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Almost Sure Convergence for Stochastically Monotone Temporally Homogeneous Markov Processes and Applications

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Abstract

A necessary and sufficient condition for a suitably normed and centered stochastically monotone Markov process to converge a.s. is given and its limit distribution is characterized. Applications to diffusions and continuous time branching processes are presented. In particular, an assumption on the conditional means and variances of a diffusion process turns out to suffice for establishing a.s. convergence.

1. Introduction

Let $P:R\times\mathcal{B}\to[0,1]$ be a transition probability function, where R is the real line and $\mathcal{B}$ is the family of Borelian subsets of R. We shall say that P is stochastically monotone (SM) if $P(x,(-\infty,y])$ is non-increasing in x for every fixed y. A Markov process $\{X(t):t\varepsilon[0,\infty)\}$ is said to be SM if its transition probability functions are SM (see [7], [9], [11], [12], [4] and [5]).

Our main concern is this paper will be to investigate the a.s. behaviour of $Y(t)=a(t)(X(t)+b(t))$, $\{a(t)\}$ and $\{b(t)\}$ being some constants, as well as to establish some properties of the limit distribution of $\{Y(t)\}$.

It was proved in [2] that if $\{X_n\}$ is a discrete-time Markov chains with stationary transition probabilities, and $\{a_n\}$ and $\{b_n\}$ are two sequences of constants such that $Y_n=a_n(X_n+b_n)$ converges in distribution to a non-degenerate limit, then under rather general conditions $\lim_{n\to\infty}a_n/a_{n-1}=\alpha$ and $\lim_{n\to\infty}a_n(b_n-b_{n-1})=\beta$ exist and are finite. If $\alpha=1$ and $\beta=0$, $\{Y_n\}$ is mixing in the sense of Renyi [17], case when a.s. convergence or convergence in probability do not hold. We note that stochastic monotonicity was not used in establishing these results. In [5] we have considered a continuous time SM Markov process $\{Y(t)\}$ and proved that $\lim_{t\to\infty}a(t+s)/a(t)=\rho^s$ and $\lim_{t\to\infty}a(t+s)(b(t+s)-b(t))=\gamma s$ exists for all $s>0$. In addition, if $\rho\neq 1$ and/or $\gamma\neq 0$ then $\{Y(t_n)\}$ converges a.s. for any $\{t_n\}$ with $\lim_{n\to\infty}=\infty$ and its limit distribution is continuous with the possible exception of $x=0$. For such results to hold some regularity conditions on $\{a(t)\}$ and $\{b(t)\}$ were required in addition to convergence in distribution for $\{Y(t)\}$. However, in many cases of interest it is rather difficult to derive convergence in distribution and therefore it would be desirable to have an a.s. convergence criterion that does not require convergence in distribution. We shall derive here such a criterion assuming only tightness and a condition on $\{P_s\}$ for $s\varepsilon(0,\delta)$ with $\delta>0$.

A random process $\{\xi(t)\}$ will be said to be tight if any subsequence thereof contains another subsequence converging in distribution to a non-identically 0 random variable. We note that our definition of tightness assumes that the limits of subsequences are not identically 0.

Further we shall consider the following conditions:

(A1) $b(t)\equiv 0$ and either $1<\liminf_{t\to\infty}a(t+s)/a(t)\leqslant\limsup_{t\to\infty}a(t+s)/a(t)<\infty$ or $0<\liminf_{t\to\infty}a(t+s)/a(t)\leqslant\limsup_{t\to\infty}a(t+s)/a(t)<1$ for some $s>0$.

(A2) $\lim_{t\to\infty} a(t+s)/a(t)=1$ for all $s>0$, and

$-\infty<\liminf_{t\to\infty} a(t+s)(b(t+s)-b(t))\leq\limsup_{t\to\infty} a(t+s)(b(t+s)-b(t))<\infty$

for some $s>0$.

(B) If $c(t)=xa(t)$ for x with $0<x<\infty$ then for $s\varepsilon(0,\delta)$ with $\delta>0$, $\rho\neq 1$ and any $\varepsilon>0$

$$\lim_{t\to\infty} P(X(t+s)\varepsilon(c(t)\rho^{s}(1-\varepsilon),c(t)\rho^{s}(1+\varepsilon))|X(t)=c(t))=1.$$

(C) The distribution function F is continuous except maybe for $x=0$, suppF is either the real line or one of its half-lines, and F is strictly increasing on suppF.

If ν and μ are two probability measures, $\nu<<\mu$ is to denote that ν is absolutely continuous with respect to μ.

Our main result is the following:

Theorem 1 Suppose that $\{X(t):t\varepsilon[0,\infty)\}$ is a temporally homogeneous, right continuous SM Markov process, $\{a(t)\}$ and $\{b(t)\}$ some constants with $\lim_{t\to\infty} a(t)=\infty$ that satisfies condition (A1). Assume further that $\nu_t<<\nu_s$ for $t>s$ where $\nu_t(.)=P(X(t)\varepsilon.)$. Then the tightness of $\{Y(t)\}$ in conjunction with condition (B) is a necessary and sufficient condition for the existence of some constants $\{a'(t)\}$ with $\lim_{t\to\infty} a'(t+s)/a'(t)=\rho^{s}$ for all s, such that $\{a'(t)X(t)\}$ converges a.s. to a limit random variable whose distribution function F satisfies condition (C).

We shall see that condition (B) is implied by the following:

Condition (B1) There exist $s>0$ and $\rho\neq 1$ such that

$$\lim_{t\to\infty} P(|X(t+s)/X(t)-\rho^{s}|>\varepsilon|X(t)\neq 0)=0 \quad \text{for any } \varepsilon>0.$$

In the case when $\{X(t)\}$ assume finite second moments it will be shown that condition (B) is implied by the following:

Condition (B2)

$$\lim_{t\to\infty}\frac{Var(X(t+s)|X(t)=c(t))}{\min^2\{[\rho^s c(t)(1+\varepsilon)-E(X(t+s)|X(t)=c(t)],-[\rho^s c(t)(1-\varepsilon)-E(X(t+s)|X(t)=c(t))\}}=0$$

As an application of Theorem 1 to the case of a Markov process with finite second moments that contains some types of diffusions (see e.g. [6],[8],[10] and [15]) we get:

Theorem 2 Suppose that $\{X(t):t\varepsilon[0,\infty)\}$ is a temporally homogeneous, right-continuous SM Markov process, $\nu_t\ll\nu_s$ for $t>s$, $E(X(t))\sim a\rho^t$ and $Var(X(t))\sim b\rho^{2t}$ for some constants a, b and ρ with $\rho\neq 1$, and condition (B2) holds. Then $\{X(t)/\rho^t\}$ converges a.s. as $t\to\infty$ to a limit random variable whose distribution function F satisfies condition (C).

The above results assume condition (A1). In the case (A2) we obtain similar results by considering $\{e^{Y(t)}\}$ instead of $\{Y(t)\}$ and using an analogous reasoning. The only difference lies in that F is continuous on the whole line as remarked in [5]. We leave it to the reader to adapt conditions (B), (B1) and (B2) to this case.

2. Preliminary results

Let q be a number with $0<q<1$ and assume that there exist a sequence $\{t_n\}$ with $\lim_{n\to\infty}t_n=\infty$ and some intervals $\{J_{t_n}\}$ such that $\lim_{n\to\infty}P(X_{t_n}\varepsilon J_{t_n})=q$ where $J_{t_n}=(-\infty,x_{t_n})$ or $(-\infty,x_{t_n}]$ for some $\{x_{t_n}\}$. Consider further the quantities $\{P(X_{t_n}\varepsilon J_{t_n}|X_{t_m}=x)\}$ for $x\varepsilon\mathrm{supp}F_{t_m}$, where

F_t is the distribution function of X_t. By stochastic monotonicity, $P(X_{t_n} \varepsilon J_{t_n} | X_{t_m}=x)$ is non-increasing in x, and by the well-known weak compactness principle (see e.g. [3] p.81) one can extract a subsequence of $\{t_n\}$, say $\{t'_n\}$, such that $G_x^{(t_m)}(q)=\lim_{n\to\infty}P(X_{t'_n} \varepsilon J_{t'_n} | X_{t_m}=x)$ exists for all $x \varepsilon \text{supp} F_{t_m}$ with m fixed. By the wellknown diagonal procedure for subsequences we can extract a further subsequence of $\{t'_n\}$, say $\{t^*_n\}$, such that $G_x^{(t_m)}(q)=\lim_{n\to\infty}P(X_{t^*_n} \varepsilon J_{t^*_n} | X_{t_m}=x)$ for any $x \varepsilon \text{supp} F_{t_m}$, $m=0,1,\ldots$ We show that $G_x(q)=\lim_{n\to\infty}P(X_{t^*_n} \varepsilon J_{t^*_n} | X_t=x)$ exists for all t and x. Indeed, choose $t>0$. Then there must be some k such that $t \leq t^*_k$ and taking the limit in the Chapman-Kolmogorov formula yields that

$$G_x^{(t)}(q)=\int G_y^{(t^*_k)}(q)P_{t^*_k-t}(x,dy)$$

exists for all x and t, where $\{P_t(x,A)\}$ are the transition probability functions of $\{X(t)\}$.

We have therefore proved the following:

<u>Lemma 1</u> Assume that for some $\{t_n\}$ with $\lim_{n\to\infty}t_n=\infty$ and left-unbounded intervals $\{J_{t_n}\}$ $\lim_{n\to\infty}P(X_{t_n} \varepsilon J_{t_n})=q$ with $0<q<1$. Then there exists a subsequence of $\{t_n\}$, say $\{t^*_n\}$, such that

(1) $$G_x^{(t)}(q)=\lim_{n\to\infty}P(X_{t^*_n} \varepsilon J_{t^*_n} | X_t=x)$$

exists for all x and t.

<u>Lemma 2</u> There exists a random variable W_q such that $W_q=\lim_{t\to\infty}G_{X_t}^{(t)}(q)$ a.s., $E(W_q)=q$ and $E(W_q|X_t)=G_{X_t}^{(t)}(q)$ a.s. for all $t>0$.

Proof The Chapman-Kolmogorov formula is easily seen to lead to

$$(2) \qquad G_x^{(t)}(q)=\int G_y^{(t+s)}(q)P_s(x,dy)$$

Further, (2) yields

$$(3) \qquad E(G_{X_{t+s}}^{(t+s)}(q)|X_t)=G_{X_t}^{(t)}(q) \text{ a.s.}$$

The Markov property in conjunction with (3) implies that $\{G_{X_t}^{(t)}(q)\}$ is a martingale. Because this martingale is bounded, $\lim_{t\to\infty}G_{X_t}^{(t)}(q)=W_q$ a.s. exists. The total probability formula yields $E(W_q)=q$, and the closure property for martingales implies $E(W_q|X_t)=G_{X_t}^{(t)}(q)$ a.s. completing the proof.

If W_q is such that $P(W_q=0)=1-P(W_q=1)$ then W_q is said to be of type I, and of type II otherwise. It was shown in [5] that type II W_q may admit at most three values with positive probability.

We shall agree to write $\lim_{t\to\infty}A_t=A$ a.s. or to say that $\lim_{t\to\infty}A_t$ a.s. exists when ever $\lim_{t\to\infty}1_{A_t}=1_A$ a.s. where 1 denotes the indicator of a set.

Lemma 3 Suppose that for some left-unbounded intervals $\lim_{t\to\infty}\{X_t\varepsilon I_t\}$ a.s. exists. Then for any real s $\lim_{t\to\infty}\{X_t\varepsilon I_{t+s}\}$ a.s. also exists.

Lemma 4 Suppose that $\{t_n\}$ is chosen such that $\{Y(t_n)\}$ converges in distribution to a limit distribution F. Then

(i) if $F(0)<1$ then there exists q with $F(0)<q<1$ such that $\lim_{n\to\infty}\{X_{t_n}\varepsilon J_{t_n}\}$ a.s. exists for some left-unbounded intervals $\{J_{t_n}\}$ with $\lim_{n\to\infty}P(X_{t_n}\varepsilon J_{t_n})=q$.

(ii) if $F(0-)>0$ then there exists q' with $0<q'<F(0-)$ such that $\lim_{n\to\infty}\{X_{t_n} \varepsilon J'_{t_n}\}$ a.s. exists for some left-unbounded intervals $\{J'_{t_n}\}$ with $\lim_{n\to\infty}P(X_{t_n} \varepsilon J'_{t_n})=q$.

We defer the proofs of Lemmas 3 and 4 as we feel at this stage necessary to explain the main idea of the paper.

3. Outline of the a.s. convergence proof

We shall confine ourselves to the case $X(t)\geq 0$ and assume condition (A1) to be in force. The other cases will turn out to be similar. By lemma 4 we know that there exists x such that $F(0)<P(W_q\geq x)<1$. Since $\{X(t)\}$ was assumed stochastically monotone, we deduce that

$$(4) \qquad \{W_q\geq x\}=\lim_{t\to\infty}\{X(t)\varepsilon J_t\} \text{ a.s.}$$

where J_t is either $(-\infty,x_t)$ or $(-\infty,x_t]$ for some numbers $\{x_t\}$. It will be shown that we may assume $J_t=(-\infty,x_t]$ such that (4) and Lemma 3 imply that $\lim_{t\to\infty}\{X(t)\leq x_{t+s}\}$ a.s. exists for any s with $-\infty<s<\infty$. Since condition (B) will turn out to entail $\lim_{t\to\infty}x_{t+s}/x_t+\rho^s$ for some ρ with $\rho\neq 1$ and any s, we get

$$(5) \qquad \lim_{t\to\infty}\{X(t)\leq x_{t+s}\}=\lim_{t\to\infty}\{X(t)\leq \rho^s x_t\} \text{ a.s.}$$

As s in (5) is arbitrary we are led to conclude that $\lim_{t\to\infty}\{x_t^{-1}X(t)\leq x\}$ a.s. exists for all x, while the assumed tightness of $\{Y(t)\}$ will ensure that $\{x_t^{-1}X(t)\}$ has a non-degenerate a.s. limit variable.

4. Proofs

Proof of Lemma 3 It is easy to see that $\Lambda=\lim_{t\to\infty}\{X_t \varepsilon I_t\}$ a.s. is an event belonging to the tail σ-field $\mathcal{T}$ of $\{X(t)\}$. Since we assumed $\nu_t<<\nu_s$ for $t>s$ we can argue as in [3] p.93 to deduce that $\theta^s\Lambda=\lim_{t\to\infty}\{X_t \varepsilon I_{t+s}\}$ a.s. also exists for all real s.

Proof of Lemma 4 We shall confine ourselves to the case of $X(t)\geqslant 0$, $b(t)\equiv 0$ and $1<\lim\inf_{t\to\infty}a(t+s)/a(t)\leqslant\lim\sup_{t\to\infty}a(t+s)/a(t)<\infty$. The other case satisfying condition (A1) is reducible to this one by taking $1/Y(t)$ instead of $Y(t)$. The proof will be carried out by assuming the contrary and reaching a contradiction. Choose x to be a continuity point of F and let $F(x)=q$. Then using the notation of Lemma 1 we get

$$(6)\qquad P(a(t_n^*)X(t_n^*+s)\leqslant x)=\int P(a(t_n^*)X(t_n^*)\leqslant x\,|\,X(0)=y)\nu_s(dy)$$

where $s>0$. Taking the limit as $n\to\infty$ yields

$$(7)\qquad F^{(s)}(x)=\int G_y^{(0)}(q)\nu_s(dy)$$

where $F^{(s)}$ is the limit distribution of $\{a(t_n^*)X(t_n^*+s)\}$. Assume now that W_q is a.s. constant, i.e. $W_q=F(x)$ a.s. By Lemma 2 $G_y^{(0)}(q)=F(x)$ a.s. with respect to ν_0 and since $\nu_s<<\nu_0$ we get $G_y^{(0)}(q)=F(x)$ a.s. with respect to ν_s as well. In view of (7) we get $F^{(s)}(x)=F(x)$ and this being true for any $s>0$ it is easily seen that $\lim\inf_{t\to\infty}a(t+s)/a(t)>1$ in conjunction with the tightness of $\{Y(t)\}$ is contradicted. Thus W_q is not a.s. constant and we may choose a point x, which is a continuity point of the distribution function of W_q, with $0<P(W_q\leqslant z)<1$.

Lemma 2 implies:

(8) $\{W_q \leq z\} = \lim_{t\to\infty}\{G_{X_t}^{(t)}(q) \leq z\}$ a.s.

Stochastic monotonicity and (8) ensure the existence of some left-unbounded intervals $\{J_t\}$ such that $\lim_{t\to\infty}\{X_t \varepsilon J_t\} = \{W_q > z\}$ a.s., and if $P(W_q > z) > F(0)$ there is nothing more to prove. Assume therefore that $P(W_q > z) \leq F(0)$. We may "split" the 0 atom of F. According to Theorem 2.9 of [5] this situation corresponds to the case of W_q of type II when W_q takes only two values k_q and 1 with positive probability. Thus $P(W_q > z) = P(W_q = 1)$ and $P(W_q = k_q) = 1 - P(W_q = 1)$. Since $\theta^s\{W_q = 1\} = \lim_{t\to\infty}\} X_t \varepsilon J_{t+s}\}$ a.s. on the account of the assumption $\lim\inf_{t\to\infty} a(t+s)/a(t) > 1$ we get that $J_{t+s} \supseteq J_t$ for t large enough and $\theta^s\{W_q = 1\} \supseteq \{W_q = 1\}$. However, as we have proved in [5] $\{W_q = 1\}$ with $0 < P(W_q = 1) < 1$ entails $0 < P(\theta^s\{W_q = 1\}) < 1$, but it is impossible that $P(\theta^s\{W_q = 1\}) > P(W_q = 1)$ from an inspection of the already mentioned Theorem 2.9 of [5]. Thus $\{W_q = 1\}$ is an invariant event, and because $\{W_q = k_q\}$ is its complementary, it must also be invariant. Therefore W_q is an invariant random variable. It follows that $E(G_{X_s}^{(0)}) = E(G_{X_s}^{(s)}) = F(x)$ and (7) implies $F_{(x)}^{(s)} = F(x)$. The proof may now be completed as in the the case when W_q=constant a.s. considered before.

<u>Proof of Theorem 1</u>

<u>Step 1</u>: We shall first show that if $\Lambda = \lim_{t\to\infty}\{X_t \varepsilon J_t\}$ a.s. where $F(0) < \lim_{t\to\infty} P(X_t \varepsilon J_t) < 1$ then $P(\theta^s\Lambda) > P(\Lambda)$ for all $s > 0$. The existence of such Λ was ensured by Lemma 4. Recall that if $\eta = \lim\inf_{t\to\infty} a(t+s)/a(t)$, then $\eta > 1$. It follows that $\lim\inf_{t\to\infty} a(t+ks)/a(t) = \eta^k$ for any positive k . Notice further that if x_t is the right end-point of J_t then, if necessary extracting a further subsequence of $\{t_n\}$, we may assume that

$a(t_n)\sim cx_{t_n}$ where c is a positive constant. The above arguments boil down to $P(\theta^{ks}\Lambda)=\lim_{n\to\infty}P(X_{t_n}\varepsilon J_{t_n+ks})=\lim_{n\to\infty}P(Y_{t_n}<c^{-1}\eta^k)\leqslant F(c^{-1}\eta^k)$. As F is a proper distribution, we can find k large enough such that $F(c^{-1}\eta^k)>q$. Thus, there is k such that $P(\theta^{ks}\Lambda)>P(\Lambda)$. However, $\theta^s\Lambda\supseteq\Lambda$ for any $s>0$ as we have already noticed in the course of the proof of Lemma 4. This makes $P(\theta^s\Lambda)>P(\Lambda)$ the only possibility and concludes the argument.

Step 2: We show now that if $\{s_n\}$ is a sequence of positive numbers with $\lim_{n\to\infty}s_n=0$ then $\lim_{n\to\infty}\theta^{s_n}\Lambda=\Lambda$ a.s. for any $\Lambda\varepsilon\mathcal{T}$. Indeed, $\{X(t)\}$ was assumed right-continuous in which case it is known that

$\mathfrak{F}_t=\lim_{n\to\infty}\mathfrak{F}_{u+s_n}$ a.s. (see e.g. [16]) where $\mathfrak{F}_t$ is the σ-algebra generated by $\{X_u:0\leqslant u\leqslant t\}$. Since $P(\theta^{s_n}\Lambda|\mathfrak{F}_{t+s_n})=P(\Lambda|\mathfrak{F}_t)$ for $t>0$ we get $P(\Lambda'|\mathfrak{F}_t)=P(\Lambda|\mathfrak{F}_t)$ with $\Lambda'=\lim_{n\to\infty}\theta^{s_n}\Lambda$ on letting $n\to\infty$. Because $\theta^s\Lambda$ is decreasing in s and $\theta^s\Lambda\supseteq\Lambda$ for all $s>0$ we conclude that $\Lambda=\Lambda'$ a.s. as stated.

Step 3: We shall next show that $\{X(t)/x_t'\}$ converges a.s. as $t\to\infty$ for some constants $\{x_t'\}$. Indeed, choose $\xi\varepsilon(0,1)$ such that $F(x_0)=\xi$ for a continuity point x_0 of F. Then x_0 must be a point of type I for $\{X(t_n/a(t_n)\}$ since otherwise Theorem 2.9 of [5] would imply that there exist Λ_1 and Λ_2 with $\Lambda_1=\lim_{n\to\infty}\{X_{t_n}\varepsilon J^1_{t_n}\}$ a.s. and $\Lambda_2=\lim_{n\to\infty}\{X_{t_n}\varepsilon J^2_{t_n}\}$ a.s. with $P(\Lambda_1)<F(x_0)<P(\Lambda_2)$ but no such set Λ with $P(\Lambda_1)<P(\Lambda)<F(x_0)$. However $P(\theta^s\Lambda_1)>P(\Lambda_1)$ for all $s>0$ by Step 1, and by Step 2 $P(\theta^s\Lambda_1)$ can be chosen such that $P(\Lambda_1)<P(\theta^s\Lambda_1)<F(x_0)$ and we reached a contradiction. Thus x_0 is a point of type I and therefore there exist some left-unbounded intervals $\{I_t\}$ with right end-points $\{x_t'\}$ such that $\Lambda=\lim_{t\to\infty}\{x_t\varepsilon I_t\}$ with $P(\Lambda)=F(x_0)$. It is further easy to see that

$$(9)\qquad \lim_{n\to\infty}P(\{X(t_n)\varepsilon I_{t_n}\}\Delta\{X(t_n)\leqslant a(t_n)x_0\})=0$$

Δ being the symbol of symmetric difference of two sets. Since $\lim_{t\to\infty}\{X(t)\varepsilon I_t\}$ a.s. exists we get

(10) $\lim_{n\to\infty}P(X(t_n)\leqslant a(t_n)x_0)=\lim_{n\to\infty}P(\{X(t_n+s)\varepsilon I_{t_n+s}\}\Delta\{X(t_n)\leqslant a(t_n)x_0\})$

On the other hand, condition (B) implies

(11) $\lim_{n\to\infty}P(a(t_n)(x_0-\varepsilon),(-\infty,a(t_n)x_0\rho^s))=1$

and

(12) $\lim_{n\to\infty}P(a(t_n)(x_0+\varepsilon),(-\infty,a(t_n)x_0\rho^s))=0$

Stochastic monotonicity applied to (11) and (12) yields

(13) $\lim_{n\to\infty}P(x,(-\infty,a(t_n)x_0\rho^s))=1$

uniformly for $x<a(t_n)(x_0-\varepsilon)$, and

(14) $\lim_{n\to\infty}P(x,(-\infty,a(t_n)x_0\rho^s))=0$

uniformly for $x>a(t_n)(x_0+\varepsilon)$.

Taking into account (10), (13) and the continuity of F at x_0 we get

(15) $F(x_0)=\lim_{n\to\infty}\int_{\{x\leqslant a(t_n)x_0\}}P_s(x,(-\infty,a(t_n)x_0\rho^s)\nu_{t_n}(dx)$

which is equivalent to

(16) $\lim_{n\to\infty}P(X(t_n)\leqslant a(t_n)x_0)=\lim_{n\to\infty}P(\{X(t_n+s)\leqslant a(t_n)x_0\rho^s\}\quad\{X(t_n)\leqslant a(t_n)x_0\})$

Proceeding in the same way as above, but using (14) instead of (13) we get

$$(17)\quad \lim_{n\to\infty} P(X(t_n)>a(t_n)x_0)=\lim_{n\to\infty} P(\{X(t_n+s)>a(t_n)x_0\rho^s\} \quad \{X(t_n)>a(t_n)x_0\})$$

It is easy to see that (10), (16) and (17) yield

$$(18)\quad \lim_{n\to\infty} P(\{X(t_n+s)\varepsilon I_{t_n+s}\}\Delta\{X(t_n+s)\leqslant a(t_n)x_0\rho^s\})=0$$

Because x_0 was chosen to be an arbitrary continuity point of F we get $\lim_{n\to\infty} x'_{t_n+s}/x'_{t_n}=\rho^s$ and since $\{t_n\}$ was assumed to be an arbitrary sequence with $\lim_{n\to\infty} t_n=\infty$ such that $\{X(t_n)\}$ converges in distribution we get $\lim_{t\to\infty} x'_{t+s}/x'_t=\rho^s$ for any $s\varepsilon(0,\delta)$. It is easy to see that the latter implies $\lim x'_{t+s}/x'_t=\rho^s$ for any real s. Recall that $\lim_{t\to\infty}\{X(t)\varepsilon I_{t+s}\}$ a.s. exists for all s and the above considerations boil down to the existence of $\lim_{t\to\infty}\{X(t)\leqslant\rho^s x'_t\}$ a.s. But ρ^s can take any value as s is at our disposal. It follows that $\{X(t)/x'_t\}$ converges a.s. as $t\to\infty$.

Step 4: To show that F satisfies condition (C) it suffices to notice that the argument used in the proof of Theorem 4.6 of [5] applies in this case as well. Indeed, although the existence of ρ was proved in [5] under assumptions different from right-continuity for $\{X(t)\}$, it is clear that $\lim_{t\to\infty} a(t+s)/a(t)=\rho^s$ for all s suffices for the proof that F satisfies condition (C). We note that unlike Theorem 4.6, Theorem 1 does not assume that a non-degenerate limit exists (for subsequences), but such limits appear non-degenerate from the properties of $\theta^s\Lambda$ described by Steps 1 and 2.

Step 5: To prove that the conditions of Theorem 1 are necessary notice first that tightness is obviously a prerequisite for convergence in distribution. On the other hand, convergence in distribution in conjunction with right-continuity for $\{X(t)\}$ may be easily shown to supercede the conditions of Theorem 4.9 of [5] preserving its conclusions. Thus $\{a'(t)X(t)\}$ converges a.s. for some $\{a'(t)\}$. This is readily seen to imply condition (B1) and a simple exercise in stochastic monotonicity along the lines of the proof of Step 3 concludes that condition (B1) implies condition (B), completing the proof.

Proof of Theorem 2 We shall show that the conditions of Theorem 1 are satisfied. Indeed, by well-known properties for sequences of distribution functions (see e.g. [16]), any subsequence of $\{X(t)/\rho^t\}$ contains another subsequence converging to a proper distribution, whereas the means and variancs of such sequences converge to the mean and the variance of the limit distribution. The positivity of the variance of the limit distribution makes such a distribution non-degenerate and tightness follows.

Condition (B2) implies condition (B) which was shown to be equivalent to condition (B1). Indeed, this can be obtained by applying the Chebyshev's inequality.

5. Applications

(i) Diffusions: It has been noticed by several authors that diffusions are SM. Indeed, the birth and death processes are SM (see e.g. [11]). Since any diffusion can be seen to be a limit of birth and death processes (see [19]) it follows that diffusions are SM. Examples of diffusions to which Theorem 2 applies may be found in Rosler's paper [18] where $\{a(t)X(t)\}$, for suitable chosen $\{a(t)\}$, is shown to converge a.s. to a limit variable X which generates the tail σ-field $\mathcal{T}$ and determines the Martin boundary of the process. Such converging processes were derived from Ornstein-Uhlenbeck processes in $\mathbb{R}^n$. For some background on Ornstein-Uhlenbeck processes see [10].

Another diffusion that falls into the category studied in this paper is a diffusion process that approximates some types of Galton-Watson processes (see [6], [8] and [15]). In this case $E(X(t)|X(0)=x_0)=x_0e^{\beta t}$ and $Var(X(t)|X(0)=x_0)=\alpha x_0/\beta e^{bt}(e^{\beta t}-1)$. It is easy to see that if we modify $E(X(t)|X(0)=x_0)$ and $Var(X(t)|X(0)=x_0)$ by allowing perturbation factors, the conditions of Theorem 2 may still hold, such that establishing a.s. convergence in such cases is no longer dependent on finding a lucky martingale trick.

2. Markov branching processes

We shall consider a supercritical Markov branching process with offspring distribution $\{p_k\}$ and life-time distribution $\beta e^{-\alpha x}dx$ (see [1] Chapter IV). Define a(t) to be a γ-quantile of the distribution function of X(t) where $q<\gamma<1$, q being the extinction probability of $\{X(t)\}$. Since

$$(19)\quad X(t+u) = \Sigma_{i=1}^{X(t)} X_{(t+u)}^{t,i}$$

where conditional on $\mathcal{F}_t$ the $X_{(t+u)}^{t,i}$ are independent and distributed as X(u). It is easy to see from (19) that $\{X(t+u)/X(t)\}$ turns out to converge in probability to $E(X(u))=\rho^u$ as $t\to\infty$ for some $\rho>1$. This property entails $\lim_{t\to\infty}a(t+u)/a(t)=\rho^u$ for u>0 such that condition (A1) is satisfied. To prove tightness for $\{X(t)/a(t)\}$ we notice that (19) and $\lim_{t\to\infty}a(t+u)/a(t)=\rho^u$ for u>0 lead us to conclude that W, the weak limit of any converging subsequence of $\{X(t)/a(t)\}$, is distributed as $1/\rho^t \Sigma_{i=1}^{X(t)} W^{t,i}$ where $\{W^{t,i}\}$ are distributed as W and are independent given $\mathcal{F}_t$. From here it is easy to see that $P(W<\infty)=1$ where the non-degeneracy of W follows from the way $\{a(t)\}$ were chosen. Thus Theorem 1 applies and provides a straightforward proof of a result which is usually proved by using properties pertaining to generation functions.

There are many models of branching processes (see e.g. [13] and [14]) that, under appropriate conditions, may be considered stochastically monotone. This observation may help relax the conditions imposed for their a.s. convergence and provide characteristics of their limit distribution functions.

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References

[1] S. Asmussen, H. Hering. Branching Processes. Boston-Basel-Stuttgart Birkhauser, 1983.

[2] H. Cohn. On the norming constants occurring in convergent Markov chains. Bull.Austral.Math.Soc. 17, 193-205, 1977.

[3] H. Cohn. On the invariant events of a Markov chain. Z. Wahrscheinlich. 48, 81-96, 1974.

[4] H. Cohn. On the convergence of stochastically monotone sequences of random variables and some applications. J.Appl.Prob. 18, 592-605, 1981.

[5] H. Cohn. Limit behaviour of stochastically monotone Markov chains. Preprint.

[6] D.R. Cox, H.D. Miller. The Theory of Stochastic Processes. Methuen, London, 1965.

[7] D.J. Daley. Stochastically monotone Markov chains. Z. Wahrscheinlich. 10, 305-317, 1968.

[8] P. Jagers. Branching Processes with Biological Applications. Wiley, London, 1975.

[9] T. Kamae, U. Krengel, G.L. O'Brien. Stochastic inequalities on partially ordered spaces. Ann.Prob. 5, 899-912, 1977.

[10] T. Kamae, U. Krengel. Stochastic partial ordering. Ann.Prob. 6, 1044-1049, 1978.

[11] S. Karlin, H.M. Taylor. A Second Course in Stochastic Processes. Academic Press, New York, 1981.

[12] J. Keilson, A. Kester. Monotone matrices and monotone Markov processes. Stoch.Proc.Appl. 5, 231-241, 1977.

[13] P. Kuster. Generalized Markov branching processes with state-dependent offspring distribution. Z. Wahrscheinlich. 64, 475-503, 1983.

[14] P. Kuster. Cooperative birth processes with linear or sublinear intensity. Stoch.Proc.Appl. 17, 313-325. 1984.

[15] T. Lindvall. Limit theorems for some functionals of certain Galton-Watson process. Adv.Appl.Prob. 2, 309-321, 1974.

[16] M. Loeve. Probability Theory I. 4th Ed. Springer, New York, 1977.

[17] A. Renyi. On mixing sequences of sets. Acta Math.Acad.Hungar. 9, 215-228, 1958.

[18] U. Rosler. The Martin boundary for time-dependent Ornstein-Uhlenbeck processes. Habilitation thesis. Gottingen, 1982.

[19] C. Stone. Limit theorems for random walks, birth and death processes, and diffusion processes. Ill.J.Math. 7, 638-660, 1963.

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