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Summary

At the same time as they solved the classical problem of points Bernoulli, de Moivre and Montmort also solved the problem of points for a game of bowls and for tennis. Using modern terminology and notation an account is given of the solutions. It is pointed out that Montmort independently of Bernoulli gave a more elegant and comprehensive solution for the game of tennis than Bernoulli's recursive solution,and that Montmort also generalized and solved de Moivre's problem on the game of bowls.

Key words:

History of probability; problem of points; game of bowls; game of tennis; Bernoulli; de Moivre; Montmort.

1 The problem of points

The problem of points, also called the division problem, is defined as follows. Two players, A and B, agree to play a specified number of games, where in each game A has probability p and B has probability q = 1-p of winning a point. If the play is interrupted when A lacks a points and B lacks b points in winning, how should the stake be divided between them?

The problem of points was discussed by Italian mathematicians in the 16th century, but they did not succeed in finding the correct solution; see Cantor (1900, Vol.2), Kendall (1956), David (1962), Coumet (1965) and Schneider (1985).

It is well known that the problem was solved by Pascal and Fermat for $p = \frac{1}{2}$ in their correspondence (1654) and by Pascal (1665); see Todhunter (1865), David (1962) and Edwards (1982, 1987). Besides combinatorial arguments Pascal also used recursion.

Denoting A's probability of winning or his expectation, when the stake is unity, by e(a,b), Pascal's recursion formula may be written as

$$e(a,b) = \tfrac{1}{2}e(a-1,b) + \tfrac{1}{2}e(a,b-1), \quad (a,b) = 1,2,\ldots,$$

with the boundary conditions $e(0,n) = 1$ and $e(n,n) = \frac{1}{2}$, $n = 1,2,\ldots$. Pascal, Huygens (1657) and Bernoulli (1713) tabulated the solution for small values of (a,b) by means of this formula. The recursion formula is the same as for the numbers in the arithmetical triangle and Pascal (1665) derived the explicit solution in his Traité.

Using the combinatorial arguments of Pascal and Fermat the general solution of the problem of points was obtained by John Bernoulli in 1710 in his correspondence with Montmort (1713, pp. 294-295) and independently by de Moivre (1712) in the form

$$e(a,b) = \sum_{i=a}^{a+b-1} \binom{a+b-1}{i} p^i q^{a+b-1-i} .$$

Using a waiting time argument Montmort (1713, pp. 245-246) gave the solution in the form

$$e(a,b) = p^a \sum_{i=0}^{b-1} \binom{a-1+i}{a-1} q^i .$$

They knew of course the generalization of the recursion formula

$$e(a,b) = pe(a-1,b) + qe(a,b-1), \quad (a,b) = 1,2,\ldots,$$

with the boundary conditions $e(a,0) = 0$, $a = 0,1,\ldots$, and $e(0,b) = 1$, $b = 1,2,\ldots$, but they did not have a general method for solving such partial difference equations; this had to wait for the contributions of Lagrange and Laplace in the 1770s.

By 1713 the classical problem of points for two players had thus been completely solved.

2 The problem of points for a game of bowls

In 1708 Francis Robartes (1650?-1718), politician, scientist and musician, after having read Montmort's Essay (1708) posed the following problem to de Moivre , who gave the solution in his De Mensura Sortis (1712), Problems 16 and 17 : "A and B, whose skills are equal between themselves, play with a given number of bowls; now after a certain number of games are completed, A lacks 1 game from coming out the winner, and B 2 : the ratio of their expectations is sought". (This is Problem 16; in Problem 17 the number of games lacking is 1 and 3, respectively.) It follows from de Moivre's solution that in each game the winner gets a number of points equal to the number of his bowls that are nearer to the jack than any of the loser's bowls. Furthermore, the problem should have been formulated as "A lacks 1 point from coming out the winner and B 2".

Montmort (1713, pp. 248-257, 366-367) generalized the problem as follows:

A and B play a game of bowls, A with m bowls and B with n. The skill of A is to the skill of B as r to s. In each game the winner gets a number of points equal to the number of his bowls which are nearer to the jack than any of the loser's. If the play is interrupted when A lacks a points and B b points in winning, how should the stake be divided equitably between them?

Montmort explicitly defines "skill" by referring to a game with one bowl for each player. A's skill, $r/(r+s)$, is then A's probability of getting nearer to the jack than B. He also points out that for $m = n = 1$ we have the classical problem of points.

Montmort assumes that the total stake is 1 so that A's expectation, $f(a,b)$ say, equals his probability of winning the stake.

To solve the problem Montmort introduces an urn with mr white chips and ns black chips representing A's and B's chances of winning. The total number of chips is $t = mr + ns$.

Let P_i be A's probability of getting at least i points, i. e. the probability of getting a run of i white chips by drawings without replacements from the urn. It follows that

$$P_i = \frac{mr}{t} \frac{mr-r}{t-r} \cdots \frac{mr-(i-1)r}{t-(i-1)r}, \quad i = 1, 2, \ldots, m.$$

Let p_i be A's probability of getting exactly i points, i. e. the probability of getting a run of i white chips followed by a black. Hence

$$p_i = \frac{mr}{t} \frac{mr-r}{t-r} \cdots \frac{mr-(i-1)r}{t-(i-1)r} \frac{ns}{t-ir}, \quad i = 1, 2, \ldots, m.$$

P_i and p_i are defined as zero otherwise.

The corresponding probabilities for B will be denoted by Q_i and q_i, $i = 1, 2, \ldots, n$, and they are obtained from P_i and p_i by interchanging (m,r) and (n,s).

Note that $p_i = P_i - P_{i+1}$ and that

$$\Sigma_1^m p_i + \Sigma_1^n q_i = P_1 + Q_1 = 1.$$

Montmort gives the solution as the recursion

$$f(a,b) = P_a + \sum_{i=1}^{a-1} p_{a-i} f(i,b) + \sum_{i=1}^{b-1} q_i f(a,b-i), \quad \begin{array}{l} a = 1, 2, \ldots \\ b = 1, 2, \ldots, \end{array}$$

and $f(a,0) = f(0,b) = 1$. The proof follows directly from the addition and multiplication theorems.

If the players have only one bowl each then $P_1 = p_1 = r/(r+s)$, $Q_1 = q_1 = s/(r+s)$ and

$$f(a,b) = p_1 f(a-1,b) + q_1 f(a,b-1),$$

which is the recursion for the classical problem of points.

Montmort states that similar results hold for any number of players and gives a numerical example for three players.

In his formulation and discussion of Robartes' problem de Moivre (1712) assumes that the players are of equal skill and have the same the same number of bowls. He then derives the formulae for $f(2,1)$ and $f(3,1)$ and states that the general formula may be found by the same method. In the Doctrine of Chances (1718, Problems 27 and 28) de Moivre acknowledges Montmort's general solution and derives $f(a,b)$ for $m = n$.

They did not give an explicit solution of the problem.

In modern terminology the problem may be described as a random walk in two dimensions, the horizontal steps being of length 1 or 2, ..., or m, and the vertical steps being of length 1 or 2, ..., or n. A wins if the random walk crosses the vertical line through $(a,0)$ before crossing the horizontal line through $(0,b)$.

3 The problem of points for the game of tennis

Bernoulli's Lettre à un Amy sur les Parties du Jeu de Paume was printed as an appendix to the Ars Conjectandi (1713). Bernoulli begins with a summary of his considerations in the Ars Conjectandi on the difference between games of chance and games depending on the skill of the players, on the corresponding determination of probabilities a priori and a posteriori and on the law of large numbers, which justifies the use of the relative frequency of winning as a measure of the probability of winning. Apart from this short introduction the letter is really an exercise in probability theory and could well have been included in Part 3 of the Ars Conjectandi.

Bernoulli writes that he will not explain the rules of the game because they are well known. The game is more complicated than tennis but with the same scoring rules; a detailed description of the game has been given by Haussner (1899).

Bernoulli gives an analysis of a large number of problems on tennis. There is, however, no new methods involved in his analysis; he keeps strictly to the methods used by Huygens, solving most of the problems by recursion between expectations. We shall confine ourselves to a discussion of the main points,leaving out most of the details. It seems that Bernoulli's results have been overlooked by modern writers on the game.

For convenience we shall give a player one point for each play he wins instead of using Bernoulli's scoring system (0,15,30,45,game). Player A's probability of winning a point will be denoted by p and players B's probability by q, p+q = 1. We shall denote the state of a game by the number of points, (i,j) say, won by the two players. The game is won by the player who scores 4 points before the other player scores more than 2 points; furthermore, if the game reaches the state (3,3) the player who first wins 2 points more than his opponent wins the game.

Using modern terminology the play may be described as a random walk in two dimensions with absorbing barriers, see Fig. 1. The random walk starts at (0,0) and moves one step to the right with probability p, if player A wins, and one step up with probability q, if player B wins. In the figure we have cut off the continuation region at the score (7,7).

Let g(i,j) denote A's probability of winning the game, given that the game is in state (i,j). Since A wins the next point with probability p and loses with probability q we have

$$g(i,j) = pg(i+1,j) + qg(i,j+1). \tag{1}$$

This is the fundamental formula which Bernoulli derives and uses to tabulate g(i,j) with the modification that he uses n = p/q as parameter.

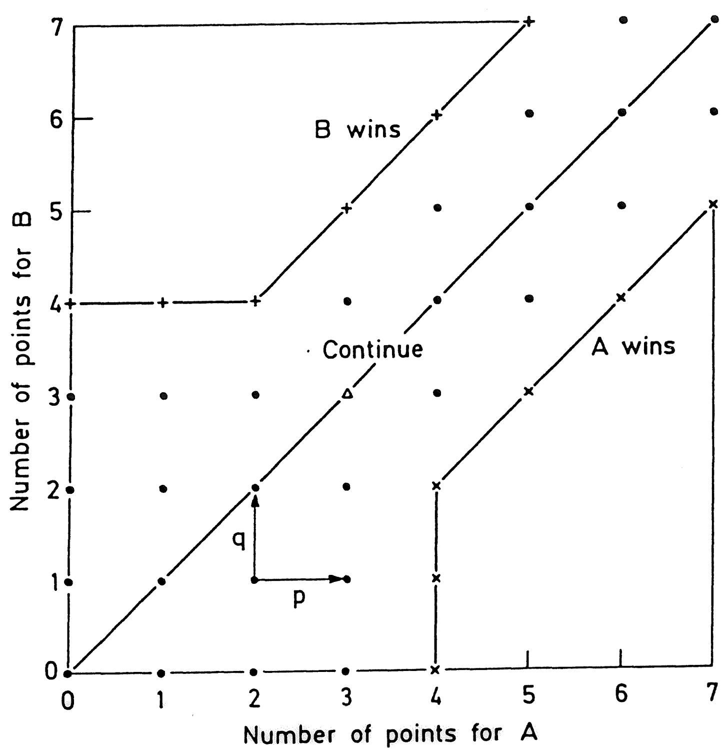

Fig.1. A random walk diagram for the game of tennis.

Beginning with the state (3,3) and using the recursion two times Bernoulli finds

$$g(3,3) = p^2 g(5,3) + 2pqg(4,4) + q^2 g(3,5).$$

Since $g(5,3) = 1$, $g(3,5) = 0$ and $g(4,4) = g(3,3)$ he gets

$$g(3,3) = p^2/(p^2+q^2) = n^2/(n^2+1). \tag{2}$$

Using (1) again he obtains

$$g(2,3) = pg(3,3) + qg(2,4) = pg(3,3) = n^3/(n^3+n^2+n+1),$$

and continuing in this manner he finds $g(i,j)$ for $i \leqq 3$ and $j \leqq 3$ and thus solves the problem completely. He tabulates $g(i,j)$ as the ratio of two polynomials of the same degree in $n = p/q$ as shown in the following example:

$$g(0,0) = (n^7+5n^6+11n^5+15n^4)/(n^7+5n^6+11n^5+15n^4+15n^3+11n^2+5n+1). \tag{3}$$

By means of these formulae Bernoulli calculates all the values of $g(i,j)$ for $p/q = 1,2,3,4$.

Bernoulli uses his results to determine the size of handicaps to get a fair game. He first asks the question: How many points should be accorded the weaker player for the game to be fair? Suppose that $p/q = 2$. Then Bernoulli's table shows that $g(0,2) = 208/405 = 0.514$ so that a handicap of two points to B will nearly equalize their chances of winning. Considering the same problem for $p/q = 3$ Bernoulli notes that $g(0,2) = 891/1280 = 0.696$ and $g(0,3) = 243/397 = 0.612$ so that handicaps of 2 and 3 are not enough to equalize the chances. He then finds that $g(1,3) = 81/160 = 0.506$, which means that a game starting with 1 point for A and 3 points for B will be nearly fair.

He next solves the inverse problem: If B has been given a handicap of j points to make the game fair what does that mean for the relative strength of the players? Obviously one has to solve the equation $g(0,j) = \frac{1}{2}$ with respect to $n = p/q$ for a given value of j. This leads to an algebraic equation in n. For $j = 2$, say, corresponding to the first example above, Bernoulli solves an equation of the 6th degree and finds $n = 1.946$.

Bernoulli also discusses the probability of winning a set of games. He remarks that because of notational difficulties he will only illustrate this problem by examples. However, his procedure is as usual very clear and easy to translate to modern notation. Let $s(u,v)$ denote A's probability of winning the set when A and B have already won u and v games, respectively. Bernoulli's procedure corresponds to the recursion formula

$$s(u,v) = g(0,0)s(u+1,v) + (1-g(0,0))s(u,v+1), \qquad (4)$$

which is analogous to (1) with $g(0,0)$ substituted for p. Bernoulli's difficulties stem from the fact that he does not have a short notation for the probabilities which we have denoted by $g(i,j)$ and $s(u,v)$.

Generalizing (4) to the case where the number of points is (i,j) in game (u,v) Bernoulli uses the formula

$$m(u,v;i,j) = g(i,j)s(u+1,v) + (1-g(i,j))s(u,v+1) \qquad (7)$$

to find A's probability of winning the set. Bernoulli uses these formulae to discuss the problem of handicaps. We shall report only one of his examples.

Suppose that B has a handicap of "half-45", which in our notation means that he in alternate games has a handicap of 2 and 3 points, respectively. The problem is to find the value of $n = p/q$ for which A's probability of winning equals $\frac{1}{2}$. Considering two games in succession A's probability of winning the first and the second, respectively, equals $g(0,2) = a/(a+b)$ and $g(0,3) = c/(c+d)$, the ratios being Bernoulli's notation. His reasoning may be illustrated by means of Fig. 2, where the states refer to the number of games.

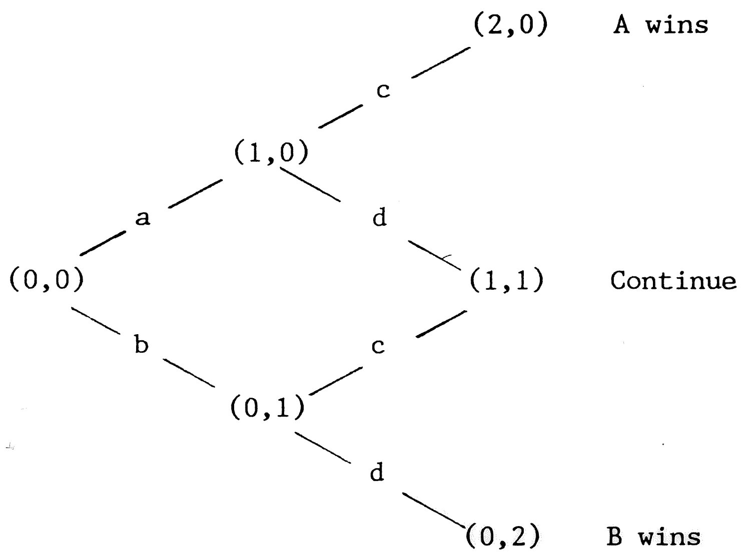

Fig. 2. The states of two games of tennis with the number of chances of winning and losing.

Let A's probability of winning in the state (0,0) and therefore also in the state (1,1) be denoted by z. By recursion Bernoulli finds that

$$\{a \frac{c+dz}{c+d} + b \frac{cz}{c+d}\}/(a+b) = z,$$

which leads to

$$z = ac/(ac+bd).$$

Setting $z = \frac{1}{2}$ Bernoulli finds that $a/b = d/c$ so that

$$g(0,2)/(1-g(0,2)) = (1-g(0,3))/g(0,3),$$

an equation of the 11th degree in n which according to Bernoulli has the root $n = 2.7$.

Bernoulli also extends his model by taking into account that a player when serving has a larger probability of winning a point than when he is not serving. Further, he discusses a game with three and four players.

The problem of points for the game of tennis is also discussed in the correspondence of Nicholas Bernoulli and Montmort (1713, pp. 333-334, 340-344, 349-350, 352-353, 371).

Nicholas Bernoulli, who knew the content of James Bernoulli's _Lettre_ before it was published in 1713, writes in a letter of the 10th November 1711 to Montmort that James has solved many interesting and useful problems on the game of tennis. He quotes four of the problems without giving the solutions or indicating James' method of solution and asks Montmort to solve the problems for comparison with James' solutions. In his answer of the 1st March 1712 Montmort does live up to the challenge; he gives an explicit formula for A's probability of winning when A lacks a points and B lacks b points.

Montmort considers the problem as a generalization of the classical problem of points. He first refers to the solution of this problem in terms of the binomial distribution and without further comment he notes that the corresponding formula for the problem of points under the rules valid for tennis becomes

$$e_t(a,b) = \sum_{i=a}^{a+b-2} \binom{a+b-2}{i} p^i q^{a+b-2-i} + \binom{a+b-2}{a-1} p^{a-1} q^{b-1} (p^2/(p^2+q^2)).$$

In the ordinary game of tennis $a = 4-i$ and $b = 4-j$, but the formula holds for any other number than 4.

It will be seen that Montmort's elegant result comprises all the formulae which Bernoulli laboriously derived by recursion since

$$g(i,j) = e_t(4-i,4-j).$$

Montmort says that $e_t(a,b)$ is obtained from $e(a,b)$ replacing $a+b-1$ by $a+b-2$ and multiplying the last term by $p^2/(p^2+q^2)$. This is also the proof of the formula since the first term gives A's probability of getting a points before B gets b-1 points and the last term gives the probability of winning after a deuce.

Noting that the first term of $e_t(a,b)$ equals $e(a,b-1)$ Montmort's formula may also be written in the form

$$e_t(a,b) = p^a \sum_{i=0}^{b-2} \binom{a-1+i}{i} q^i + \binom{a+b-2}{a-1} p^{a-1} q^{b-1} (p^2/(p^2+q^2)).$$

For example

$$e_t(4,4) = p^4(1 + 4q + 10q^2) + 20p^5q^3/(p^2+q^2).$$

Montmort does not mention this alternative form of his formula in the published part of his letter, but according to Henny (1975) the corresponding form of $e(a,b)$ is given in the letter. Presumably Montmort left it out of the letter and transferred it to his general discussion of the problem of points (1713, pp. 245-246) for systematic reasons.

To solve the problem of handicaps for a given relative strength of the players Montmort solves the equation $1-e_t(a,b) = \frac{1}{2}$ with respect to $m = a+b-2$ for given values of a and p/q. He considers the example with $a = 4$ and $p/q = 2$. First he solves the corresponding equation for the problem of points; as pointed out by Todhunter (1865, p. 125) the equation given by Montmort (1713, p. 342) is wrong but the root is correct so that he must have had the correct equation.For the game of tennis Montmort just gives the solution that B's handicap should be $j = 2\frac{11}{224}$ without giving the equation, which obviously is

$$1 + 2\binom{m}{1} + 4\binom{m}{2} + (8/5)\binom{m}{3} = 3^m/2 ,$$

or

$$8m^3 + 36m^2 + 16m + 30 = 5 \times 3^{m+1}$$

with the approximate solution $m = b+2 = 3\frac{213}{224}$.

As a check Montmort inserts $a = 4$ and $b = 1\frac{213}{224}$ in the equation $e_t(a,b) = \frac{1}{2}$ which leads to a homogeneous algebraic equation of the 6th degree in p and q from which he finds that $p/q = 2$.

As the handicap normally will lie between two integers Montmort remarks that the solution requires randomization to be carried out in practice. In the example above a chip is drawn from a bag with 213 white and 11 black chips, and if the chip drawn is white B gets a handicap of 2 points, if black only 1 point.

In his reply Nicholas Bernoulli acknowledges Montmort' solution and makes a further generalization. He assumes that A's probability of winning a point in odd and even numbered games equals p_1 and p_2, respectively, thereby taking into acccount that A's probability of winning depends on whether he is serving or not. Let $a+b-1 = m+n$, $m = n$ if $m+n$ is even and $m = n+1$ if $m+n$ is odd. Bernoulli then gives A's probability of winning for the problem of points as

$$\sum_{i=0}^{b-1} \sum_{j=0}^{i} \binom{m}{i-j} p_1^{m-i-j} q_1^{i-j} \binom{n}{j} p_2^{n-j} q_2^{j} .$$

He adds that for the game of tennis $m+n$ should be replaced by $m+n-1$ and that the term for $i = b-1$ should be multiplied by $p_1p_2/(p_1p_2+q_1q_2)$.

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