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# A Markov Chain Approach to Periodic Queues

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A MARKOV CHAIN APPROACH TO PERIODIC QUEUES

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Abstract. We consider GI/G/1 queues in an environment which is periodic in the sense that the service time of the nth customer and the next interarrival time depend on the phase $\Theta_n$ at the arrival instant. Assuming Harris ergodicity of $\{\Theta_n\}$ and a suitable condition on the traffic intensity, various Markov chains related to the queue are then again Harris ergodic and provide limit results for the standard customer- and time-dependent processes like waiting times and queue lengths. As part of the analysis, a result of Nummelin (1979) concerning Lindley processes on a Markov chain is reinspected.

Keywords: Periodicity, queue, Harris recurrence, Lindley process, waiting time, queue length, coupling.

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## 1. Introduction

We consider $GI/G/1$ queues in an environment which is periodic (without loss of generality with period one) in the following sense: at the arrival of customer $n$, say at time $t$, his service time $U_n$ and the next interarrival time $T_n$ are drawn according to distributions $B_x, A_x$ depending on the phase $x = \Theta_n = t \bmod 1$ at the arrival instant. We use the obvious notation like $GI_{per}/G_{per}/1$ for the general case, $GI/G_{per}/1$ if $A_x$ is independent of $x$ and so on.

Such models are obviously well motivated from a number of phenomena exhibiting a marked variation according to the time of the day, the day of the week, the season of the year and so on. As examples from different fields of applied probability, we mention [13], [14], [7], [4], whereas the basic reference within the framework of queues is a paper by Harrison and Lemoine [6]. They treat the $M_{per}/G/1$ case by assuming that the arrival process is a Poisson process with periodic intensity $\alpha(t) = \alpha(t+1)$ (and that all $B_x = B$). Letting $\{W_n\}_{n \in \mathbb{N}}$, $\{V_t\}_{t \geq 0}$, $\{Q_t\}_{t \geq 0}$ be the processes of actual waiting times, resp. virtual waiting times, resp. queue lengths, they show that under natural conditions $W_n$ has a weak limit $W$ as $n \to \infty$, that for each $s \in [0,1)$ $V_{n+s} \overset{\mathcal{D}}{\to} V^{(s)}$ as $n \to \infty$, and they also extend some classical formulas for the $M/G/1$ queue to relations between the distributions of $W$ and the $V^{(s)}$ (see in that connection also [8]).

Our aims are here twofold, to extend the results of [6] to the $GI_{per}/G_{per}/1$ case and to present a rather different approach which provides strikingly simple and short proofs (but also relies on methods which are rather advanced compared to the ones in current use in queueing theory). Whereas regenerative processes provide the key tool in [6], we exploit here the ergodic theory for Harris recurrent Markov chains on a general state space. This topic is to some extent classical (see e.g. [12], [14]), but has only rather recently become more applicable in view of the regeneration techniques presented in [9], [3].

We refer to [11] for a recent text book treatment (see also [2] VI.3 for a more condensed survey) and state in Section 2 for the sake of easy reference a few simple lemmas not explicitly given in that form in [11].

The basic assumption is Harris ergodicity of $\{\Theta_n\}$, the process of phases at consecutive arrivals, and also that the traffic intensity in some appropriate sense is less than one. Thus the waiting time process $\{W_n\}$ becomes a Lindley process on a Markov chain, and in fact $\{(\Theta_n,W_n)\}$ inherits the Harris ergodicity of $\{\Theta_n\}$. This result, given in [10], is basic for our purposes and briefly reinspected in Section 3 since the arguments of [10] require some amendments.

The body of material for periodic queues is then given in Section 4. The $GI_{per}/G_{per}/1$ model is formulated, examples showing the versatility of the set-up are given and the existence of limit distributions is established. For $\{W_n\}$, this last fact comes out immediately from Section 3, whereas for $\{V_t\}$ and $\{Q_t\}$ we need to introduce some auxiliary Markov chains (e.g. for $\{V_t\}$ we look at $\{\xi_n\}$ of the form $\xi_n = (R_n, (V_{n+s})_{0 \le s \le 1})$ with $R_n$ a suitable supplementary variable).

## 2. Preliminaries on Harris chains

Let $\{\Theta_n\}_{n\in\mathbb{N}}$ be a Markov chain taking values in the measurable space $(\mathbb{E},\mathcal{E})$ and let in a standard manner $P_x, P_\varphi$ etc. correspond to $P_x(\Theta_0\in A) = I(x\in A)$, $P_\varphi(\Theta_0\in A) = \varphi(A)$. It is assumed throughout that $\mathcal{E}$ is countably generated and all subsets $A\subseteq\mathbb{E}$ considered are tacitly assumed to be in $\mathcal{E}$.

The classical ergodic theory of such chains under the assumption of Harris recurrence is given, e.g. in [12] or [14]. More recent references of relevance for the present paper are [3], [9], [11], where regenerative processes are shown to be of basic importance, and we shall need in particular the following result:

**Lemma 2.1** *Let $\{\Theta_n\}$ be Harris recurrent with stationary measure $\pi$ and let $A\subseteq\mathbb{E}$, $\pi(A)>0$. Then there exist $C\subseteq A$ with $\pi(C)>0$ and randomized stopping times $\tau(0)<\tau(1)<\ldots$ such that:*

(i) *$\{\tau(k)\}$ is a discrete renewal process;*

(ii) *the $\Theta_{\tau(k)}$ are i.i.d. with the common distribution concentrated on $C$;*

(iii) *for each $k$, the post-$\tau(k)$ chain $\{\Theta_{\tau(k)+n}\}_{n\in\mathbb{N}}$ is independent of $\{\tau(0),\ldots,\tau(k)\}$;*

(iv) *if $\{\Theta_n\}$ is positively recurrent, then $\tau(k)/k\to\beta$ $P_x$- a.s. with $\beta\in(0,\infty)$ independent of $x\in\mathbb{E}$.*

Proof We apply Orey's C-set theorem ([12] Theorem 2.1 or [14] p. 160-161; historically, the theorem is associated also with the names of Doob and Jain & Jamison, cf. [11]). This implies the existence of $C\subseteq A$ with $\pi(C)>0$ and $P_x(\Theta_r>F)\geqq\varepsilon\pi(FC)$ for some $\varepsilon>0$, some $r\in\{1,2,\ldots\}$ and all $x\in C$. If $r=1$, the relevant construction is then described in [3] or [11]. The general case can then either be reduced to this by constructing the chain first in steps of length $r$ and next conditioning in the missing values, or one may proceed by an easy modification of the construction spelled out in [2]. It should be noted that if $r=1$, (iii) can be strengthened to the post-$\tau(k)$ chain being independent of $\{\Theta_n, n<\tau(k);\ \tau(0),\ldots,\tau(k)\}$, whereas if $r>1$, both of the approaches just sketched only lead to independence of $\{\Theta_n, n \leqq \tau(k) - r;$ $\tau(0),\ldots,\tau(k)\}$. This difference is, however, immaterial for the present purposes.

We shall say that a Markov chain $\{\Theta_n\}$ <u>admits coupling</u> if for any two initial distributions $\varphi,\psi$ there exist versions $\{\Theta_n^{(\varphi)}\}$, $\{\Theta_n^{(\psi)}\}$ of the $P_\varphi$-, resp. $P_\psi$-, chain defined on the same probability space and an a.s. finite random time $T = T(\varphi,\psi)$ with the property $\Theta_n^{(\varphi)} = \Theta_n^{(\psi)}$, $n \geq T$.

<u>Lemma 2.2</u> <u>A Markov chain</u> $\{\Theta_n\}$ <u>having a stationary distribution</u> $\pi$ <u>admits coupling if and only if it is Harris ergodic</u>.

<u>Proof</u> That a Harris ergodic chain admits coupling follows from its regenerative properties (a general discussion of this topic is in [15]; more elementary, one can simply take $T$ as the coupling epoch of the imbedded renewal processes. See also [5], where a coupling is constructed under conditions entailed by t.v. convergence). Conversely, $\Theta_n^{(\varphi)} = \Theta_n^{(\pi)}$, $n \geq T$, and the stationarity of $\{\Theta_n^{(\pi)}\}$ implies t.v. convergence of the distribution of $\Theta_n^{(\varphi)}$ to $\pi$, hence Harris ergodicity ([1] Prop.6.3).

<u>Lemma 2.3</u> <u>If a Harris recurrent Markov chain</u> $\{\Theta_n\}$ <u>has the property that for some probability measure</u> $\varphi$ <u>on</u> $\mathbb{E}$ <u>and some</u> $\varepsilon > 0$

$$(2.1)\qquad \liminf_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N} I(\Theta_n \in A) \geq \varepsilon\, \varphi(A)$$

$P_x$- <u>a.s. for all</u> $A \subseteq E$, <u>then</u> $\{\Theta_n\}$ <u>is positively recurrent</u>.

<u>Proof</u> Suppose the stationary measure $\pi$ has infinite mass. Then by $\sigma$-finiteness, we can find $A$ with $\varphi(A) > 0$ and $\pi(A) < \infty$. By [12] p. 36, $P_x(\Theta_n \in A) \to 0$, and taking expectations in (2.1) and using Fatou's lemma, a contradiction comes out from

$$\varepsilon\,\varphi(A) \le \liminf_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N} P_x(\Theta_n \in A) = 0.$$

Note, conversely, the standard fact that if $\{\Theta_n\}$ is positively recurrent with stationary distribution $\pi$, then $P_x$- a.s.

(2.2) $$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N} I(\Theta_n \in A) = \pi(A).$$

## 3. Lindley processes on a Markov chain

We consider throughout this Section a Harris ergodic Markov chain $\{\Theta_n\}$ in the set-up of Section 2 and let $\pi$ denote the stationary distribution. Then by a Lindley process on $\{\Theta_n\}$ we understand a process $\{W_n\}$ of the form $W_0 = w$, $W_{n+1} = (W_n + X_n)^+$ where $\{(\Theta_n, X_n)\}$ is a bivariate Markov chain on $\mathbb{E} \times \mathbb{R}$ with transition function depending only on the first coordinate [obviously, $\{W_n\}$ may be interpreted as the zero-reflected version of the corresponding random walk (or Markov additive process) $S_n = X_0 + \ldots + X_{n-1}$ on $\{\Theta_n\}$]. Clearly, $\{(\Theta_n, W_n)\}$ and $\{(\Theta_n, X_n, W_n)\}$ are Markov chains on $\mathbb{E} \times [0,\infty)$, resp. $\mathbb{E} \times \mathbb{R} \times [0,\infty)$ and have been studied by Nummelin [10], to whom we refer for further formalism and background material. We shall here briefly reinspect the parts of [10], which are relevant for our purposes, in particular the following result ([10] Lemma 4.2):

Theorem 3.1 The chains $\{(\Theta_n, W_n)\}$, $\{(\Theta_n, X_n, W_n)\}$ are Harris ergodic if $\{\Theta_n\}$ is so and $E_\pi X < 0$. In particular, $W_n$ has a limit $W$ in the sense of t.v. convergence of distributions.

Whereas the assertion of the theorem seems correct, so is not the case for the proof of [10] More precisely, with reference to the waiting time paradox the vital assertion on p. 668, that sampling of a stationary version of $\{(\Theta_n, X_n)$ at a certain sequence $\{\tau(i)\}$ of random times yields the same distribution of $(\Theta_{\tau(i)}, X_{\tau(i)})$ for all $i$, is erroneous. [This has been acknowledged by Nummelin and, in fact, a counterexample has been given by M. Jacobsen. Also the way in which this (unvalid) stationarity is used in the proof of relation (4.8) and the aperiodicity is not clear to us]. In view of the basic importance of Theorem 3.1 for the rest of this paper (as well as of the intrinsic interest of the result) we shall therefore give an alternative (and in fact shorter) proof.

It is easily seen that $\{(\Theta_n, X_n)\}$ inherits the Harris ergodicity of $\{\Theta_n\}$, cf. [10], and hence it is obviously sufficient to consider $\{(\Theta_n, X_n, W_n)\}$. We proceed from Lemma 2.2 by first showing the existence of a stationary version and next constructing a coupling. The first of these steps is given in [10], but may be reformulated slightly in a manner more familiar from standard queueing theory:

Proposition 3.2 Suppose that $\{\Theta_n\}$ has a stationary distribution $\pi$ and that $E_\pi X < 0$. Then, with $\{(\Theta_n^*, X_n^*)\}_{-\infty}^{\infty}$ a doubly infinite stationary version of $\{(\Theta_n, X_n)\}$ and

$$W_n^* = \sup_{-\infty < k \leq n} X_k^* + \ldots + X_{n-1}^* = \Big(\sup_{-\infty < k \leq n-1} X_k^* + \ldots + X_{n-1}^*\Big)^+,$$

it holds that $\{(\Theta_n^*, X_n^*, W_n^*)\}$ is a stationary version of $\{(\Theta_n, X_n, W_n)\}$.

Proof The stationarity is obvious. Further, it is well-known that $E X^* = E_\pi X < 0$ implies $X_k^* + \ldots + X_{n-1}^* \to -\infty$ as $k \to -\infty$ and thus $W_n^* < \infty$ a.s.. Finally,

$$W_{n+1}^* = \Big(\sup_{-\infty < k \leq n} X_k^* + \ldots + X_n^*\Big)^+ = (W_n^* + X_n^*)^+.$$

Remark It is of some interest to note that (with some minor topological assumptions) one may interpret the distribution of $W_n^*$ as that of the maximum of a random walk governed by the time-reversed transition probabilities, cf. [14] p. 123-124.

Proceeding to the coupling, write $\tilde{\Theta}_n = (\Theta_n, X_n)$ for brevity and $\tilde{\mathbb{E}} = \mathbb{E} \times \mathbb{R}$. Let $\varphi, \psi$ be two initial distributions on $\tilde{\mathbb{E}} \times [0,\infty)$ for $\{(\tilde{\Theta}_n, W_n)\}$ and $\mu, \lambda$ their marginals in the first component. Then since $\{\tilde{\Theta}_n\}$ is Harris ergodic, we can couple $\{\tilde{\Theta}_n^{(\mu)}\}$, $\{\tilde{\Theta}_n^{(\lambda)}\}$, i.e. $\tilde{\Theta}_n^{(\mu)} = \tilde{\Theta}_n^{(\lambda)}$, $n \geq T$. Then also $S_{n+T}^{(\mu)} - S_T^{(\mu)}$ and $S_{n+T}^{(\lambda)} - S_T^{(\lambda)}$ have a common value for $n = 0,1,2,\ldots$, say $S_n^{(T)}$. Also by choosing $W_0^{(\varphi)}$, $W_0^{(\psi)}$ such that $(\tilde{\Theta}_0^{(\mu)}, W_0^{(\varphi)})$, $(\tilde{\Theta}_0^{(\lambda)}, W_0^{(\psi)})$ have distributions $\varphi$, resp. $\psi$, and letting

$$(3.3) \qquad W_{n+1}^{(\varphi)} = (W_n^{(\varphi)} + X_n^{(\mu)})^+, \quad W_{n+1}^{(\psi)} = (W_n^{(\psi)} + X_n^{(\lambda)})^+,$$

$\{(\tilde{\Theta}_n^{(\mu)}, W_n^{(\varphi)})\}$ is a $P_\varphi$-version of $\{(\tilde{\Theta}_n, W_n)\}$ and $\{(\tilde{\Theta}_n^{(\lambda)}, W_n^{(\psi)})\}$ a $P_\psi$-version. To complete the proof, it is thus sufficient to show that $W_n^{(\varphi)} = W_n^{(\psi)}$ eventually. Now it is well-known that $S_n$ and hence $S_n^{(T)}$ tends a.s. to $-\infty$. In particular, there is a $T_1$ such that $S_n^{(T)} \leq -\max(W_T^{(\varphi)}, W_T^{(\psi)})$, $n \geq T_1$.

Now it is a standard fact that (3.3) implies

$$W_{T+n}^{(\varphi)} = \max\{W_T^{(\varphi)} + S_n^{(T)},\ S_n^{(T)} - S_1^{(T)}, \ldots, S_n^{(T)} - S_{n-1}^{(T)},\ 0\},$$

$$W_{T+n}^{(\psi)} = \max\{W_T^{(\psi)} + S_n^{(T)},\ S_n^{(T)} - S_1^{(T)}, \ldots, S_n^{(T)} - S_{n-1}^{(T)},\ 0\}.$$

For $n \geq T_1$, the first term in these maxima may be cancelled and thus $W_{T+n}^{(\varphi)} = W_{T+n}^{(\psi)}$ so that $T+T_1$ is the desired coupling epoch. Finally for the t.v. convergence of $W_n$ we just remark that t.v. convergence of a Markov chain entails that of any measurable functional.

## 4. Limit theorems for periodic queues

We number the customers $n = 0,1,2,\ldots,$ denote their service times by $U_0, U_1, U_2, \ldots$ and let $T_n$ be the time between the arrival of customers $n$ and $n+1$. The phase space is $\mathbb{E} = [0,1)$ and we let $\Theta_n$ be the phase at the nth arrival. Thus for $n = 1,2,\ldots$

$$\Theta_n = (\Theta_{n-1} + T_{n-1}) \bmod 1 = (\Theta_0 + T_0 + \ldots + T_{n-1}) \bmod 1$$

and we let $P_x, E_x$ etc. refer to the case $\Theta_0 = x$. With $\mathcal{F}_n = \sigma(T_0,\ldots,T_{n-1}; U_0,\ldots,U_{n-1})$, the basic periodicity assumption then means that for suitable families $(A_x)_{x\in\mathbb{E}}$, $(B_x)_{x\in\mathbb{E}}$ of distributions on $(0,\infty)$ it holds that

(4.1) $$P_x(T_n \le t,\ U_n \le u \mid \mathcal{F}_n) = A_{\Theta_n}(t) B_{\Theta_n}(u)$$

(in fact, the conditional independence between $T_n$ and $U_n$ is not essential for the following).

We shall need:

Assumption I $\{\Theta_n\}$ is Harris ergodic on $\mathbb{E}$, with stationary distribution (say) $\pi$

Assumption II $\rho = \dfrac{E_\pi U}{E_\pi T} < 1$

Here $E_\pi T$ means $\int_{\mathbb{E}} E_x T\, \pi(dx)$ with $E_x T = \int_0^\infty y\, A_x(dy)$ and similarly for $E_\pi U$. By the law of large numbers (2.2) for Harris ergodic chains, one also has for each $x \in \mathbb{E}$ that $P_x$ – a.s.

(4.2) $$E_\pi T = \lim_{n\to\infty} \frac{T_0 + \ldots + T_n}{n},\quad E_\pi U = \lim_{n\to\infty} \frac{U_0 + \ldots + U_n}{n}.$$

Example 4.1 Assume as in [6] that the arrival process is a Poisson process with a periodic intensity $\lambda(t) = \lambda(t+1)$ which is measurable and bounded. It is then easy to see that $\{\Theta_n\}$ has a transition density $p(x,y)$ satisfying

(4.3) $$p(x,y) = \varphi(x,y)\lambda(y) I(x \le y) + \varphi(x,1) p(0,y)$$

where $\varphi(x,y) = \exp\{-\int_x^y \lambda(u)du\}$ is the probability of no arrivals in $[x,y)$ and $p(0,y)$ coincides with $\psi(y) = \varphi(0,y)\lambda(y)/(1-e^{-\lambda})$ where $\lambda = \int_0^1 \lambda(t)dt$ [that $p(0,y) = \psi(y)$ follows by noting that for any $k$ $\psi(y)$ is the conditional $P_0$-density of $T_0$ given $\{k \leq T_0 < k+1\}$]. In particular, $p(x,y) \geq \varphi(0,1)p(0,y)$ for all $x \in \mathbb{E}$ implies that $\{\Theta_n\}$ is Harris ergodic, see e.g. [3] for references and background (going back to Doeblin). It is readily guessed in various ways that the stationary distribution $\pi$ should be given by the density $\lambda(y)/\lambda$, $y \in \mathbb{E}$, i.e. that

$$\frac{\lambda(y)}{\lambda} = \int_0^1 \frac{\lambda(x)}{\lambda} p(x,y)dx$$

and this is also easily checked by (4.3). Finally the first limit in (4.2) is $\lambda^{-1}$, cf. [6], and thus if all $B_x = B$, Assumption II is the same as the basic requirement $\lambda EU < 1$ of [6].

Example 4.2 (cf. also [16]). Assume that all $A_x$ contain a common component $H$ which is spread-out. I.e., $A_x \geq \varepsilon H$ for all $x \in \mathbb{E}$ and some $\varepsilon > 0$, and for some $k$ $H^{*k}$ is absolutely continuous. Then also for some $\delta > 0$ and some integers $\ell, m$ the density of $H^{*\ell}$ is bounded below by $\delta$ on $[m, m+2]$ so that for all $x \in E$ and all Borel sets $A \subseteq E$

$$(4.4) \qquad P_x(\Theta_\ell \in A) \geq \varepsilon^\ell H^{*\ell}(\{A+m-x\}) \geq \delta\varepsilon^\ell \int_A dy.$$

Again, this implies Harris recurrence and positivity. Also aperiodicity can be seen to follow from the fact that (4.4) holds for two consecutive integers ($\ell+1$ as well as $\ell$).

In just the same way as for the standard GI/G/1 queue, it now follows that the waiting time of customer $n+1$ is given by $W_{n+1} = (W_n + X_n)^+$ with $X_n = U_n - T_n$. Here $W_0 = 0$ if customer 0 enters an empty queue, whereas otherwise $W_0$ is to be interpreted as the remaining work in the system at the arrival of customer 0. In view of (4.1), $\{W_n\}$ is a Lindley process on $\{\Theta_n\}$ and since Assumption II is equivalent to $E_\pi X < 0$, Th. 3.1 yields:

Corollary 4.3 *For a $GI_{per}/G_{per}/1$ queue satisfying Assumptions I, II, it holds that $\{(\Theta_n, W_n)\}$ is Harris ergodic. In particular, the actual waiting time $W_n$ has a limit $W$ in the sense of t.v. convergence of distributions.*

We shall consider the behaviour of $\{W_n\}$ as settled by Cor. 4.3, but of course an essential problem is to say more about the distribution of $W$, at least in some particular cases. Some information is provided by Prop. 3.2 and the remark following it, and thereby also investigations like those of [1] (for a discrete state space for $\{\Theta_n\}$) become relevant at least for approximation purposes. We shall, however, not go into this but pass on to the virtual waiting time process $\{V_t\}_{t\geq 0}$ and the queue length process $\{Q_t\}_{t\geq 0}$ in continuous time.

In continuous time, the periodicity excludes the existence of limits as $t\to\infty$, and we must instead restrict attention to sequences of the form $s, 1+s, 2+s, \ldots$ with $s\in[0,1)$.

It seems natural first to look for discrete renewal processes making $\{V_{n+s}\}$ and $\{Q_{n+s}\}$ regenerative, but in fact we did not manage to come up with such ones. In particular, it is not sufficient as in [6] to look at integers $n$ with $Q_n = 0$ (neither do the $\sigma(k)$ below suffice though this requires a little more reflection). However, Markov chain methods turn out to apply surprisingly well, and we proceed by considering Markov chains $\{\xi_n\}$, $\{\eta_n\}$ containing the relevant information on $\{V_t\}_{t\geq 0}$, resp. $\{Q_t\}_{t\geq 0}$.

We may assume that the paths of $\{V_t\}$, $\{Q_t\}$ are in $D[0,\infty)$ and for the study of $\{V_t\}$ we can then define a random element $Z_n = (Z_n(s))_{0\leq s\leq 1}$ of $D[0,1]$ by $Z_n(s) = V_{n+s}$. The process $\{Z_n\}$ is not Markovian since $Z_n$ alone does not determine the arrival process in $[n+1, n+2]$. We therefore let $\{R_t\}_{t\geq 0}$ be the forwards recurrence time process of the arrival process and look at $\{(R_n, Z_n)\}$. It is then easy to see that $\{\xi_n\} = \{(R_n, Z_n)\}$ is indeed a Markov chain taking values in $(0,\infty)\times D$ and that also $\{R_n\}$ itself is Markovian on $(0,\infty)$. A small problem arises when trying to fit $\{\xi_n\}$ into the framework of

Harris chains, since we need to specify the transitions from all $\xi_0 = (r,z) \in (0,\infty) \times D[0,1]$, including also some $z$ which do not at all look like the paths of the virtual waiting time, some couples $(r,z)$ not consistent with $z(t)$ to have a jump at $t = r$ if $r < 1$ and so on. This may be done, e.g., by letting the $P_{r,z}$-distribution of $\xi_1$ be that of $(R_0,(V_s)_{0 \le s \le 1})$ corresponding to a queue with starting values $R_0, V_0$ determined by $V_0 = |z(1)|$, $R_0 = r-1$ if $r > 1$, and finally if $r \le 1$, $R_0$ should correspond to an arrival process with the last arrival before time zero occurring at time $w(z)-1$ where $w(z) = \sup\{s \le 1: z(s-0) \neq z(s)\}$.

In the discrete time scale indexed by the customers, Assumption I on Harris ergodicity of $\{\Theta_n\}$ may be characterized as a regularity property of the arrival process. It seems reasonable to expect that something similar must be set up in the physical time scale (where we look at consecutive integers). Noting that $\{R_t\}$ is in one-one correspondance with the arrival process, one may more precisely be lead to the following condition:

Assumption III $\{R_n\}$ is Harris ergodic on $(0,\infty)$

In fact, this condition will be in force in our theorems, but it might be noted that all that need to be checked is in fact aperiodicity (the requirement $E_\pi T < \infty$ is in general innocent in examples):

Proposition 4.4 Suppose that Assumptions I, II are in force. Then $\{R_n\}$ and $\{\xi_n\}$ are Harris recurrent, and positively recurrent if and only if $E_\pi T < \infty$.

Proof Let $\tilde{\Theta}_n = (\Theta_n, T_n, U_n)$. Then since $\{(\Theta_n, W_n)\}$ is Harris ergodic (with $\mathbb{E} \times \{0\}$ recurrent), it is easy to see that so is $\{(\tilde{\Theta}_n, W_n)\}$ with $\mathbb{E} \times (0,\infty)^2 \times \{0\}$ recurrent. Thus by Lemma 2.1, we can find a sequence $\{\tau(k)\}$ of random times such that $W_{\tau(k)} = 0$, that $\tau(k)/k \to \beta \in (0,\infty)$ and that the sequence $B_1, B_2, \ldots$ of blocks

$$B_k = (\tau(k+1) - \tau(k); (\tilde{\Theta}_{\tau(k)}, W_{\tau(k)}), \ldots, (\tilde{\Theta}_{\tau(k+1)-1}, W_{\tau(k+1)-1}))$$

is strictly stationary with one-dependent components. In particular, $\{B_k\}$ is metrically transitive, since one-dependent sequences have even trivial tail-$\sigma$-field as may be seen by a minor modification of Kolmogorov's zero-one law. Let $r$ be the time of the first arrival and

$$\sigma(k) = [r + T_0 + \ldots + T_{\tau(k)-1}] + 1.$$

Then for some suitable functional $\psi$ (the explicit form of which needs not concern us) it holds that

$$\xi_{\sigma(k)} = \psi(B_k, B_{k+1}, \ldots)$$

Therefore also $\{\xi_{\sigma(k)}\}_{k=1,2,\ldots}$ is metrically transitive, and hence by the pointwise ergodic theorem

$$(4.5) \qquad \lim_{K\to\infty} \frac{1}{K} \sum_{k=1}^{K} I(\xi_{\sigma(k)} \in A) = P(\xi_{\sigma(1)} \in A) = \varphi(A) \text{ (say) a.s.}$$

no matter initial conditions and thus $\{\xi_n\}$ is $\varphi$-recurrent. Further if $E_\pi T < \infty$, then $\sigma(k)/k \to \alpha = \beta E_\pi T < \infty$. Thus letting $K_N = \sup\{k : \sigma(k) \leqq N\}$, it follows by an argument familiar from renewal theory that $K_N/N \to \alpha^{-1}$. Hence by (4.5)

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} I(\xi_N \in A) \geqq \liminf_{N\to\infty} \frac{K_N}{N} \frac{1}{K_N} \sum_{n=1}^{K_N} I(\xi_{\sigma(k)} \in A) = \alpha^{-1}\phi(A)$$

so that by Lemma 2.3 we have positive recurrence. Finally the necessity of $E_\pi T < \infty$ for positive recurrence of $\{R_n\}$ alone follows easily from (2.2). Alternatively see Theorem 3 in [15].

Remark It is tempting to assert that the $\xi_{\sigma(k)}$ are so i.i.d. since the $(\tilde{\Theta}_{\tau(k)}, W_{\tau(k)})$ are so. However, if is not apriori obvious even that $\sigma(k+1) \neq \sigma(k)$, and also dependence between $\xi_{\sigma(k+1)}$ and $\xi_{\sigma(k)}$ arises if $\sigma(k+1) = \sigma(k) + 1$.

We can now state and prove our main result on the virtual waiting time:

Theorem 4.5 Suppose that Assumptions I, II are in force and also that Assumption III holds (or just that $\{R_n\}$ is aperiodic and $E_\pi T < \infty$). Then $\{(R_n, (V_{n+s})_{0 \le s \le 1})\}$ is Harris ergodic on $(0,\infty) \times D[0,1]$. In particular, for each $s \le 1$ $V_{n+s}$ has a t.v. limit $V^{(s)}$ as $n \to \infty$.

Proof Since $\{\xi_n\}$ has a stationary distribution (Proposition 4.4), it is sufficient for Harris ergodicity to show that $\{\xi_n\}$ admits coupling. This is done almost in the same way as in the proof of Theorem 3.1. Since $\{R_n\}$ is assumed Harris ergodic and hence admits coupling, it is sufficient to show that two chains $\{\xi_n'\}$, $\{\xi_n''\}$ with the same initial distribution of $R_0$ can be coupled. To this end, we may assume that $T_n' = T_n''$, $U_n' = U_n''$ for all $n$ and that $R_0' = R_0''$. Then $\{\xi_n'\}$, $\{\xi_n''\}$ have the same input process (in particular $R_n' = R_n''$ for all $n$) and differ only through the values $Z_0', Z_0''$. Now if say $Z_0'(1) \le Z_0''(1)$, there is a $T \ge 1$ with $V_T'' = 0$ and hence $V_T' = 0$. Similarly, we let $T \ge 1$ satisfy $V_T' = 0$ if $Z_0'(1) > Z_0''(1)$ and get in both cases that $\xi_n' = \xi_n''$ for $n \ge [T] + 1$.

The queue length can be treated in an entirely similar way. As auxiliary Markov chain we can take, e.g., $\{(R_n, Y_n, \eta_n)\}$ defined on the state space $(0,\infty) \times D[0,1] \times \mathbb{R}^{\mathbb{N}}$ by $Y_n = (Q_{n+s})_{0 \le s \le 1}$ and $\eta_n = (M_1, \ldots, M_k, 0, 0, \ldots)$ if $Q_n = k$, $M_1$ is the residual service time of the customer being served at time $n$ and $M_2, \ldots, M_k$ the service times of the remaining $k-1$ customers (in their order of service). We remark that whereas in Prop.4.4 it is not essential that $W_{\tau(k)} = 0$ (only certain stationarity properties are used), then so is not the case here: if $W_{\tau(k)} \neq 0$, we can not represent say $Q_{\sigma(k)}$ as a functional of $B_k, B_{k+1}, \ldots$ alone but most invoke also the $B_\ell$ with $\ell < k$. Otherwise just the same argument applies, and we get:

Theorem 4.6 Under the conditions of Theorem 4.5, $\{(R_n, Y_n, \eta_n)\}$ is Harris ergodic as well. In particular, $Q_{n+s}$ has a t.v. limit $Q^{(s)}$ as $n \to \infty$.

Also several further generalizations are possible. For example, along similar (even easier) lines it may be shown that the processes of queue lengths just before arrivals and just after departures have t.v. limits, one may consider simultaneous convergence of $V_{n+s}$ and $Q_{n+s}$ and so on. Also since the law of large numbers holds for Harris ergodic chains, cf. (2.2), one would expect that time-average considerations like in [6], [8] should apply to deduce relations between the various limit distributions, but we have not carried this out. As a limitation to the methods of this paper one may note, however, that a model with a server working periodically seems to require additional work. We finally remark that Assumption III (or equivalently the aperiodicity) can be shown almost trivially to hold in Examples 4.1, 4.2: In Example 4.1, we just need to note that two arrival processes with $R_0' \neq R_0''$ can be coupled by letting them coincide after $\max(R_0', R_0'')$, and in Example 4.2, the assertion follows easily from $H^{*\ell}$ having a density bounded below on an interval of length $\geq 2$ (alternatively, a proof is in [16]).

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