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Notes on Right-(Left-) Continuous Functions

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§1. Measurability

If $\mathcal{I}_r = \{[a,b): a,b \in \mathbb{R}\}$, then $\mathcal{I}_r$ is a base for the topology $\mathcal{T}_r = \{\cup\mathcal{C} : \mathcal{C} \subseteq \mathcal{I}_r\}$. If $\mathcal{T}$ denotes the natural toplogy on $\mathbb{R}$, then a function $f : (\mathbb{R}, \mathcal{T}_r) \to (\mathbb{R}, \mathcal{T})$ is continuous iff $f : (\mathbb{R},\mathcal{T}) \to (\mathbb{R},\mathcal{T})$ is right-continuous.

To prove that a right-continuous function on $(\mathbb{R}, \mathcal{T})$ is Borel-measurable, it suffices to prove $\sigma(\mathcal{T}_r) \subseteq \sigma(\mathcal{T}) = \mathcal{B}$, which amounts to prove $\sigma(\mathcal{T}_r) = \mathcal{B}$, since obviously $\mathcal{T} \subset \mathcal{T}_r$. We shall now prove this [and a little more] by proving $\mathcal{T}_r \subset \mathcal{B}$.

Proposition If $T \in \mathcal{T}_r$, then T has a unique representation $T = \bigcup_{j\in\mathcal{J}} I_j$, where I_j is an interval of the type (a,b) or $[c,b)$; $a,b \in \overline{\mathbb{R}}$, $c \in \mathbb{R}$. $\mathcal{J}$ is countable and if $\#\mathcal{J} \neq \{0,1\}$ then the intervals are pairwise disjoint and maximal in the sense that $\sup I_j \notin T$. [If $T = \emptyset$, then $\mathcal{J} = \emptyset$].

Proof Suppose $\emptyset \neq T \in \mathcal{T}_r$ and write $x \sim y$ if $x,y \in T$ and $[x,y] \subseteq T$. Obviously $\sim$ is an equivalence relation and the equivalence classes are intervals of the stated types and therefore each of them contains a rational number. Hence $\mathcal{J}$ is at most countable, and the rest is obvious.

Remark If we are only interested in proving $T \in \mathcal{B}$ then the properties of the rationals could be used more effectively as suggested by S. Tolver Jensen. Here is his beautiful proof: If $\mathcal{C} \subseteq \mathcal{I}_r$, then $\cup\mathcal{C} = \bigcup_{q\in\mathbb{Q}} \cup\{I \in \mathcal{C}: q \in I\}$, and clearly $\cup\{I \in \mathcal{C}: q \in I\}$ is an interval.

An easy and more conventional way to prove the measurability of the right-continuous functions is to construct a proper sequence of step-functions. Suppose $f : \mathbb{R} \to \mathbb{R}$ is right-continuous and define the following functions on $\mathbb{R}$ for $n = 1,2,\ldots$

$$\phi_n = \sum_{k=-\infty}^{\infty} f((k+1)2^{-n}) \cdot 1_{[k2^{-n},(k+1)2^{-n})} \tag{1}$$

$$\psi_n = \sum_{k=-n}^{n} f((k+1)2^{-n}) \cdot 1_{[k2^{-n},(k+1)2^{-n})} \tag{2}$$

The sequence $(\phi_n)_{n\geqq 1}$ of elementary step-functions will then converge pointwise to f on $\mathbb{R}$, and the same holds for the simple functions (2).

Note that the functions (1) and (2) are right-continuous and therefore the pointwise convergence is uniform on any compact set in the space $(\mathbb{R}, \mathcal{T}_r)$.

§2. The space $(\mathbb{R}, \mathcal{T}_r)$.

The space $(\mathbb{R}, \mathcal{T}_r)$ has already been defined in §1. The main result in this section is a complete characterization of the $\mathcal{T}_r$-compact sets.

Compactness is an ambiguous concept in the mathematical litterature. We shall say that a set in a topological space is compact iff every open covering has a finite subcovering. When dealing with Hausdorff spaces this terminology conforms to that of Bourbaki. Some authors would prefer the term bicompact [1,2].

From only a rudimentary knowledge of $(\mathbb{R}, \mathcal{T})$ much can be said about $(\mathbb{R}, \mathcal{T}_r)$ because of the relation $\mathcal{T} \subset \mathcal{T}_r$. This will become apparent in the following proposition, which is easily proved.

<u>Proposition 1</u> Let the set X be equipped with two topologies $\mathcal{T}_i$, $i = 1,2$, and suppose $\mathcal{T}_1 \subset \mathcal{T}_2$. Then the following holds.

a) Every $\mathcal{T}_1$-closed set is $\mathcal{T}_2$-closed.

b) Every $\mathcal{T}_2$-compact set is $\mathcal{T}_1$-compact.

c) Every $\mathcal{T}_2$-limit point of $A \subseteq X$ is a $\mathcal{T}_1$-limit point of A.

<u>Proposition 2</u> The space $(\mathbb{R}, \mathcal{T}_r)$ has the following properties:

a) It is a Hausdorff space.

b) It satisfies the First Axiom of Countability, but does not have a countable base.

c) It is a Lindelöf space.

d) Q is everywhere dense.

e) The base $\mathcal{I}_r$ consists of clopen sets (i.e. every [a,b) is both open and closed.)

<u>Proof</u> a), b), d), and e) are obvious. An easy proof of c) is the following. Suppose $\mathcal{C} \subseteqq \mathcal{T}_r$ and $\mathbb{R} \subseteqq \cup\mathcal{C}$. Since $\mathbb{R}$ is a countable union of intervals of the

form $[a,b)$, $a \in \mathbb{R}$, then it is sufficient to prove that every such interval can be covered by a countable subcover of $\mathcal{C}$. Suppose $[x,y)$ is given and define $C = \{z \in [x,y) : \exists\, \tilde{\mathcal{C}} \subseteq \mathcal{C},\ \tilde{\mathcal{C}}$ countable, and $[x,z] \subseteq \cup\tilde{\mathcal{C}}\}$. Obviously $x \in C$ and C is an interval, and it is seen that $y = \sup C$. The rest is easy. □

<u>Definition</u> $x \in \mathbb{R}$, $A \subseteq \mathbb{R}$. A is said to have x as a left-limit point [right-limit point] if $A \cap [y,x) \neq \emptyset$ $[A \cap (x,y] \neq \emptyset]$ for all $y < x$ $[x < y]$.

Of course x is a right-limit point of A iff x is a $\mathcal{T}_r$-limit point of A, but a left-limit point of A is not necessarily a $\mathcal{T}_r$-limit point of A.

<u>Lemma 1</u> If $A \subseteq \mathbb{R}$ has a left-limit point, then A is not $\mathcal{T}_r$-compact.

<u>Proof</u> Suppose A has x_0 as a left-limit point. Then A contains a strictly increasing sequence $(x_n)_{n \geq 1} \uparrow x_0$. The class of sets $\{[x_n, x_{n+1}) : n \in \mathbb{N}\} \cup \{(-\infty, x_1), [x_0, \infty)\}$ is a $\mathcal{T}_r$-open cover of A which does not contain a finite subcover. □

<u>Lemma 2</u> If $A \subseteq \mathbb{R}$ is uncountable, then A has at least one left-limit point and one right-limit point.

<u>Proof</u> Since $A = \cup\{A \cap (-\infty, n) : n \in \mathbb{N}\}$ we may and shall assume in the following that A is bounded above. Define $C = \{z \in \mathbb{R} : A \cap [z,\infty)$ is countable$\}$. Then $C \neq \emptyset$ and bounded below. Hence $x = \inf C \in \mathbb{R}$, and we claim that $A \cap [y,x) \neq \emptyset$ for all $y < x$. The existence of right-limit point follows by considering $-A = \{x : -x \in A\}$ and using the result already proved. □

Of course much stronger result could be proved about limit-points of uncountable sets, but this is all we need.

<u>Proposition 3</u> $A \subseteq \mathbb{R}$ is $\mathcal{T}_r$-compact iff A is countable, bounded, contains all its right-limit points, but does not have any left-limit point.

Proof The necessity of the conditions follows immediately from L.1 and L.2 and the fact that a $\mathcal{T}_r$-compact set is $\mathcal{T}$-compact (See P.1).

To prove the sufficiency of the conditions we proceed as follows. Suppose A has the stated properties and let $\mathcal{C}$ denote a $\mathcal{T}_r$-open covering of A. Define $x_1 = \inf A$.

Then $x_1 \in A$ and $x_1 \in C_1$ for some $C_1 \in \mathcal{C}$. The set $A \setminus C_1$ is $\mathcal{T}_r$-closed and satisfies the conditions stated in P.3. Define $x_2 = \inf(A \setminus C_1)$. Then $x_2 \in A \setminus C$ and $x_1 < x_2$. Proceeding as above we either reach the state where $A \setminus \bigcup_{i=1} C_i = \emptyset$ for some $n \in \mathbb{N}$ or we get a strictly increasing sequence $(x_n)_{n \geqq 1}$ of points in A. Since the latter case would lead to a contradiction the proposition is proved. □

Remark 1 From P.2.c., the Proposition in §1, and the fact that $\mathcal{T}$ has a countable base follows easily the following theorem (A.P. Moore): "Let $\mathcal{I}$ denote the class of all open or closed intervals containing more than one point, and suppose $\mathcal{C} \subseteqq \mathcal{I} \cup \mathcal{I}_\ell \cup \mathcal{I}_r$. Then there is a countable $\tilde{\mathcal{C}} \subseteqq \mathcal{C}$ such that $\bigcup \tilde{\mathcal{C}} = \bigcup \mathcal{C}$."

From Moore's theorem it easily follows that every subspace of $(\mathbb{R}, \mathcal{T}_r)$ is a Lindelöf space. ([1],pp.58-59)

Remark 2 We shall finish this section with a curious example showing that a product space of two Lindelöf spaces, with the usual product topology is not necessarily a Lindelöf space. Consider $(\mathbb{R}, \mathcal{T}_r) \times (\mathbb{R}, \mathcal{T}_r)$. If this space is a Lindelöf space then every closed subset with the relative topology will be a Lindelöf space. The line $y = -x$ is closed and intersects an open set $[z,\infty) \times [-z,\infty)$ in a single point $(z, -z)$. Hence the line with the relative topology is a discrete space and therefore not a Lindelöf space. ([1],p.59).

§3. On the discontinuity of RC-functions

Before formulating our main results concerning discontinuities [T-discontinuities to be more specific] of right-continuous functions [also called RC-functions in the following] we shall define some auxiliary concepts.

Definition 1 Suppose $f: \mathbb{R} \to \mathbb{R}$ and $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$, then

$$\sigma_\ell(f;x) = \begin{cases} f(x) - f(x-), & \text{if } f(x-) \overset{D}{=} \lim_{z \uparrow\uparrow x} f(z) \text{ exist in } \overline{\mathbb{R}}, \\ 0 & \text{otherwise} \end{cases}$$

$$\omega_\ell(f;x) = \limsup_{z\uparrow\uparrow x} f(z) - \liminf_{z\uparrow\uparrow x} f(z) \text{ in } \overline{\mathbb{R}},$$

where $\infty - \infty$ is taken to be zero.

The functions $\sigma_\ell(.;.)$ and $\omega_\ell(.;.)$ might be called the left-saltus and the left oscillation, respectively.

Proposition 1 Suppose $f: \mathbb{R} \to \mathbb{R}$ is right-continuous. Define $\Sigma_r^{(1)}(f) = \{x: \sigma_\ell(f;x) \neq 0\}$ and $\Sigma_r^{(2)}(f) = \{x: \omega_\ell(f;x) > 0\}$. Then the following holds:

a) $\Sigma_r^{(1)}(f) \cap \Sigma_r^{(2)}(f) = \emptyset$ and $\Sigma_r^{(1)}(f) \cup \Sigma_r^{(2)}(f)$ is the set of T-discontinuity points of f.

b) $\lim_{z\downarrow\downarrow x} \sigma_\ell(f;x) = \lim_{z\downarrow\downarrow x} \omega_\ell(f;x) = 0$ for all $x \in \mathbb{R}$.

c) The set $\{x: |\sigma_\ell(f;x)| + \omega_\ell(f;x) > \varepsilon\}$ has no right-limit points, if $\varepsilon > 0$, and in that case is countable.

Proof a,b) are obvious. c) follows from b) and §2.L.2. □

Since the set $\Sigma_r^{(1)}(f) \cup \Sigma_r^{(2)}(f)$ in P.1.a. can obviously be written as $\bigcup_{n \geqq 1}^{\infty} \{x: |\sigma_\ell(f;x)| + \omega_\ell(f;x) > \frac{1}{n}\}$ we have:

Corollary Every right-continuous function $f:\mathbb{R}\to\mathbb{R}$ has at most countably many discontinuity points.

Example 1 Let $(x_n)_{n\geqq 1}$ be any strictly increasing bounded sequence of real numbers and $(c_n)_{n\geqq 1}$ any sequence of real numbers. Consider the right-continuous function

$$f = \sum_{n=1}^{\infty} c_n \cdot 1_{[x_n, x_{n+1})} \quad . \tag{1}$$

If $c = \lim_n c_n \neq 0$, then $x_0 = \lim_n x_n$ is a saltus point and $\sigma_\ell(f;x_0) = -c$. c may belong to $\overline{\mathbb{R}}$, and if $c \in \{-\infty, \infty\}$, then f is unbounded on the closed interval $[x_n, x_0]$. If $\limsup_n c_n > \liminf_n c_n$, then $\omega_\ell(f;x_0) > 0$.

Example 2 Let $A \subseteqq \mathbb{R}$ be any countable subset and $x \to c_x \neq 0$ any function from A into $\mathbb{R}$, such that $\sum_{x\in A} |c_x| < +\infty$. Consider the right continuous function

$$f = \sum_{z\in A} c_z \, 1_{[z,\infty)} \tag{2}$$

Using the notation above, we have $A = \Sigma_r^{(1)}(f)$ and $\Sigma_r^{(2)}(f) = \emptyset$.

Example 3 Let A and $x \to c_x$ be as in Ex.2. Define

$$g(x) = \begin{cases} \sin(\frac{1}{x}), & \text{if } x \in [-\frac{1}{\pi}, 0) , \\ 0 & \text{otherwise on } \mathbb{R} \end{cases} \tag{3}$$

Then f is right-continuous and in the notation above: $\Sigma_r^{(1)}(g) = \emptyset$, $\Sigma_r^{(2)}(g) = \{0\}$ and $\omega_\ell(g;0) = 2$.

Consider now the right-continuous function

$$h(x) = \sum_{z\in A} c_z \cdot g(x-z), \quad x \in \mathbb{R} . \tag{4}$$

We now have $\Sigma_r^{(1)}(h) = \emptyset$, $\Sigma_r^{(2)}(h) = A$, and $\omega_\ell(h,x) = 2|c_x|$, if $x \in A$. □

It will be convenient in the sequel to have special names for the sets $\Sigma_r^{(1)}(f)$ and $\Sigma_r^{(2)}(f)$, and so we shall call them the left-jump spectrum and the left-noise spectrum of f, respectively, where $f:\mathbb{R}\to\mathbb{R}$ is not necessarily right-continuous. The definition of $\Sigma_\ell^{(1)}(f)$ and $\Sigma_\ell^{(2)}(f)$ should be clear. Of course, for every RC-function f, $\Sigma_\ell^{(1)}(f)=\Sigma_\ell^{(2)}(f)=\emptyset$.

<u>Proposition 2</u> Suppose $\Gamma_i\subseteq\mathbb{R}$, $i=1,2$, then necessary and sufficient conditions for the existence of a right-continuous [left-continuous] function $f:\mathbb{R}\to\mathbb{R}$ with left-jump [right-jump] spectrum Γ_1 and left-noise [right-noise] spectrum Γ_2, are that Γ_1 and Γ_2 are disjoint and countable.

<u>Proof</u> Let f_1 be the function defined in Ex.2 with $A=\Gamma_1$, and let f_2 be the function (4) defined in Ex.3 with $A=\Gamma_2$ and consider $f=f_1+f_2$. This proves the sufficiency. The necessity follows from P.1.a and the Corollary above. □

<u>Remark</u> It is somewhat unsatisfactory to work with limit-values $\pm\infty$ outside the space $(\mathbb{R},\mathcal{T})$. This implies that for a RC-function f with an empty left-noise spectrum, i.e. pure left-jump spectrum, the "left-version" $f(x-)\overset{D}{=}\lim_{z\uparrow\uparrow x} f(x)$ is not necessarily a LR-function in our sense. There are at least two possibilities to avoid this. The one is to consider only bounded functions, which we do not find attractive, and the other possibility is to extend $(\mathbb{R},\mathcal{T})$ to $(\overline{\mathbb{R}},\overline{\mathcal{T}})$, where $\overline{\mathcal{T}}$ is the smallest topology containing $\mathcal{T}\cup\{[-\infty,a):a\in\mathbb{R}\}\cup\{(a,\infty]:a\in\mathbb{R}\}$, which is essentially that adopted by Jacobsen in his recently published book on counting processes. [3].

§4. On the variation of RC-functions

The main topic in this section is the variation of a right-continuous function in the vicinity of a point in its noise spectrum. For this purpose we shall introduce the following concept for an arbitrary real function.

Definition If $f : \mathbb{R} \to \mathbb{R}$ and $x \in \mathbb{R}$, then the set

$$\Omega_\ell(f;x) = \{y \in \overline{\mathbb{R}} : \exists (x_n)_{n \geq 1} \uparrow\uparrow x \text{ and } y = \lim_n f(x_n)\}$$

will be called the left-noise of f at the point x.

In an analogous way the right-noise of f at x is defined and denoted by $\Omega_r(f;x)$

If f is a RC-function, then of course $\Omega_r(f;x) = \{f(x)\}$ for all $x \in \mathbb{R}$, and $\Omega_\ell(f;x)$ is equal to $\{f(x)\}$ or $\{f(x-)\}$ if x is a $\mathcal{T}$-continuity point of f or a saltus point respectively. The situation is quite different, when $x \in \Sigma_r^{(2)}(f)$, i.e. x belongs to the left-noise spectrum of f. Note that $\Omega_\ell(f;x)$ is never empty.

Proposition 1 Suppose $f : \mathbb{R} \to \mathbb{R}$, then $\Omega_\ell(f;x)$ and $\Omega_r(f;x)$ are $\overline{\mathcal{T}}$-closed sets for all $x \in \mathbb{R}$.

Proof It suffices to consider the left-noise. Suppose y is a $\overline{\mathcal{T}}$-limit point of $\Omega_\ell(f;x)$. [For the definition of $\overline{\mathcal{T}}$ see the last lines on p.8]. Then there is a sequence $(y_n)_{n \geq 1}$ of points in $\mathbb{R} \cap \Omega_\ell(f;x)$, such that $y = \lim_n y_n$ and $y_n \neq y$ for all n. Since $y_n \in \Omega_\ell(f,x)$ there is a sequence $(x_{nm}, f(x_{nm}))_{m \geq 1}$, such that $(x_{nm})_{m \geq 1} \uparrow\uparrow x$ and $y_n = \lim_m f(x_{nm})$.

Now construct a sequence $(a_n)_{n \geq 1}$ in the following way. Choose $a_1 = (x_{1m_1}, f(x_{1m_1})$, then $a_2 = (x_{2m_2}, f(x_{2m_2}))$, where $x_{2m_2} > (x + x_{1m_1})/2$ etc. ..., $a_n = (x_{nm_n}, f(x_{nm_n}))$, where $x_{nm_n} > (x + x_{(n-1)m_{n-1}})/2, \ldots$. Obviously $a_n \to (x,y)$, hence $\Omega_\ell(f,x)$ is $\overline{T}$-closed since it contains its $\overline{T}$-limit points.

Proposition 2 Suppose F is a non-empty $\overline{T}$-closed set and $x_0 \in \mathbb{R}$, then there exist a RC-function $f:\mathbb{R} \to \mathbb{R}$ such that $\Omega_\ell(f;x_0) = F$ and $\Sigma_r^{(2)}(f) = \{x_0\}$.

Proof There exist a sequence $(y_n)_{n \geqq 1}$ of points in F, such that the $\overline{T}$-closure of $\{y_n : n \in \mathbb{N}\}$ is F. Note that the y_n's are not necessarily all different. Let $(x_n)_{n \geqq 1}$ be strictly increasing sequence, such that $(x_n)_{n \geqq 1} \uparrow x_0$. Define a sequence of functions $g_n : \mathbb{R} \to \overline{\mathbb{R}}$, $n \in \mathbb{N}$, such that

$$g_1(x) = \sum_{n=1}^{\infty} y_n 1_{[x_n, x_{n+1})}(x) , \quad x \in \mathbb{R} , \tag{1}$$

and

$$g_n(x) = n \wedge (- n \vee g_1(\frac{x_0 - x_1}{x_{n+1} - x_n} (x - x_n) + x_1)) , \quad x \in \mathbb{R} , \ n \geqq 2. \tag{2}$$

Note that g is zero outside $[x_n, x_{n+1})$, and consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \sum_{n=2}^{\infty} g_n(x) , \quad x \in \mathbb{R} \tag{3}$$

f is right-continuous and takes the value $n \wedge (- n \vee y_n)$ at the point $z_n = x_n + (x_{n+1} - x_n)(x_n - x_1)/(x_0 - x_1) \in [x_n, x_{n+1})$, $n = 2,3,\ldots$, since

$$f(z_n) = g_n(z_n) = n \wedge (- n \vee g_1(x_n) . \tag{4}$$

Hence $\lim_n f(z_n) = y_n$, $(z_n)_{n \geqq 1} \uparrow\uparrow x_0$ and P.2 now follows from P.1. □

Remark 1 Suppose $\Sigma \subseteqq \mathbb{R}$ is countable and has no limit-point. Let (x_n) be an enumeration of the elements in Σ. It is then easy by the aid of P.2 to prove the existence of a right-continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $\Omega_\ell(f;x_n) = F_n, n = 1,2,\ldots$, where (F_n) are a family of arbitrary prefixed non-empty $\overline{T}$-closed sets. This is of course not the case, if Σ has one or more limit-points. If x_0 is a right-limit point, $z_n \in \Sigma$, and $(z_n)_{n \geqq 1} \downarrow\downarrow x_0$, then, because of §3. P.1.b, $d(F_n) \to 0$, as $n \to \infty$, where $d(F_n)$ stands for the diameter of F_n. Note that $d(F_n) = \omega_\ell(f;z_n)$. If on the other hand x_0 is a left-limit point and $(z_n)_{n\geq 1} \uparrow\uparrow x_0$, then $\limsup_n F_n \subseteq \Omega_\ell(f;x_0)$.

If $f : \mathbb{R} \to \mathbb{R}$, then the graph of f, denoted here by Graph(f), is $\{(x,f(x)) : x \in \mathbb{R}\}$. Writing $(\mathbb{R}^2, \mathcal{T}^2)$ for the plane equipped with the natural product topology we can state the following proposition, which is obvious from what has been said so far.

Proposition 3 Suppose $f : \mathbb{R} \to \mathbb{R}$ is a RC-function. Then

a) $\text{Graph}(f) = \cup\{\Omega_r(f;x) : x \in \mathbb{R}\}$

b) $\text{Graph}(f) \cup \cup\{\Omega_\ell(f;x) : x \in \mathbb{R}\}$ is the $\mathcal{T}^2$-closure of the graph of f. The closure is nowhere dense in $(\mathbb{R}^2, \mathcal{T}^2)$ and so has Lebesgue-measure zero.

We shall close this section by an example showing that a closure of a Borel-function can be the whole plane.

Example Consider the countably infinite set $D = \{x_{nm} \in \mathbb{R} : n,m \in \mathbb{N}\} = \cup\{D_n : n \in \mathbb{N}\}$, where $x_{nm} \neq x_{k\ell}$ if $(n,m) \neq (k,\ell)$, $D_n = \{x_{km} \in D : k = n\}$ is dense in $(\mathbb{R}, \mathcal{T})$ for all $n \in \mathbb{N}$. Note that $D_n \cap D_k = \emptyset$ if $n \neq k$. Let $A = (a_n)_{n\geq 1}$ denote an enumerated dense set in $(\mathbb{R}, \mathcal{T})$, and consider the following Borel-function $f : \mathbb{R} \to \mathbb{R}$,

$$f = \sum_{n=1}^{\infty} a_n 1_{D_n} \tag{5}$$

We claim that $\mathbb{R} = \Omega_\ell(f;x) = \Omega_r(f;x)$ for all $x \in \mathbb{R}$, and $\overline{\text{Graph}(f)} = \mathbb{R}^2$. Of course $f = 0$ a.e. $[\lambda_1]$, where λ_1 is the Lebesgue-measure on the line.

§5. Digression

In this section we shall digress somewhat from our main topic and make some comments on M. Jacobsen's book referred to in §3.R. We shall restrict ourselves here to the first section "ONE-DIMENSIONAL COUNTING PROCESSES", pp. 1-52.

One of the basic concepts introduced by Jacobsen is that of "smooth density". Using the notation $\mathbb{R}^+ = (0,\infty)$, $\overline{\mathbb{R}}^+ = \mathbb{R}^+ \cup \{\infty\}$, and $\overline{\mathcal{B}} = \sigma(\overline{\mathcal{T}})$, the concept can be defined as follows:

Definition 1 A probability P on $(\overline{\mathbb{R}}^+, \overline{\mathcal{B}} \cap \overline{\mathbb{R}}^+)$ is said to have a smooth density $f:(\mathbb{R}^+, \mathcal{T} \cap \mathbb{R}^+) \to (\overline{\mathbb{R}}_0^+, \overline{\mathcal{T}} \cap \overline{\mathbb{R}}_0^+)$, if

a) f is right-continuous;

b) $f(0+) \overset{D}{=} \lim_{t \downarrow\downarrow 0} f(t)$ exist;

c) $f(t-)$ exist for every $t \in \mathbb{R}^+$;

d) $P(0,t] = \int_0^t f(s)\,d\lambda_1$ for all $t \in \mathbb{R}^+$, where λ_1 is the Lebesgue measure on the line.

Remark 1 We shall say that a function $f:\mathbb{R} \to \overline{\mathbb{R}}_0^+$ is a smooth density function, if it satisfies the conditions in D.1. above. We shall see in the next section that the integral in D.1.d. may be written $\int_0^t f(s)ds$ and considered as a Riemann integral.

Following Jacobsen we shall denote by F and G the distribution function $F(t) = P(0,t]$ defined on $\mathbb{R}^+$ and the survivor function $G = 1 - F$, respectively The point $t^\dagger = \inf\{t > 0 : G(t) = 0\}$ is called the termination point of P.

Definition 2 Suppose P has a smooth density f. The intensity or hazard for P is the function $\mu:(\mathbb{R}^+, \mathcal{T} \cap \mathbb{R}^+) \to (\overline{\mathbb{R}}_0^+, \overline{\mathcal{T}} \cap \overline{\mathbb{R}}_0^+)$ defined by

$$\mu = (f/G) \cdot 1_{(0,t^\dagger)} . \tag{1}$$

From D.2 follows, that

$$G(t) = \exp(-\int_0^t \mu(s)dx) = G(\infty -) + \int_t^\infty f(s)ds. \qquad (2)$$

Proposition 1 Suppose P has a smooth density, then the intensity for P $\mu : (\mathbb{R}^+, \mathcal{T} \cap \mathbb{R}^+) \to (\overline{\mathbb{R}}_0^+, \overline{\mathcal{T}} \cap \overline{\mathbb{R}}_0^+)$ is characterized by the following properties:

a) μ is right-continuous;

b) $\mu(0+)$ exist;

c) $\mu(t-)$ exist for every $t \in \mathbb{R}^+$ except possibly at $t^\dagger$;

d) $\int_0^h \mu(s)ds < \infty$ for some $h > 0$;

e) $\mu(t) = 0$ whenever $\int_0^t \mu(s)ds = \infty$.

f) $\mu(t) = D^+(-\ln G(t)) \overset{D}{=} -\lim_{h \downarrow\downarrow 0} (\ln G(t+h) - \ln G(t))/h$, if $t < t^\dagger$.

At this stage we want to point out the following: i) Jacobsen's class of smooth densities and the corresponding class of intensity or hazard functions are rather restrictive and very elementary functions are excluded; (ii) the fact that the intensity functions do not necessarily have a left-limit at the termination point (and only at this point) is rather annoying and the space of these functions becomes rather awkward; iii) the concept of "intensity function" is much more fundamental for the whole theory than that of "smooth density". Hence we shall introduce below a different smoothness requirements. Of course this will be done in such a way that all the main results in [3] hold under these requirements, which is indeed the fourth and main reason for writing this section.

Definition 1 A probability P on $(\overline{\mathbb{R}}^+, \mathcal{B} \cap \overline{\mathbb{R}}^+)$ is said to have a left-smooth intensity $\mu : (\mathbb{R}^+, \mathcal{T} \cap \mathbb{R}^+) \to (\overline{\mathbb{R}}_0^+, \overline{\mathcal{T}} \cap \overline{\mathbb{R}}_0^+)$, if

a) μ is left-continuous;

b) $P(0,t] = 1 - e^{-\int_0^t \mu(s)ds}$ for all $t \in \mathbb{R}^+$;

c) $\mu = 0$, if $t > t^\dagger$.

Remark 2 Note that P.1.d. is satisfied, since

$$\lim_{t \downarrow\downarrow 0} G(t) = \lim_{t \downarrow\downarrow 0} \exp\{-\int_0^t \mu(s)ds\} = 1. \tag{3}$$

After we have decided on the class of intensity functions, specific properties of the corresponding density functions are of little or no concern regarding the exposition of the theory of multiplicative intensity models for counting processes in Jacobsen's book [3]. On the other hand it is of importance for practical purposes that the class of density functions (or distribution functions) is reasonably extensive and contains all the well known functions, which have proved useful (this is probably the reason why Jacobsen prefers to start with "smooth density" as a basic concept). Let this be the justification for the following proposition, which is easily proved.

Proposition 1' A probability P on $(\overline{\mathbb{R}}^+, \mathcal{B} \cap \overline{\mathbb{R}}^+)$ has a left-smooth intensity iff the corresponding distribution function F (or survival function G) is left-differentiable with left-continuous derivative, except possibly at $t^\dagger$, $f:(\mathbb{R}^+, \mathcal{B} \cap \mathbb{R}_0^+) \to (\overline{\mathbb{R}}_0^+, \overline{\mathcal{B}} \cap \overline{\mathbb{R}}_0^+)$, such that for $t \neq t^\dagger$

$$f(t) = D^- F(t) = \mu(t) \cdot \exp\{-\int_0^t \mu(s)ds\} \tag{4}$$

Remark 3 Note that F can be recovered from (4) by partial integration in Riemannian sense. Note also that the anomaly that can arise at the termination points in Jacobsen's class of intensity functions occur now in our class of density functions.

Two other basic concepts in Jacobsen's book and of great concern here are the "canonical counting process of class H" and the corresponding "intensity process". Let us briefly sketch what leads up to these concepts.

Definition 3 If $\overline{\mathbb{N}}_0 = \mathbb{N}_0 \cup \{\infty\}$, then the full counting process path-space is the set $\overline{W}$ of functions $w: (\mathbb{R}_0, T \cap \mathbb{R}_0) \to (\overline{\mathbb{N}}_0, T \cap \overline{\mathbb{N}}_0)$, which are right-continuous, non-decreasing, increasing only in jumps of size 1, and $w(0) = 0$. The stable counting process path-space is the subset $W = \{w \in \overline{W} : w(t) < \infty$ for all $t \geq 0\}$.

Using the terminology introduced in §3.p.6, which is easily adapted to $\overline{W}$, the following proposition concerning the structure of the paths should be obvious [See §3.P.1.c].

Proposition 2 If $w \in \overline{W}[W]$, then $\Sigma_r^{(2)}(w) = \emptyset$, $\Sigma_r^{(1)}(w)$ is countable and has at most one limit point. In that case it is a left-limit point and does not belong to the jump spectrum $\Sigma_r^{(1)}(w)$.

Let $\overline{F}_s$ and $\overline{F}$ denote the smallest σ-algebras over $\overline{W}$ w.r.t. which the projection $N_t : \overline{W} \to (\overline{\mathbb{N}}_0, \sigma(T \cap \mathbb{N}_0))$ is measurable for all $t \in [0,s]$, all $t \in \mathbb{R}_0$, respectively. For the restrictions to W we have $F_s = \overline{F}_s \cap W$ and $F = \overline{F} \cap W$.

Definition 4 A canonical one-dimensional counting process in short CCP, is a probability on $(\overline{W}, \overline{F})$ and a stable canonical one-dimensional counting process is a probability on (W, F).

We now come to the basic idea of Jacobsen. From the exceedingly simple structure of the jump spectrum $\Sigma_r^{(1)}(w)$, $w \in \overline{W}$' [P.2] and the fact that it determines completely the path w, he is led to consider the distribution of the first jumping time τ_1, the distribution of the second jumping time τ_2 given τ_1 etc. This results in his basic theorem, which we shall now quote ([3], p. 16).

Theorem Suppose given for $n \in \mathbb{N}_0$ and any $0 < t_1 < \ldots < t_n < \infty$ a probability concentrated on the interval $(t_n, \infty]$ with survivor function $G_{nt_1 \ldots t_n}$ such that the collection of probabilities satisfies that for every $t > 0$ the mapping $(t_1,\dots,t_n) \to G_{nt_1\dots t_n}(t)$ is measurable. Then there is a unique canonical counting process P such that for $n \in \mathbb{N}_0$, $t > 0$

$$P(\tau_{n+1} > t \mid \overline{F}_{\tau_n}) = G_{n\tau_1\dots\tau_n}(t) \quad \text{P-a.s. on} \quad [\tau_n < \infty] \tag{5}$$

Remark 2 The probabilities concentrated on $(t_n,\infty]$ are probabilities on $\overline{\mathbb{R}}^+$ with $G_{nt_1\dots t_n}(t_n) = 1$. $\tau_0 \overset{D}{\equiv} 0$ and $\overline{F}_{\tau_n}$ $[F_{\tau_n}]$ is the σ-algebra generated by $\tau_0,\dots,\tau_n$. In particular $\overline{F}_0 = \overline{F}_{\tau_0} = \{\emptyset,\overline{W}\}$ and $F_0 = \{\emptyset,W\}$.

Definition 5 A CCP,P, is said to belong to the class H, if the members of a corresponding family of G-functions, $\{G_{nt_1\dots t_n}\}$ have smooth densities (See D.1).

Suppose $P \in H$ and a corresponding family of G-functions has $\{\mu_{nt_1\dots t_n}\}$ as a class of intensities (See D.2). Then Jacobsen defines on $(\overline{W},\overline{F})[(W,F)]$ the stochastic process $(\lambda_t)_{t>0}$ given by

$$\lambda_t = \begin{cases} \mu_{N_t\tau_1\dots\tau_{N_t}}(t) & \text{on } [N_t < \infty] \\ 0 & \text{on } [N_t = \infty]. \end{cases} \tag{6}$$

Obs. On (W,F) $[N_t = \infty] = \emptyset$.

We are now able to define a very important concept. We quote Jacobsen ([3], p.28).

Definition 6 For a canonical counting process P of class H, the intensity process $\lambda_- = (\lambda_{t-})_{t>0}$ is given by $\lambda_{t-} = \lim_{s\uparrow\uparrow t} \lambda_s$ for $t < \tau_\infty$, $\lambda_{t-} = 0$ for $t \geqq \tau_\infty$.

Remark 4 In D.6 $\tau_n(w) = \infty$ for $n > m$ if w only jumps m times, and $\tau_\infty \overset{D}{=} \lim_{n\uparrow\uparrow\infty} \tau_n$. The intensity process is determined module indistinguishability ([3],p.28).

Jacobsen gives explicitly two reasons for using λ_- as intensity process rather than λ ([3],p.26),viz. i) λ_- is predictable, i.e. λ_{t-} is measurable w.r.t. $\overline{F}_{t-} \overset{D}{=} \bigvee_{s<t} \overline{F}_s$;ii) for all $n \in \mathbb{N}_0$, $\lambda_{\tau_n -} > 0$ P-a.s. on $[\tau_n < \infty]$.

Consider now the following two definitions regarding left-smoothness to be compared with D.5 and D.6 above.

<u>Definition 5'</u> A CCP,P, is said to belong to the class H_ℓ, if the members of corresponding family of G-functions $\{G_{nt_1 \ldots t_n}\}$ have left-smooth intensities. (See D.1').

<u>Definition 6'</u> If P is a CCP of class H_ℓ, then the corresponding intensity process $\lambda = (\lambda_t)_{t>0}$ is defined by

$$\lambda_t = \begin{cases} \mu_{N_{t-}\tau_1 \ldots \tau_{N_{t-}}} & \text{on } [N_t < \infty] \\ 0 & \text{on } [N_t = \infty] \end{cases} \tag{7}$$

where $\mu_{N_t \tau_1 \ldots \tau_{N_t}}$ is the intensity of the G-function $G_{N_t \tau_1 \ldots \tau_{N_t}}$ (See D.2').

We now claim that (7) is predictable and $\lambda_{\tau_n} > 0$ on $[\tau_n < \infty]$ P-a.s. We also have, that

$$\lambda_t = \lim_{h \uparrow\uparrow 0} P(N_t - N_{t-h} \geqq 1 | F_{t-h})/h, \tag{8}$$

P-a.s. on $[N_t < \infty]$, and Jacobsens propositions and theorem become valid after proper and trivial modifications. It therefore seems clear that the "extension" to left-smoothness is a natural one and will result in a more elegant exposition of the subject.

Remark 5 In his definition of a martingale ([3]p.38), Jacobsen requires that the sample paths are right-continuous having left-limits everywhere, and in his proposition p.46 he proves that the likelihood process, $(\ell_t)_{t\geq 0}$ there considered is a martingale. In the mathematical setting just suggested above, we should prefer open intervals, $[0,t)$, instead of closed ones in the construction of the process. Then ℓ_t becomes

$$\ell_t = (e^{-\Lambda_t} \prod_{k=1}^{N_{t-}} \lambda_{\tau_k})/\mu^{N_{t-}} e^{-\mu t}), \tag{9}$$

which is left-continuous, predictable, and satisfies

$$E\{|\ell_t|\} < +\infty, \quad E\{\ell_t | F_s\} = \ell_s \quad \text{for} \quad s < t.$$

Hence (9) is a martingale in the "classical" sense, - see for example [4](J. Doob, 1953).

Remark 6 It is regrettable that the proofreading of the first section of Jacobsens inspiring "Lecture Notes" had not been very successful. There are more than dozen minor "misprints" and inaccuracies. In 3.12. Th.(p.23) and its proof the condition $[N_\sigma < \infty]$ is missing, and the proof needs an amendment. As it stands the theorem and its proof is valid for stable CCP's. The comments following the definition of the integrated intensity p.33 should be compared with §3. C.p.7 and R.1.p.12 above.

§6. On the Riemann integrability of RC-functions

In this section we shall briefly collect some useful facts concerning the classical Riemann integral and study its natural extension to unbounded functions on unbounded intervals. For convenience we begin defining two function-spaces, which are of special concern in this and subsequent sections.

Definition 1 Let $I \subseteqq \mathbb{R}$ denote an interval.

i) By $RC(I)$ we denote the vector space over $\mathbb{R}$ of all right-continuous functions from $(I, \mathcal{T} \cap I)$ into $(\mathbb{R}, \mathcal{T})$.

ii) By $\overline{RC}(I)$ we denote the space of all right-continuous functions from $(I, \mathcal{T} \cap I)$ into $(\overline{\mathbb{R}}, \overline{\mathcal{T}})$.

iii) By $LC(I)$ and $\overline{LC}(I)$ we denote the spaces of left-continuous functions corresponding to i) and ii) respectively.

iv) By $RC(I)^+$ $[LC(I)^+]$ and $\overline{RC}(I)^+$ $[\overline{LC}(I)^+]$ we denote the cones over $\mathbb{R}_0^+$ and $\overline{\mathbb{R}}_0^+$, respectively, of non-negative functions in the corresponding spaces above.

Proposition 1 The space $RC(I)$ $[\overline{RC}(I)]$ contains all the constant functions on I and is closed under the binary operations $\vee$ and $\wedge$, i.e. "max" and "min".

Remark 1 Since P.1. is obvious we shall not prove it, but note the following consequences. If $f \in RC(I)$ $[\overline{RC}(I)]$ then $f^+ \overset{D}{=} f \vee 0$ and $f^- \overset{D}{=} -(f \wedge 0)$ belong to $RC(I)$ $[\overline{RC}(I)]$, $f = f^+ - f^-$, and $|f| = f^+ + f^-$.

We shall now give two equivalent definition of sets of measure zero on the real line, and we shall call such sets null-sets without any reference to Lebesgue-measure, because as noted by F. Riesz: "In fact, the idea of a set of measure zero does not depend essentially on the general theory of measure,and the main properties of these sets can be established in a few words." ([5], p.5).

Definition 2 A subset N of $\mathbb{R}$ will be called a null-set, if it can be covered by a finite number or by a denumerable sequence of intervals whose total length is arbitrarily small.

Definition 2' A subset N of $\mathbb{R}$ will be called a null-set, if it can be covered by a sequence of intervals of finite total length in such a way that every point of N is an interior point of an infinite number of these intervals.

Remark 2 The equivalence of D.2 and D.2' is easily proved ([5],p.6). D.2 is Lebesgue's definition, but D.2' seems to be Riesz's idea. For our purpose D.2 is the proper one.

The following theorem contains one of the main results of the classical theory of Riemann integrals. We assume the reader is acquainted with the rudiments of this theory.

Theorem 1 Suppose $f:(I,\mathcal{T}\cap I)\to(\mathbb{R},\mathcal{T})$ is bounded and $I\subseteq\mathbb{R}$ is a bounded proper interval, i.e. $a=\inf I<b=\sup I\in\mathbb{R}$, then f is Riemann integrable over I iff the set of discontinuity points is a null-set, and in that case

$$\int_a^b f(x)\,dx = \lim_n \sum_{i=1}^n f(\xi_i)(x_i - x_{i-1}) \,, \qquad (1)$$

where $a=x_0<x_1<\ldots<x_n=b$, $x_{i-1}\leqq\xi_i\leqq x_i$, and $\max_i(x_i-x_{i-1})\to 0$ as $n\to\infty$.

Remark 3 A proof of Th.1 can easily be found in the mathematical literature. For those who read Danish a lucid account of the relation of Riemann integration to that of Lebesgue is given in [6], which also contains a proof of Th.1.

From Th.1 and §3.C follows directly the following proposition.

Proposition 2 Suppose $f \in RC(I)$ $[LC(I)]$, where I denotes a bounded proper interval. Then f is Riemann integrable in the classical sense iff f is bounded.

The next proposition is of interest and will be used later in this section.

Proposition 3 Suppose $f_i : (I, \mathcal{T} \cap I) \to (\mathbb{R}, \mathcal{T})$ is Riemann integrable (I and f bounded!), $i = 0,1$, then $f_0 \vee f_1$, $f_0 \wedge f_1$ are Riemann integrable on I.

In particular $(f_i \wedge a) \vee b$, $a,b \in \overline{\mathbb{R}}$, are Riemann integrable.

Proof Let N_i denote the null-set, where f_i is discontinuous, $i = 0,1$. If $x \in I \setminus N_0 \setminus N_1$, then $f_0 \vee f_1$ is continuous of x. Hence the set of discontinuity points of $f_0 \vee f_1$ is a null-set, it is a subset of $N_0 \cup N_1$. The rest is obvious.

We shall now extend the notion of Riemann integrability guided by the wish to preserve the middel-sum property (1) and P.3 if possible. Note that the ingredients of the constructive definition of Riemann integral are: i) bounded proper interval I, ii) bounded function $f: I \to \mathbb{R}$; iii) finite partition of I into subintervals; iv) areal of a rectangle. We want to remove the boundedness requirements in i-ii), and we shall first be concerned with ii).

Definition 3 Suppose $f: (I, \mathcal{T} \cap I) \to (\overline{\mathbb{R}}, \overline{\mathcal{T}})$, where $I, I \subseteqq \mathbb{R}$, denotes a bounded proper interval with endpoints $\alpha < \beta$. Then f is said to be Riemann integrable in extended sense if for all $a,b \in \mathbb{R}$ the functions $(f \wedge a) \vee b$ are Riemann integrable on I (in the classical sense), and

$$\int_\alpha^\beta f(x)dx \overset{D}{=} \lim_{\substack{a \to \infty \\ b \to -\infty}} \int_\alpha^\beta (f \wedge a) \vee b(x)dx \quad \text{exist in } \overline{\mathbb{R}}. \tag{2}$$

We shall say that f is Riemann summable if the integral (2) is finite.

Remark 4 The "integral operator" defined in D.3 is of course linear over $\mathbb{R}$. If f is integrable in the sense of D.3, then f^+ and f^- are. f is summable iff f^+ and f^- are (or if $|f|$ are). Note that the integrability of f^+ and f^- only implies the integrability of f, if either f^+ or f^- is summable.

Remark 5 D.3 does not preserve the middel-sum property (1) for unbounded functions, but P.3 is preserved in the sense that the spaces of Riemann integrable functions in the extended sense are lattices as follows from P.4

Lemma Suppose $f:(I,\mathcal{T}\cap I)\to(\overline{\mathbb{R}}_0^+,\overline{\mathcal{T}}\cap\overline{\mathbb{R}}_0^+)$ and for all $a\in\mathbb{E}$, $f\wedge a$ is continuous except possibly on a null-set N_a, then f is continuous except possibly on a null-set $N_f = \bigcup_{a\in\mathbb{R}^+} N_a = \bigcup_{n\in\mathbb{N}} N_k$.

Proof If N_f denotes the set of discontinuity points of f then it is easy to see, as in the proof of P.3, that $N_a \subseteq N_f$. It is also clear that $a<b$ implies $N_a\subseteq N_b$. Hence $\bigcup_{a\in\mathbb{R}^+} N_a = \bigcup_{k=\mathbb{N}} N_k \subseteq N_f$. On the other hand if $x\in N_f$ and $f(x)<+\infty$, then $x\in N_k$ for $k>f(x)$, and if $f(x)=\infty$ then $x\in N_a$ for some a. Hence $N_f\subseteq \bigcup_{k\in\mathbb{N}} N_k$ and $\bigcup_{k=1}^{\infty} N_k$ is a null-set so the proof is completed.

From L.1 above follows immediately the following proposition by considering separately f^+ and f^-.

Proposition 4 If f is Riemann integrable in the extended sense, then f is continuous everywhere except possibly on a null-set. If g is also Riemann integrable in the extended sense (over the same interval!) then $f\vee g$ and $f\wedge g$ are also.

We are now able to extend further the classical Riemann integral to functions on unbounded intervals.

Definition 4 Suppose $f:(I,\mathcal{T}\cap I)\to(\overline{\mathbb{R}},\overline{\mathcal{T}})$, where I denotes a proper interval on $\mathbb{R}$, then $f \geqq 0$ is said to be Riemann integrable, if f is continuous except possibly on a null-set and its Riemann integral is

$$(R)\int_I f(x)\,dx = (R)\int_\alpha^\beta f(x)\,dx \overset{D}{=} \lim_{\substack{\gamma\downarrow\alpha \\ \delta\uparrow\beta}} \lim_{a\uparrow\uparrow\infty} \int_\gamma^\delta f \wedge a(x)\,dx, \tag{3}$$

where $\alpha = \inf I < \gamma < \delta < \beta = \sup I$ and $a \in \mathbb{R}^+$. f is said to be Riemann summable if (3) is finite.

If $f \geqq 0$ does not hold, then f is said to be Riemann integrable if f^+ and f^- are [implying that f is continuous except possibly on a null-set] and at least one of them are Riemann summable. In this case

$$(R)\int_I f(x)\,dx = (R)\int_\alpha^\beta f(x)\,dx \overset{D}{=} (R)\int_\alpha^\beta f^+(x)\,dx - (R)\int_\alpha^\beta f^-(x)\,dx. \tag{4}$$

If (4) is finite, f is said to be Riemann summable.

<u>Remark 6</u> D.4 is quite natural so we prefer just to speak of Riemann integrable functions. It is also convenient to define $(R)\int_I f(x)dx = 0$, if I is empty or a singleton. It should be clear that the middel-sum property (1) does not hold if f is unbounded and does not necessarily hold if f is bounded on an unbounded interval.

<u>Example 1</u> Define $f : \mathbb{R}_0^+ \to \mathbb{R}$ as follows,

$$f = \begin{cases} 0 \text{ on } [0,\tfrac{1}{2}] \text{ on } [0,\tfrac{1}{2}], \\ n^{-1} - 2|xn^{-1} - 1| \text{ on } [n-\tfrac{1}{2}, n+\tfrac{1}{2}], \text{ if } n \in \mathbb{N} \text{ is even} \\ -n^{-1} + 2|xn^{-1} - 1| \quad \text{ - " - } \qquad \text{ - " - } \quad \text{odd}. \end{cases}$$

Then f, f^+ and f^- are all continuous and bounded. f^+ and f^- are therefore both Riemann integrable, but not summable,

$$(R)\int_0^\infty f^+(x)\,dx = \sum_{\substack{n\in\mathbb{N} \\ \text{even}}} \frac{1}{2n} = \infty = \sum_{\substack{n\in\mathbb{N} \\ \text{odd}}} \frac{1}{2n} = (R)\int_0^\infty f^-(x)\,dx .$$

Hence f is not Riemann integrable, but the "classical" improper Riemann integral exists. Its value is $\frac{1}{2} \cdot \sum_{n=1}^{\infty} \frac{(-1)^n}{n} = - \ell n\sqrt{2}$.

We now come to the following question. What is the class of Riemann integrable functions on unbounded interval such that a middel-sum property like (1) holds for these functions? The answer depends on how the concept "finite partition of the (bounded) interval I into disjoint subintervals" is extended.

What makes (1) interesting and computationally useful is the single requirements on the sequence of partition-points $\max_i (x_i - x_{i-1}) \to 0$ as their number increases. In particular the partition points may be equidistant (!). The consequence of this single requirement is that a complete knowledge of the function is not needed to calculate in a simple way a good estimate of its integral. From this point of view the following definition of W. Feller and quoted from S. Asmussen seems to be the only proper one embrasing important classes of functions. [7]

Definition 5 Suppose $f:(I,\mathcal{T} \cap I) \to (\overline{\mathbb{R}},\overline{\mathcal{T}})$ is bounded, and I a proper interval on $\mathbb{R}$, bounded or unbounded. If $h > 0$ define $I_h^n = I \cap [nh,(n+1)h)$, $n \in Z$, and the two following step-functions on I:

$$\overline{f}_h(x) = \sup_{y \in I_h^n} f(y), \quad \text{if } x \in I \cap I_h^n \; ; \tag{4}$$

$$\underline{f}_h(x) = \inf_{y \in I_h^n} f(y), \quad \text{if } x \in I \cap I_h^n \; . \tag{5}$$

Then $f \geqq 0$ is said to be directly Riemann integrable, if

$$\int_I \overline{f}_h(x)\,dx < +\infty \quad \text{for all } h > 0,$$

and $$\int_I \overline{f}_h(x)\,dx - \int_I \underline{f}_h(x)\,dx \to 0 \quad \text{as } h \to 0.$$

If $f \geqq 0$ does not hold then f is said to be directly Riemann integrable if f^+ and f^- are.

Remark 7 It is obvious that the direct Riemann integral is an extention of the classical one, but to rather restrictive classes of bounded functions on unbounded intervals, and this extension is the only natural one preserving the middel-sum property (1). That the classes of directly Riemann integrable functions on unbounded intervals is rather restrictive is caused by the equidistance of the partition points as will be demonstrated in the following example. Roughly speaking, the functions must go rather fast and smoothly to zero, say at least like $|x|^{-1-\delta}$ $(\delta > 0)$ as $|x| \to \infty$.

Example 2 Consider the function $f: \mathbb{R}_0^+ \to \mathbb{R}$ determined by

$$f = \begin{cases} n^{-1} - |x-n| \text{ on } [n-n^{-1}, n+n^{-1}), & n \in \mathbb{N} \setminus \{1\}, \\ 0 \quad \text{otherwise} . \end{cases}$$

Since f is non-negative and continuous it is Riemann integrable in our sense, even summable, and $(R)\int_0^\infty f(x)\,dx = \sum_{n=2}^{\infty} n^{-2} = \frac{\pi^2}{6} - 1$.

f is also bounded and $f(x) \to 0$ as $x \to \infty$. Choose $h \in (0,1)$ and consider $\overline{f}_h$ (see(4)). Whatever the value of h $\overline{f}_h$ takes all the values n^{-1}, $n \in \mathbb{N} - \{1\}$, and hence $\int_0^\infty \overline{f}_h(x)\,dx \geqq \sum_{n=2}^{\infty} hn^{-1} = h \sum_{n=2}^{\infty} n^{-1} = \infty$, i.e. f is not directly Riemann integrable.

Remark 8 Note that the space of all directly Riemann integrable functions on some unbounded interval I is a vectorspace over $\mathbb{R}$, which is a proper subspace of the vectorspace of all bounded Riemann summable functions on I.

We have now extended the classical Riemann integral without invoking Lebesgue's general theory of measure and integration, but of course every Riemann integrable [summable] function f is Lebesgue integrable [summable] and these integrals of f are the same number.

We finish this section with an important (but obvious!) proposition concerning our main subject, and recall §5.R.1.

<u>Proposition 5</u> If $f \in \overline{RC}(I)$ then f is locally integrable in sense that to every $x \in I$ there exists a $h \in \mathbb{R}^+$ such that $\int_x^{x+h} f(x)dx$ exist. If $f \in RC(I)$ then h can be so chosen that $\int_x^{x+h} f(x)dx$ is finite.

<u>Acknowledgement</u>

During my stay at the Institute of Mathematical Statistics, University of Copenhagen, (12/3-4/8) this year I had the opportunity to attend M. Jacobsen's lectures on counting processes. Studying his "Lecture Notes" I found it necessary to clear up some points for my own benefit.

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