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ABSTRACT

In this paper we characterize simple transformation models by means of the functional form of the densities. We discuss sufficiency of the pair (t,π) where t is an equivariant estimator and π is a maximal invariant. Furthermore, we introduce and discuss the algebraic concept of structural sufficiency. This gives rise to an example of a simple transformation model where (t,π) is non-sufficient.

1. INTRODUCTION

In the analysis of statistical models it is sometimes convenient to make use of invariance properties of the model in question. For instance, the *invariance principle* (see Lehmann [20] or Hall et al [16]) is a widely accepted and frequently used statistical tool. Closely related to this concept is the notion of transformation models. Let E be a sample space, Θ a parameter set and G a group acting on E and Θ. In our set-up a transformation model is a family of probability measures $(P_\theta)_{\theta\in\Theta}$ with the property

(1.1) $$\forall\theta\in\Theta\ \forall g\in G:\ P_{g\theta} = gP_\theta .$$

Though much attention has been given to the study of particular transformation models (see e.g. Andersson et al [5], Andersson and Perlman [4], Eriksen [14] or Jensen [18]) a more general treatment of transformation models has only been given in some special cases (see e.g. Barndorff-Nielsen et al [8], Eaton [12], Eriksen [13], Fraser [15], Roy [22] and Rukhin [23]) using different set-ups. The aim of this paper is to introduce a basic set-up for general transformation models. In this set-up we will characterize the models (1.1) by means of their densities in the case where G acts transitively on Θ. Furthermore we will discuss the concept of unique maximum likelihood estimation. If $t: E\to\Theta$ is a MLE and π is a maximal invariant it is sometimes assumed that (t,π) is sufficient (see e.g. Barndorff-Nielsen [6],[7] and Barndorff-Nielsen et al [8]). We will give conditions ensuring (t,π) to be sufficient and, by a non-trivial example, show that (t,π) is indeed not always sufficient.

In this paper we will make some apparently harmless topological regularity assumptions. These assumptions are nevertheless strong enough to imply that the results, proofs etc. almost only depend on the algebraic structure of the groups and actions involved. We will rely heavily on the theory of invariant measures and group theory at a fairly elementary level. For an extensive exposition of the theory of invariant measures see Bourbaki [10] or Reiter [21]. For a more introductory exposition see Andersson [2]. In the theory of invariant measures the notion of a proper action appears naturally. For more comments on proper actions see Andersson [3] and Wijsman [25].

2. TRANSFORMATION MODELS: TWO APPROACHES

The definition (1.1) of transformation models goes back to, at least, Lehmann [20]. Recent treatments of transformation models see e.g. Barndorff-Nielsen [6],[7], Barndorff-Nielsen et al [8], Fraser [15], Jensen [18], Roy [22] and Rukhin [23] use a slightly different approach: let P_0 be a probability measure on E then $\mathcal{P} = \{gP_0 | g \in G\}$ is a (simple) transformation model, or more generally: an *invariant* family of probability measures $\mathcal{P}$, i.e. $P \in \mathcal{P}, g \in G \Rightarrow gP \in \mathcal{P}$, is called a (composite) transformation model. Now, if $\mathcal{P} = (P_\theta)_{\theta \in \Theta}$ satisfies (1.1) then it is an invariant family of probability measures and if G acts transitively on Θ then it is the form $\mathcal{P} = (P_{g\theta_0})_{g \in G} = (gP_{\theta_0})_{g \in G}$. Conversely, if $\mathcal{P}$ is an invariant family of probability measures we can parametrize $\mathcal{P}$ by itself $(\Theta = \mathcal{P})$ which obviously defines a transformation model as in (1.1). If $\mathcal{P} = \{gP_0 | g \in G\}$ we define $P_g = gP_0$ so $\mathcal{P} = (P_g)_{g \in G}$ is a transformation model in the sense of (1.1) (with $\Theta = G$). Note that if we let $K = G_{P_0} = \{g \in G | gP_0 = P_0\}$ and $P_{gK} = gP_0$ then $\mathcal{P} = (P_{gK})_{gK \in G/K}$ and this parametrization is one-to-one.

In a statistical context it seems to be most natural to use the concept defined by (1.1) since any statistical analysis is intimately connected with concepts such as parameter estimation, sufficiency, ancillarity etc. Using the other approach one is forced to introduce, say, G/K as a parameterset which seems to be both artificial and unsatisfactory.

3. PRELIMINARIES AND REGULARITY ASSUMPTIONS

In this section we will state the basic assumptions used throughout this paper. We will first introduce some notation.

Definition 3.1 A locally compact topological space (group) with a denumerable basis for the topology is called a *LCD space* (group). □

Remark A LCD space is in fact a locally compact Polish space so it is indeed σ-compact, metrizable with a complete metric and there exists a countable dense subset. □

Let, as usual, E denote the sample space, Θ the parameter set and G a group.

Assumption T G is a LCD group, Θ and E are LCD spaces. □

We will asume that G acts *continuously* on both E and Θ by

$$(3.1)\qquad \begin{array}{ccc} G\times E & \to & E \\ (g,x) & \to & \gamma(g)(x) = gx \end{array}$$

$$(3.2)\qquad \begin{array}{ccc} G\times \Theta & \to & \Theta \\ (g,\theta) & \to & \gamma(g)(\theta) = g\theta \end{array}$$

(Both actions being *leftactions*). The action (3.1) induces an action of G on $\Pi(E)$, the set of all probability measures on E, by

$$(3.3)\qquad \begin{array}{ccc} G\times \Pi(E) & \to & \Pi(E) \\ (g,P) & \to & \gamma(g)(P) = gP \end{array}$$

Sometimes we will consider sample spaces with a particular simple algebraic structure.

Definition 3.2 E is a *TT-space* if E is isomorphic and homeomorphic to a product space $E_1 \times E_2$ so that

(3.4) G acts *trivially* on E_1 i.e. $\forall g \in G \forall x_1 \in E_1 : gx_1 = x_1$

(3.5) G acts *transitively* on E_2 i.e. $E_2 = \{gx_2 | g \in G\}$. □

Remark TT is an abbreviation for Trivial, Transitive. □

Note that a TT-space E is a LCD space if and only if E_1 and E_2 are LCD spaces.

Proposition 3.1 Let $E \simeq E_1 \times E_2$ be a TT-space. Then

(3.6) E_2 is a *homogeneous* space

(3.7) E_1 is homeomorphic to $G \backslash E$.

(3.8) G acts properly on E (i.e. the mapping $(g,x) \to (x,gx)$ is proper) if and only if G acts *properly* on E_2.

(3.9) If μ is a relatively invariant measure on E with multiplier χ then $\mu \simeq \kappa \otimes \nu$ where ν is a relatively invariant measure on the homogeneous space E_2 with multiplier χ and κ is a measure on E_1. κ and ν are determined uniquely up to a norming factor.

Proof Omitted. □

Remark G is σ-compact so the action on E_2 is proper if and only if the isotropic groups $G_x = \{g \in G | gx = x\}$ are compact (see Bourbaki [10]). □

In general, if there is no risk of confusion, we will use π to denote an orbitprojection e.g. $\pi : E \to G \backslash E$, $\pi : E \times \Theta \to G \backslash E \times \Theta$ is respectively the orbitprojection under G's action on E and the orbitprojection under G's *diagonal* action on $E \times \Theta$ (i.e. $(g,(x,\theta)) \to (gx, g\theta)$). We will equip the orbitspaces with the finest topology making π continuous. If G acts properly then the orbitspace is a LCD space as well.

We will restrict our attention to families $\mathcal{P} = (P_\theta)_{\theta\in\Theta}$ which are particular nice:

<u>Assumption D</u> $\mathcal{P}$ is dominated by a *relatively* invariant measure μ with a continuous multiplier χ and $\mathrm{supp}(\mu) = E$. We can choose Radon-Nikodym derivates $f_\theta = \frac{dP_\theta}{d\mu}$ so that the mapping

$$(3.10) \qquad \begin{aligned} E \times \Theta &\to \mathbb{R}_+ \\ (x,\theta) &\to f_\theta(x) \end{aligned}$$

is continuous. □

Finally we will impose assumptions on the parameterset Θ as follows

<u>Assumption P</u>

(3.11) G acts *transitively* on Θ

(3.12) There exists a modulator for χ on Θ i.e. a strictly positive continuous function $m:\Theta \to \mathbb{R}_+$ with the property

$$\forall g \in G \forall \theta \in \Theta: \; m(g\theta) = \chi(g) m(\theta).$$ □

<u>Remark</u> (3.11) means that we only consider *simple* transformation models, i.e. Θ is a homogeneous space. If G acts *properly* on Θ then (3.12) is automatically fulfilled (see Bourbaki [10]); this is also the case if μ can be chosen to be invariant i.e. $\chi \equiv 1$. □

While Assumptions T and D seem quite harmless Assumption P is more restrictive. Some of the results in this paper hold under less restrictive assumptions but Assumption T, D and P were made to define a *basic* setup for transformation models.

4. CHARACTERIZATION OF TRANSFORMATION MODELS

First, we need an easy but fundamental lemma.

Lemma 4.1 $(P_\theta)_{\theta\in\Theta}, P_\theta = f_\theta\mu$, is a transformation model if and only if

(4.1) $\quad \forall\theta\in\Theta\forall g\in G\forall x\in E : f_\theta(x) = f_{g\theta}(gx)\chi(g)$

Proof

$$gP_\theta = \gamma(g)(f_\theta\mu) = \gamma(g)f_\theta\ \gamma(g)(\mu) = \gamma(g)f_\theta\ \chi(g)^{-1}\mu$$

so (1.1) is satisfied if and only if $f_{g\theta}(x) = f_\theta(g^{-1}x)\chi(g)^{-1}$ which is equivalent to (4.1). □

The following theorem gives the basic structure of transformation models.

Theorem 4.1 If $(P_\theta)_{\theta\in\Theta}, P_\theta = f_\theta\mu$, is a tranformation model then there exists a continuous function $p : {}_G\backslash^{E\times\Theta} \to \mathbb{R}_+$ with

(4.2) $\quad \forall\theta\in\Theta\forall g\in G\forall x\in E : f_\theta(x) = p(\pi(x,\theta))/m(\theta)$

(4.3) $\quad \forall\theta\in\Theta : {}_E\int p(\pi(x,\theta))d\mu(x) = m(\theta)$.

On the other hand, if $p : {}_G\backslash^{E\times\Theta} \to \mathbb{R}_+$ is a continuous function so that

(4.4) $\quad {}_E\int p(\pi(x,\theta))d\mu(x) < +\infty$

then, possibly after a normalization of p, (4.2) defines a transformation model.

Proof Let $(P_\theta)_{\theta\in\Theta}$ be a transformation model. Lemma 4.1 shows that $f_\theta(x) = f_{g\theta}(gx)\chi(g) = f_{g\theta}(gx)m(g\theta)/_{m(\theta)}$ so the mapping $\psi : E\times\Theta \to \mathbb{R}_+$ defined by $\psi(x,\theta) = f_\theta(x)m(\theta)$ is invariant under the diagonal action of G on $E\times\Theta$ i.e. ψ factorizes through the orbitprojection π, $\psi = p\circ\pi$, where p is continuous. This establishes (4.2). (4.3) is trivial. On the other hand, if $p : {}_G\backslash^{E\times\Theta} \to \mathbb{R}_+$ is G continuous function f_θ's defined by (4.2) obviously satisfy (4.1) and

$\int f_{g\theta}(x)d\mu(x) = \int f_\theta(g^{-1}x)\chi(g^{-1})d\mu(x) = \int f_\theta(x)d\mu(x)$ which shows that p can be normalized to make $(f_\theta\mu)_{\theta\in\Theta}$ a transformation model. □

Definition 4.1 If $(P_\theta)_{\theta\in\Theta}$ is a transformation model we will denote p *the associated modelfunction.* □

If G acts properly on Θ (4.3) and (4.4) can be formulated in a more natural way. If G acts properly on Θ then G acts properly on $E\times\Theta$ (see Bourbaki [9], Ch.3, §4, exercise 10, c) i.e. $G\backslash^{E\times\Theta}$ is locally compact and the orbitprojection $\pi : E\times\Theta \to G\backslash^{E\times\Theta}$ is proper. π_θ is a composition of the two proper mappings $x \to (x,\theta)$ and π and hence a proper mapping. Therefore $\pi_\theta(\mu)$ is a well-defined measure on $G\backslash^{E\times\Theta}$ and the $\pi_\theta(\mu)$'s are proportional because

$$\text{(4.5)} \qquad \pi_{g\theta}(\mu) = \pi_\theta \circ \gamma(g^{-1})(\mu) = \chi(g)\pi_\theta(\mu)$$

so (4.3) and (4.4) can be reformulated as

$$\text{(4.3')} \qquad \forall\theta\in\Theta : \int_{G\backslash^{E\times\Theta}} p d\pi_\theta(\mu) = m(\theta)$$

$$\text{(4.4')} \qquad \exists\theta\in\Theta : \int_{G\backslash^{E\times\Theta}} p d\pi_\theta(\mu) < +\infty \ .$$

If E is a TT-space $G\backslash^{E\times\Theta}$ can be represented in a particularly nice way. Fix $\theta_0\in\Theta$ and set $L = G_{\theta_0} = \{g\in G | g\theta_0 = \theta_0\}$.

Proposition 4.2 If $E = E_1\times E_2$ is a TT-space $G\backslash^{E\times\Theta}$ is homeomorphic to $L\backslash^{E}(= E_1 \times {}_L\backslash^{E}2)$. ($_L\backslash^E$ denotes the orbit space under L's action on E).

Proof Since E_2 and Θ are homogeneous spaces it is obviously enough to show that, say, $G\backslash^{G/K\times G/L}$ is homeomorphic to $L\backslash^{G/K}$ where K and L are subgroups of G. Define

$$\text{(4.6)} \qquad \psi : G/K\times G/L \to L\backslash^{G/K},\ \psi(gK,\tilde{g}L) = L\tilde{g}^{-1}gK$$

ψ is easily seen to be well defined, invariant, onto and continuous (using the relevant quotient topologies). To see that ψ is maximal invariant let $g,\tilde{g},h,\tilde{h}\in G$ with $\psi(gK,\tilde{g}L)=\psi(hK,\tilde{h}L)$. Then $L\tilde{g}^{-1}gK=L\tilde{h}^{-1}hK \Leftrightarrow \tilde{g}^{-1}g\in L\tilde{h}^{-1}hK$ i.e. $\exists \ell\in L\ \exists k\in K$ with $\tilde{g}^{-1}g=\ell\tilde{h}^{-1}hk$. This implies

(4.7) $\qquad gK=\tilde{g}\ell\tilde{h}^{-1}hkK=\tilde{g}\ell\tilde{h}^{-1}hK$

(4.8) $\qquad \tilde{g}L=\tilde{g}\ell\tilde{h}^{-1}\tilde{h}\ell^{-1}L=\tilde{g}\ell\tilde{h}^{-1}\tilde{h}L$

showing that $(gK,\tilde{g}L)\underset{G}{\sim}(hK,\tilde{h}L)$ and hence that ψ is maximal invariant. To see that ${}_{L}\backslash^{G/K}$ and ${}_{G}\backslash^{G/K\times G/L}$ are homeomorphic it remains to show that the mapping $LgK\to\pi(gK,L)$ is continuous but this is trivial. □

Remark Fix $x_0\in E$ and set $K=G_{x_0}$. By symmetry we have ${}_{G}\backslash^{E\times\Theta}\simeq E_1\times{}_{K}\backslash^{\Theta}$. This is a useful observation. □

Now we can formulate Theorem 4.1 for TT-spaces.

Theorem 4.2 Assume that G acts properly on Θ and that $E\simeq E_1\times E_2$ is a TT-space. Fix $\theta_0\in\Theta$ and set $L=G_{\theta_0}$. If $(P_\theta)_{\theta\in\Theta}, P_\theta=f_\theta\mu$, $\mu=\kappa\otimes\nu$, is a transformation model then there exists a continuous function $p:E_1\times{}_{L}\backslash^{E_2}\to\mathbb{R}_+$ with

(4.9) $\qquad f_{g\theta_0}(x_1,x_2)=p(x_1,Lg^{-1}x_2)/\chi(g)$

(4.10) $\qquad \int p\,d\kappa\otimes\pi_L(\nu) \quad <+\infty$

($\pi_L: E_2\to{}_{L}\backslash^{E_2}$ is the orbitprojection). On the other hand, if $p:E_1\times{}_{L}\backslash^{E_2}\to\mathbb{R}_+$ is continuous fulfilling (4.10) then, possibly after a normalization of p, (4.9) defines a transformation model. □

Remark Under the assumptions in the theorem one can construct transformation models ad libitum as soon as $L=G_{\theta_0}$, $L\backslash^{E_2}$ and $\pi_L(\nu)$ have been identified.

□

We will comment a little bit on unique maximum likelihood estimation. First, a well known result:

Proposition 4.3 If $(P_\theta)_{\theta\in\Theta}$ is a transformation model admitting unique maximum likelihood estimation then the maximum likelihood estimator (MLE) $t : E \to \Theta$ is equivariant i.e.

(4.11) $\quad \forall x \in E\ \forall g \in G: \quad t(gx) = gt(x).$ □

If t is equivariant we have $G_x \subseteq G_{t(x)}$ so if G acts properly on Θ the G_x's are compact so if E is a TT-space then G acts properly on E. In this situation it is no restriction to assume that $\mu = \kappa \otimes \nu$ is *invariant*. Let $(P_\theta)_{\theta\in\Theta}$ be a transformation model. Fix $\tilde{x}_2 \in E_2$ and set $K = G_{\tilde{x}_2}$. According to Theorem 4.2 and the remark to Proposition 4.2 the densities have the form $f_\theta(x_1, g\tilde{x}_2) = p(x_1, Kg^{-1}\theta)$ where $p : E_1 \times {}_K\backslash^{\Theta} \to \mathbb{R}_+$ is continuous.

Proposition 4.4 $(P_\theta)_{\theta\in\Theta}$ admits unique maximum likelihood estimation if and only if - for each $x_1 \in E_1$ - the mapping $K\theta \to p(x_1, K\theta)$ has a unique maximum at, say, $K\tilde{\theta}(x_1)$ with $K\tilde{\theta}(x_1)$ *degenerate* i.e. $K\tilde{\theta}(x_1) = \{\tilde{\theta}(x_1)\}$.

Proof Straight forward. □

We will close this section with some applications of Theorem 4.2 and Proposition 4.4.

Example 4.1 (Multivariate location- and scaleparameter models)
Take $E = \mathbb{R}^d$, $\Theta = H^+(d) \times \mathbb{R}^d$ and $G = AG(d)$. $AG(d) = \{[A,\alpha] \mid A \in GL(d), \alpha \in \mathbb{R}^d\}$ is the affine group of order d and $H^+(d)$ is the set of positive definite $p \times p$-matrices. The composition rule in $AG(d)$ is defined as follows $[A,\alpha][B,\beta] = [AB, A\beta + \alpha]$, $[A,\alpha]^{-1} = [A^{-1}, -A^{-1}\alpha]$ the unity being $[I,0]$. The actions are given by

$$(4.12)\qquad \begin{aligned} AG(d) \times \mathbb{R}^d &\to \mathbb{R}^d \\ ([A,\alpha],x) &\to Ax+\alpha \end{aligned}$$

$$(4.13)\qquad \begin{aligned} AG(d)\ x(\mathcal{H}^+(d) \times \mathbb{R}^d) &\to \mathcal{H}^+(d) \times \mathbb{R}^d \\ ([A,\alpha],(\Sigma,\xi)) &\to (A\Sigma A^*, A\xi+\alpha) \end{aligned}$$

$\theta = (\Sigma,\xi)$ should be thought of as the covariance and the mean respectively. Both actions are transitive, (4.13) is proper whereas (4.12) is non-proper. There exists no invariant measure on $\mathbb{R}^d$ under $AG(d)$ but taking μ as Lebesgue measure μ is relatively invariant with multiplier $\chi(A,\alpha) = |\det(A)|$ and $m:\mathcal{H}^+(d) \times \mathbb{R}^d \to \mathbb{R}_+$, $m(\Sigma,\xi) = \det(\Sigma)^{\frac{1}{2}}$ is a modulator. We are thus covered by Theorem 4.2. Take $\theta_0 = (I,0)$. We thus have to identify $L = G_{\theta_0}$, $L\backslash\mathbb{R}^d$, the mapping $(x,g\theta_0) \to Lg^{-1}x$ and finally the measure $\pi_L(\mu)$. Now, $L \simeq \mathcal{O}(d)$ and $\mathcal{O}(d)\backslash\mathbb{R}^d$ is well known to be homeomorphic to $[0,+\infty[$ via the identification $Lx \simeq \|x\|^2 = x^*x$. Let $(\Sigma,\xi) = (AA^*,\xi) = [A,\xi](I,0) \in \Theta$ then $L[A,\xi]^{-1}x \sim \|[A,\xi]^{-1}x\|^2 = \|A^{-1}x - A^{-1}\xi\|^2 = (x-\xi)^*(A^{-1})^*A^{-1}(x-\xi) = (x-\xi)^*\Sigma^{-1}(x-\xi)$. It thus remains to identify $\pi_L(\mu)$. Letting γ denote the left-translation on the group $(\mathbb{R}_+,\cdot)$ we see that $\gamma(s^{-1})\ \pi_L(\mu) = s^{d/2}\pi_L(\mu)$ so $\pi_L(\mu)$ is relatively invariant with multiplier $s \to s^{d/2}$ and hence having density $s^{\frac{d}{2}-1}$ w.r.t. Lebesgue measure on $\mathbb{R}_+$.

We can thus conclude that the transformation models on $\mathbb{R}^d$ with parameterset $\mathcal{H}^+(d) \times \mathbb{R}^d$ are exactly those of the form $P_{\Sigma,\xi} = f_{\Sigma,\xi}\mu$, μ Lebesgue measure, where

$$(4.14)\qquad f_{\Sigma,\xi}(x) = p((x-\xi)^*\Sigma^{-1}(x-\xi))/\det(\Sigma)^{\frac{1}{2}}$$

and $p : [0,+\infty[\to \mathbb{R}_+$ is a continuous function with

$$(4.15)\qquad \int_0^\infty p(s)s^{\frac{d}{2}-1}\, ds < +\infty .$$

This is a well known result (see e.g. Kelker [19]) and distributions with densities of the form (4.14) are called *elliptic distributions*. Note finally that if $(P_{\Sigma,\xi})$ is a statistical model parametrized by the covariance and the mean then it is a transformation model under the affine group and hence of the form (4.14). Conversely, it is possible to show that if $(P_{\Sigma,\xi})$ is a transformation model with finite expectation and covariance then the expectation equals ξ and the covariance is proportional to Σ. □

Example 4.2 Take $E = \Theta = H^+(d)$ and $G = GL(d)$ the general linear group of order d. The action is given by

$$(4.16)\qquad \begin{aligned} GL(d) \times H^+(d) &\to H^+(d) \\ (A,\Sigma) &\to A\Sigma A^* . \end{aligned}$$

This action is transitive and proper, the invariant measure on $H^+(d)$ has density $S \to (\det S)^{-\frac{1}{2}(d+1)}$ w.r.t. Lebesgue measure on $H^+(d)$. We are thus covered by Theorem 4.2. Take $\theta_0 = I$ then $L = G_I = O(d)$ and $O(d)\backslash H^+(d)$ can be represented by $\Lambda_d = \{(\lambda_1,\ldots,\lambda_d) \in \mathbb{R}^d \mid \lambda_1 \geqq \ldots \geqq \lambda_d > 0\}$ using the identification $O(d)S \simeq$ "the vector of ordered eigenvalues of S". (see e.g. Bourbaki [9]). Let $\Sigma = AA^* \in H^+(d)$ then $O(d)A^{-1}S(A^{-1})^* \simeq$ "the vector of ordered eigenvalues of S w.r.t. Σ" which we will denote $E(S;\Sigma)$. According to Anderson [1], Theorem 3.3 , $\pi_L(\mu)$ has density w.r.t. Lebesgue measure on Λ_d and the density is given by

$$(4.17)\qquad \delta(\lambda_1,\ldots,\lambda_d) = \prod_{i=1}^{d} \lambda_i^{-\frac{1}{2}(d+1)} \prod_{i<j} (\lambda_i - \lambda_j).$$

We can thus conclude that the transformation models on $H^+(d)$ are those of the form $P_\Sigma = f_\Sigma \mu, \mu$ Lebesgue measure on $H^+(d)$, with

$$(4.18)\qquad f_\Sigma(S) = p(E(S;\Sigma))(\det S)^{-\frac{1}{2}(d+1)}$$

where $p: \Lambda_d \to \mathbb{R}_+$ is a continuous function with

(4.19) $$\int_{\Lambda_d} p(\lambda_1,\dots,\lambda_d) \prod_{i=1}^{d} \lambda_i^{-\frac{1}{2}(d+1)} \prod_{1\leqq i<j\leqq d} (\lambda_i - \lambda_j) d(\lambda_1,\dots,\lambda_d) < +\infty .$$

Since the only degenerate $\mathcal{O}(d)$ orbits of $H^+(d)$ are those corresponding to $\lambda I_d, \lambda > 0$, (see Lemma 4.2 below) we get, according to Proposition 4.3, that $(P_\Sigma)_{\Sigma\in H^+(d)}$ admits unique maximum likelihood estimation if and only if the associated modelfunction p has a unique maximum at a point of the form $(\lambda,\dots,\lambda) \in \Lambda_d$ and the MLE is then given by $t(S) = \lambda S$. Letting

(4.20) $$p(\lambda_1,\dots,\lambda_d) = \prod_{i=1}^{d} \lambda_i^{\frac{1}{2}m} e^{-\frac{1}{2}\lambda_i}, \quad m \geqq d$$

we see that p has an unique maximum at $(m,\dots,m)$ and p satisfies (4.19) so p is the associated modelfunction of a transformation model with unique MLE $t(S) = mS$ - namely the d-dimensional Wishart distribution with m degrees of freedom and unknown parameter Σ. □

Lemma 4.2 Consider the action of $GL(d)$ on $H^+(d)$ in (4.16). If $\mathcal{O}(d) \subseteqq G_\Sigma$ then Σ is of the form λI_d, $\lambda > 0$.

Proof Assume that $\mathcal{O}(d) \subseteqq G_\Sigma$. The action (4.16) is transitive so $\Sigma = AA^*$ for a $A \in GL(d)$ i.e. $\mathcal{O}(d) \subseteqq A\mathcal{O}(d)A^{-1}$. Now, $\mathcal{O}(d)$ is a maximal compact subgroup of $GL(p)$ (see e.g. Bourbaki [10] and $A\mathcal{O}(d)A^{-1}$ is compact so $\mathcal{O}(d) = A\mathcal{O}(d)A^{-1}$ (this can also be seen using Proposition 5.5 in Section 5). This implies

$$\forall\, U \in \mathcal{O}(d) : AUA^{-1} \in \mathcal{O}(d)$$

$$\Leftrightarrow$$

$$\forall\, U \in \mathcal{O}(d) : (AUA^{-1})^{-1} = (AUA^{-1})^*$$

$$\Leftrightarrow$$

$$\forall\, U \in \mathcal{O}(d) : AUA^{-1} = (A^{-1})^* UA^*$$

$$\Leftrightarrow$$

(4.21) $$\forall\, U \in \mathcal{O}(d) : A^*A = UA^*AU^*$$

Now, $A^*A \in H^+(d)$ so there exists an orthogonal matrix U with UA^*AU^* diagonal i.e. A^*A is diagonal by (4.21). Let $A^*A = \mathrm{diag}(\lambda_1,\dots,\lambda_d)$. Letting

$U = \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$ we get $UA^*AU^* = \mathrm{diag}(\lambda_2, \lambda_1, \ldots, \lambda_d)$ so, again by (4.21), $\lambda_1 = \lambda_2$.

Repeating this argument we get $A^*A = \lambda I_d$ but $\Sigma = AA^* = A(A^*A)A^{-1} = \lambda I_d$. □

<u>Remark</u> This lemma in fact shows that the only equivariant mappings $t : H^+(d) \to H^+(d)$ are those of the form $t(S) = \lambda S$. □

<u>Example 4.3</u> <u>(Transformation models on the unithyperboloid)</u>
Let $\underline{\underline{\phi}}_d = \mathrm{diag}(1,-1,-1,\ldots,-1)$ be a $d \times d$ matrix and let ϕ_d denote the corresponding bilinearform on $\mathbb{R}^d$. The unithyperboloid is defined as $H_d = \{(x_1,\ldots,x_d)^* \in \mathbb{R}^d \mid x_1 > 0, \phi_d(x,x) = 1\}$ and the group of hyperbolic transformations is $SH_d = \{A \in GL(d) \mid a_{11} > 0, \det(A) = 1, A^*\underline{\underline{\phi}}_d A = \underline{\underline{\phi}}_d\}$. SH_d acts transitively and properly on H_d by

$$(4.22) \qquad \begin{aligned} &SH_d \times H_d \to H_d \\ &(A,x) \to Ax \ \text{(matrix multiplication)} \end{aligned}$$

(see Vilenkin [24] or Jensen [18]). The invariant measure μ is given by

$$(4.23) \qquad \mu(C) = \lambda_d(\{\underline{x} \in \mathbb{R}^d \mid 0 < \phi_d(\underline{x},\underline{x}) \leqq 1, x_1 > 0, \frac{x}{\sqrt{\phi_d(\underline{x},\underline{x})}} \in C\})$$

for C a compact subset of H_d.

We will consider transformation models with $E = \Theta = H_d$ and $G = SH_d$ for $d \geqq 3$. The above considerations imply that we are covered by Theorem 4.2. Let $\theta_0 = (1,0,\ldots,0)^* \in H_d$, then $L = \{\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \mid A \in SO(d-1)\}$ where $SO(d-1)$ is the special orthogonal group of order $d-1$. One can readily check that $\underline{x} \underset{L}{\sim} \underline{y}$ if and only if $x_1 = y_1$ since $SO(d-1)$ acts transitively on every sphere in $\mathbb{R}^{d-1}$. Therefore $L \backslash H_d$ can be identified with $[1,+\infty[$ using the identification $L\underline{x} \sim x_1 = \phi_d(x,\theta_0)$. If $\theta = A\theta_0$ with $A \in SH_d$ then $LA^{-1}\underline{x} \sim \phi_d(A^{-1}x,\theta_0) = (A^{-1}x)^* \underline{\underline{\phi}}_d\theta_0 = x^*(A^{-1})^* \underline{\underline{\phi}}_d\theta_0 = x^*\underline{\underline{\phi}}_d A\theta_0 = x^*\underline{\underline{\phi}}_d\theta = \underline{\underline{\phi}}_d(x,\theta)$ which shows that the transformation models have the form, $P_\theta = f_\theta \mu$

(4.24) $\quad f_\theta(x) = p(\phi(x,\theta))$

where $p : [1,+\infty[$ is a continuous function. We will now identify the measure $\pi_L(\mu)$. For $t > 1$ we find - using Fubini's theorem -

$$\pi_L(\mu)([1,t]) = \lambda_d(\{x \in \mathbb{R}^d \mid 0 < x_1^2 - \ldots - x_d^2 \leqq 1, x_1 > 0, x_1^2 \leqq t^2(x_1^2 - \ldots - x_d^2)\})$$

$$= \lambda_d(\{x \in \mathbb{R}^d \mid x_1^2 - 1 \leqq x_2^2 + \ldots + x_d^2 < x_1^2, x_2^2 + \ldots + x_d^2 \leqq (1 - \frac{1}{t^2})x_1^2, x_1 > 0\})$$

$$= \int_0^1 \lambda_{d-1}(\{x \in \mathbb{R}^{d-1} \mid 0 \leqq x_1^2 + \ldots + x_{d-1}^2 \leqq (1 - \frac{1}{t^2})y^2\})dy$$

$$+ \int_1^t \lambda_{d-1}(\{x \in \mathbb{R}^{d-1} \mid y^2 - 1 \leqq x_1^2 + \ldots + x_{d-1}^2 \leqq (1 - \frac{1}{t^2})y^2\})dy$$

$$= c\left[\int_0^t (1 - \frac{1}{t^2})^{\frac{d-1}{2}} y^{d-1} dy - \int_1^t (y^2 - 1)^{\frac{d-1}{2}} dy\right]$$

where c is a constant depending on d. This shows that $\pi_L(\mu)$ has density with respect to Lebesgue measure on $[1,+\infty[$ given by

(4.25) $\quad \delta(t) = \frac{\partial}{\partial t}\pi_L(\mu)([1,t]) = c\,\frac{d-1}{d}(t^2-1)^{\frac{d-3}{2}}.$

This means that the model functions in (4.24) have to satisfy

(4.26) $\quad \int_1^\infty p(s)\, s^{d-3}\, ds < +\infty.$ $\square$

5. STRUCTURAL SUFFICIENCY

Let $(P_\theta)_{\theta\in\Theta}$ be a transformation model admitting unique maximum likelihood estimation, $t : E \to \Theta$. In this section we will discuss sufficiency of the pair $(t,\pi) : E \to \Theta \times {}_G\backslash^E$. Assume that (t,π) is sufficient. For a moment we will ignore problems with null-sets, continuity, measurability etc. According to Neymann's theorem $f_\theta(x) = a_\theta(t(x),\pi(x))b(x)$. Then

$$(5.1) \qquad f_\theta(x) = \frac{f_\theta(x)}{f_{t(x)}(x)} f_{t(x)}(x) = \frac{a_\theta(t(x),\pi(x))}{a_{t(x)}(t(x),\pi(x))} f_{t(x)}(x).$$

Now, $f_{t(gx)}(gx)m(t(gx)) = f_{gt(x)}(gx)m(gt(x)) = f_{t(x)}(x)\chi(g)^{-1}\chi(g)m(t(x)) =$ $f_{t(x)}(x)m(t(x))$ (according to (4.1)) so $f_{t(x)}(x)$ is of the form $g(\pi(x))/m(t(x))$ which inserted in (5.1) gives

$$(5.2) \qquad f_\theta(x) = \frac{a_\theta(t(x),\pi(x))}{a_{t(x)}(t(x),\pi(x))} \frac{g(\pi(x))}{m(t(x))}$$

showing that the density factorizes through (t,π). This fact together with the structure theorem in Section 4 should motivate the following definition.

Definition 5.1 Let $t : E \to \Theta$ be an equivariant mapping, $\pi : E \to {}_G\backslash^E$ the orbit-projection. (t,π) is *structural sufficient* if - for each $\theta \in \Theta$ - the mapping $\pi_\theta : E \to {}_G\backslash^{E\times\Theta}$, $\pi_\theta(x) = \pi(x,\theta)$, factorizes through (t,π).

Remark If (t,π) is structural sufficient it is in fact a sufficient reduction in all transformation models.

We can give a simple necessary and sufficient condition for structural sufficiency.

Proposition 5.1 (t,π) is structurally sufficient if and only if

$$(5.3) \qquad \forall\theta \in \Theta \ \forall x \in E : G_{t(x)} \subseteqq G_\theta G_x$$

Proof (t,π) is structurally sufficient if and only if

$$\forall\theta\in\Theta\ \forall g\in G\ \forall x\in E : t(gx) = t(x) \Rightarrow \pi_\theta(x) = \pi_\theta(gx)$$

$$\Leftrightarrow$$

$$\forall\theta\in\Theta\ \forall g\in G\ \forall x\in E : g\in G_{t(x)} \Rightarrow [\exists h\in G : h\theta = \theta, hx = gx]$$

$$\Leftrightarrow$$

$$\forall\theta\in\Theta\ \forall g\in G\ \forall x\in E : g\in G_{t(x)} \Rightarrow [\exists h\in G_\theta : h^{-1}g\in G_x]$$

$$\Leftrightarrow$$

$$\forall\theta\in\Theta\ \forall g\in G\ \forall x\in E : g\in G_{t(x)} \Rightarrow g\in G_\theta G_x$$

which is exactly (5.3). □

Remark t is equivariant so $G_x \subseteqq G_{t(x)}$. (5.3) says that even though $G_{t(x)}$ is larger than G_x it should not be to large. □

Corollary 5.1 If the G_θ's are *normal* subgroups of G then (t,π) is structurally sufficient.

Proof If the G_θ's are normal then they are all equal so $G_{t(x)} = G_\theta \subseteqq G_\theta G_x$. □

Corollary 5.2 If G acts freely on E, i.e. $G_x = \{e\}\ \forall x$, then (t,π) is structurally sufficient if and only if the G_θ's are normal subgroups of G.

Proof If $G_x = \{e\}$ (5.3) reads $\forall x\in E\ \forall\theta\in\Theta : G_{t(x)} \subseteqq G_\theta$ which is equivalent to $\forall g\in G\ \forall\theta\in G_\theta : gG_\theta g^{-1} \subseteqq G_\theta$. □

We will now introduce (see e.g. Barndorff-Nielsen [6],[7])

Definition 5.2 E and Θ are *of the same orbittype* if the G_x's and G_θ's are conjugates of one another i.e. $\forall x\in E\ \forall\theta\in\Theta\ \exists g\in G : G_x = gG_\theta g^{-1}$. □

Remark If E is a TT-space and E_2 is isomorphic to Θ then E and Θ are of the same orbittype. □

In the rest of this section we will assume that E and Θ are of the same orbittype. In this case the concept of structural sufficiency turns out to be rather trivial.

Proposition 5.2 (t,π) is structurally sufficient if and only if (t,π) is one-to-one and onto.

Proof Assume that (t,π) is structurally sufficient. Let $x \in E$ then $G_x \subseteq G_{t(x)}$ and we can find θ with $G_\theta = G_x$. According to (5.3) $G_x \subseteq G_{t(x)} \subseteq G_\theta G_x = G_x$ so $G_x = G_{t(x)}$ showing that (t,π) is one-to-one. (t,π) is obviously onto. □

The above proposition motivates the following definition.

Definition 5.3 A subgroup $H \subseteq G$ is *regular* if

(5.4) $\forall g \in G : \ H \subseteq gHg^{-1} \Rightarrow H = gHg^{-1}$. □

Remark If H is regular any conjugate group gHg^{-1} is regular. □

We then obtain

Proposition 5.3 If the G_θ's are regular then (t,π) is structurally sufficient i.e. one-to-one and onto.

Proof Let $x \in E$ and choose $\theta \in \Theta$ with $G_x = G_\theta$. Now $t(x)$ is of the form $g\theta$ so $G_\theta = G_x \subseteq G_{t(x)} = gG_\theta g^{-1}$ which by the regularity of G_θ implies $G_x = G_{t(x)}$. □

This suggests a study of the concept of regularity. The following proposition is easily proved.

Proposition 5.4

(5.5) A normal subgroup is regular.

(5.6) A maximally compact subgroup is regular.

Proof Omitted. □

Example 5.1 Consider example 4.2. We then have $E = \Theta = \mathcal{H}^+(d)$ so E and Θ are of the same orbittype. Now, $G_I = \mathcal{O}(d)$ which is known to be maximally compact so by (5.6) it is regular and by Proposition 5.3 we see that t has to be one-to-one and onto. This is in accordance with example 4.2 in which we showed that $t(S) = \lambda S$ for some $\lambda > 0$. □

We will now state a widely applicable result.

Proposition 5.5 Every compact subgroup of a Lie group of non-zero dimension is regular. □

For the notion of Lie groups see e.g. Bourbaki [11] or Hochschild [17]. The proposition is an easy corollary of the following result.

Lemma 5.1 Let H be a compact Lie group of non-zero dimension. If $\phi : H \to H$ is a continuous, injective homomorphism then ϕ is onto.

Proof Let H_e denote the connected component containing e. H_e is a closed, normal subgroup of H. Since ϕ is a continuous homomorphism $\phi(H_e) \subseteqq H_e$. Let $L(H_e)$ denote the Lie algebra associated with H_e. Then ϕ in a canonical way induces an algebra homomorphism $L(\phi) : L(H_e) \to L(H_e)$. ϕ being one-to-one implies that $L(\phi)$ is one-to-one (see Bourbaki [11], Ch.III, §6). $L(H_e)$ is finite dimensional so $L(\phi)$ is onto i.e. $L(\phi)(L(H_e)) = L(H_e)$. According to Bourbaki [11], Ch.III, §6 we then have $H_e = \phi(H_e)$. Since H is locally connected H_e is open so H being compact implies that H/H_e is finite. ϕ defines in a canonical way a mapping $\overline{\phi} : {}^{H}/_{H_e} \to {}^{H}/_{H_e}$ by $\overline{\phi}(hH_e) = \phi(h)H_e$. $\overline{\phi}$ is easily seen to be one-to-one so the finiteness of ${}^{H}/_{H_e}$ then imply that $\overline{\phi}$ is onto as well. Let $h \in H$, choose $\tilde{h} \in H$ with $\overline{\phi}(\tilde{h}H_e) = hH_e$ i.e. $\phi(\tilde{h})H_e = hH_e$. Choose now $k \in H_e$ with $\phi(\tilde{h}) = hk$ and $\tilde{k} \in H_e$ with $\phi(\tilde{k}) = k^{-1}$. Then $\phi(\widetilde{hk}) = hkk^{-1} = h$ showing that ϕ is onto. □

Proof of Proposition 5.5 Assume that H is a compact subgroup of G with $gHg^{-1} \subseteqq H$. Define $\phi : H \to H$ by $\phi(k) = ghg^{-1}$. Now, ϕ is a continuous, injective homomorphism and H is a compact Lie group so by the lemma we indeed have that ϕ is onto i.e. $gHg^{-1} = \phi(H) = H$. □

Remark It is not true that every *closed* subgroup of a Lie group of non-zero dimension is regular. □

We will finally state a result for TT-space.

Proposition 5.6 Let E be a TT-space. (t,π) is structurally sufficient for *all* equivariant mappings $t: E \to \Theta$ if and only if the G_θ's are regular. □

We will close this section with an example of a transformation model which admits unique maximum likelihood estimation t with (t,π) non-sufficient.

Example 5.2 Introduce $M = \{((x_k)_{k=N+1}^{\infty};N) \mid N \in \mathbb{Z}, x_k \in \{0,1\}, k = N+1, N+2, \ldots\}$. We equip M with the topology making $\iota: \{0,1\}^{\mathbb{N}} \times \mathbb{Z} \to M$, $\iota((X_k)_{k=1}^{\infty};N) = ((X_{k-N})_{k=N+1}^{\infty};N)$, a homeomorphism. Let $G = \{[\phi,a] \mid \phi \in \{0,1\}^{\mathbb{Z}}, a \in \mathbb{Z}\}$, G is the semiproduct of $\{0,1\}^{\mathbb{Z}}$ and $\mathbb{Z}$, with composition rule

$$(5.7) \qquad [\phi,a][\psi,b] = [\phi(a\psi), a+b]$$

where $(a\psi)(k) = \psi(k-a)$ and $(\phi\psi)(k) = \phi(k)\psi(k)$, where the unit is $(\underline{0},0)$ and the inverse is given by

$$(5.8) \qquad [\phi,a]^{-1} = [(-a)\phi, -a]$$

G acts on M by

$$(5.9) \qquad \begin{array}{ccc} G \times M & \to & M \\ ([\phi,a],((X_k)_{k=N+1}^{\infty};N)) & \to & ((\phi(k)X_{k-a})_{k=a+N+1}^{\infty};a+N) \end{array}$$

(5.9) is transitive and proper. The invariant measure on M is given by $\mu = (\bigotimes_{i=1}^{\infty} \mu_i) \otimes \tau$ where $\mu_i(\{0\}) = \mu_i(\{1\}) = \frac{1}{2}$ and τ is counting measure on Z. Notice that G is an LCD group, M is an LCD space and the isotropic group for $(\underline{0},0)$ is $G_{(\underline{0},0)} = K = \{[\phi,0] \mid \phi(k) = 0 \ \forall k > 0\}$ which is homeomorphic to $\{0,1\}^{\mathbb{N}}$ and hence compact but it is *non-regular*. Define $E = \Theta = M$. We will introduce a transformation model on E with parameterset Θ. Let $(p_k)_{k=1}^{\infty}$, $p_k \in]0,1[$ be known reals. For $\theta = ((\theta_k)_{k=M+1}^{\infty};M) \in \Theta$ we define the conditional distribution of

$(X_k)_{k=N+1}^{\infty}$ given N as follows

(5.10) $\quad X_{N+1}, X_{N+2}, \ldots$ are independent

If $M \leqq N$ then

$$(5.11) \qquad X_{N+k} \sim \begin{cases} \mathrm{Bin}(1,p_k) & \text{if } \theta_{k+N} = 0 \\ \mathrm{Bin}(1,1-p_k) & \text{if } \theta_{k+N} = 1 \end{cases}$$

If $M \geqq N$ then

$$X_{N-1}, X_{N+2}, \ldots, X_M \sim \mathrm{Bin}(1,\tfrac{1}{2})$$

$$X_{M+k} \sim \begin{cases} \mathrm{Bin}(1,p_k) & \text{if } \theta_{k+M} = 0 \\ \mathrm{Bin}(1,1-p_k) & \text{if } \theta_{k+M} = 1 \ . \end{cases}$$

The marginal distribution of N has density $q(M-\cdot)$ w.r.t. counting measure on Z.

If

(5.12) $\quad \forall k \in \mathbb{N} : p_k < \frac{1}{2}, \ p_1 = \frac{1}{4}$

(5.13) $\quad \sum_{k=1}^{\infty} (1-2p_k) < +\infty$

and, say,

(5.14) $\quad \forall k \geqq 1 : q(k) = 0$

(5.15) $\quad < \ldots < q(-1) < q(0) < \frac{2}{3} q(1)$

then the above probability distributions on E give rise to a transformation model with an unique maximum likelihood estimator $t : E \to \Theta, t((X_k)_{k=N+1}^{\infty};N) = ((X_k)_{k=N+2}^{\infty};N+1)$ which is non-sufficient (details are left to the reader). This is thus an example of a transformation model where E and Θ are of the same orbittype, the maximum likelihood estimator exists uniquely but (t,π)

is non-sufficient. As pointed out above this relies on the fact that the isotropic groups of M are non-regular.

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