number of data sets. Here we will as examples show the (still somewhat preliminary) results from two of his sets.

<u>Example 1</u> The first data are from Bader & Priest [1], and consist of strength measurements on about 60 carbon fibres of each of four different lengths $\ell_1 = 1$, $\ell_2 = 10$, $\ell_3 = 20$, and $\ell_4 = 50$ mm's. As of now we unfortunately have only had access to the ordered values and have hence not yet tested for independence. In Figure 2 the empirical s.f.'s for each of the four samples is plotted. The scales are choosen so that if the Weibull model (4) holds then they should, except for random fluctuations, yield parallel straight lines of slope α, and with the i-th and j-th line a vertical distance $\log(\ell_j/\ell_i)$ apart. The plot roughly agrees with this expected behaviour. Nevertheless, the Cox estimator for β in the model (5) is

$\hat{\beta}$ = .83 which is significantly different from zero (p = 0.04).

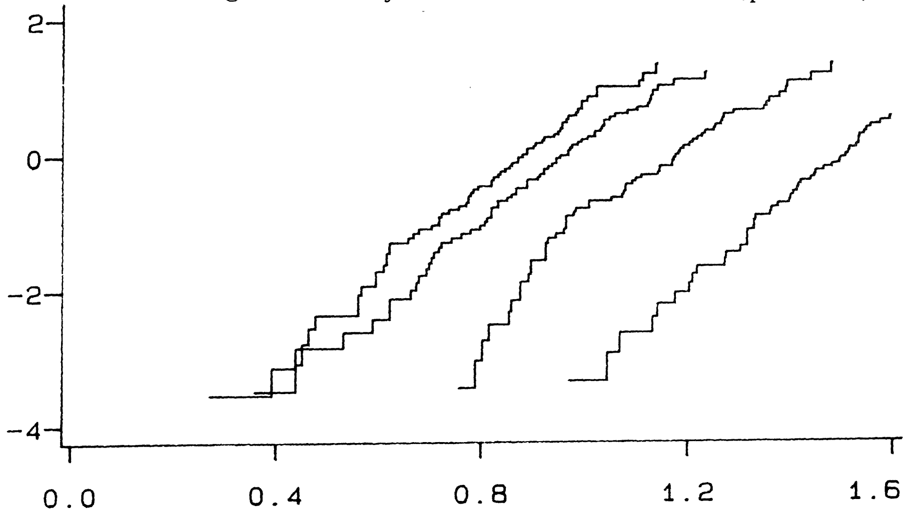

Figure 2 Plots of log {- log S(x)} against log x for glass fibres of four different lengths, in order from left to right 50 mm's, 20 mm's, 10 mm's, and 1 mm's.

Figure 3 contains a plot of the estimated survival function S in (5), using the method of Breslow to combine all four samples. There is a rather clear deviation from the straight line which would result if S(x) were the Weibull s.f. (4). We have not yet performed any formal tests of this deviation.

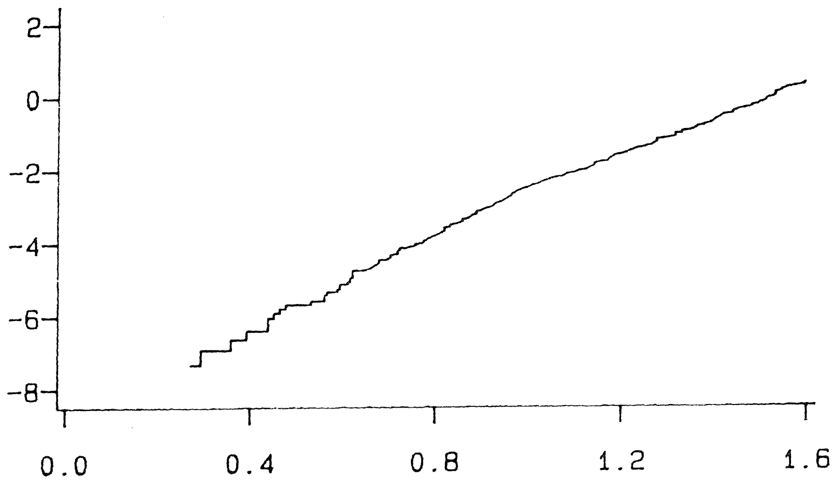

Figure 3 Plot of log (- log (underlying survival function) against log (strength), estimated using data from all four samples.

Example 2 The data here have been provided by L. Nilsson and

S. Uvell, Umeå University. The part we will discuss contains two strata (I and II) of measurements of strengths of optical fibres, which differ in experimental conditions (rate of increase of tension). Both contain 40 measurements on each of the two lengths $\ell_1 = 40$ and $\ell_2 = 80$ cm's. Correllations and partial autocorrellations were small in three of the four samples and formal tests did not indicate deviations from independence. Figure 4 contains plots of the survival functions for each set.

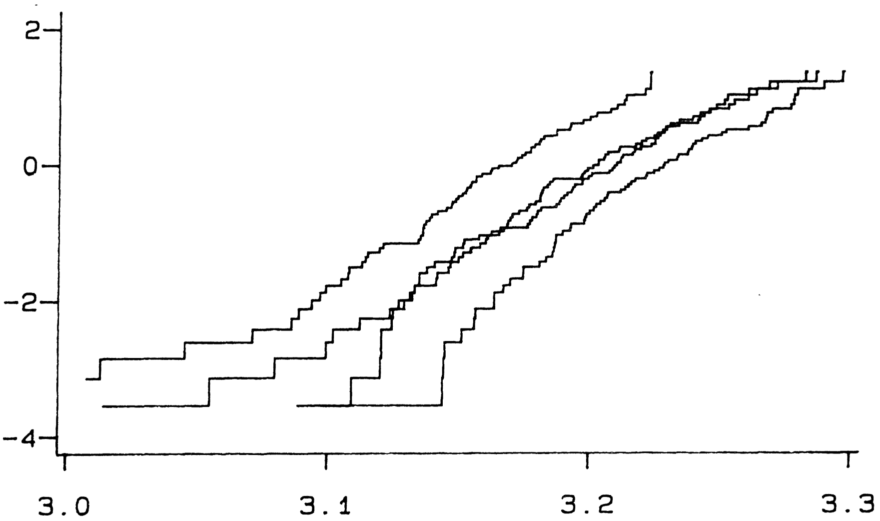

Figure 4 Plot of log $\{-\log S(x)\}$ against log x. The first and third line from the left are lengths 80 and 40 cm's from Stratum I, and the second and fourth are the lengths 80 and 40 cm's from Stratum II.

The Cox estimates for the model (5) are $\hat{\beta}=1.04$ and $\hat{\beta} = 0.76$. With the size of the random variation taken into account, both estimates agree well with the hypothesis that $\beta = 1$. Figure 5 shows Breslow's estimate for the underlying s.f.'s for the two strata. They are fairly linear, as predicted by the Weibull model (4).

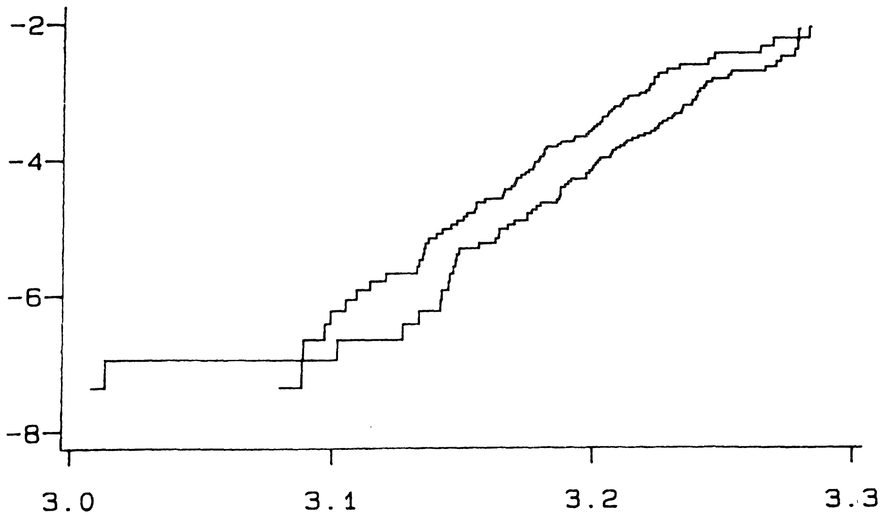

Figure 5 Plot of log {- log S(x)} against log x. Left Stratum I, right Stratum II. □

Hence Example 2 is seen to agree with the macroscopic model, while for Example 1 not even assumptions (i) and (ii) seem to be satisfied, since β is significantly different from 1. One would be inclined to believe that it is the homogeneity assumption (ii) which is violated, e.g. due to randomly varying diameters or changes in experimental conditions. This would then lead to a mixture model

$$(6) \qquad S_\ell(x) = \int S(x;\alpha)^\ell dF(\alpha),$$

where $F(\alpha)$ represents, say, variations in diameter or composition of the material.

Methods for analysing such models (sometimes called frailty models) are being developed, see [6].

If one tries to force the model (5) onto data which really come from (6), i.e. tries to find β such that

$$S_\ell(x) = \int S(x,\alpha)^\ell dF(\alpha) \approx S_1(x)^{\ell^\beta} = (\int S(x;\alpha)dF(\alpha))^{\ell^\beta}$$

then, as an easy consequence of Jensens's inequality, this leads to β - values less than 1. A further unfortunate consequence is that using (5) then leads to an overestimate of the strengths of large specimens.

In many situations one would (as for Examples 1 and 2) be rather convinced that (i) holds, and then the main practical use of the test for $\beta = 1$ is as a means to find inhomogeneities in the material or experimental setup, as mentioned above. This agrees with the experiences of Nilsson and Uvell. Initially their experiments showed very marked departures from (3) and (4), but after they had eliminated a number of causes for inhomogeneity in the material and experimental conditions, they were consistently able to obtain data agreeing with the model (4).

Acknowledgement I would like to thank Richard Smith for some valuable suggestions, in particular on the connection with the Cox model, and Bader & Priest and Nilsson & Uvell for the permission to use their data.

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