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INSTITUTE OF MATHEMATICAL STATISTICS
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ATTAINABLE RATES OF CONVERGENCE OF MAXIMA

by

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Abstract Any exponential rate of convergence can be obtained for maxima of i.i.d. random variables, while faster than exponential convergence implies that the variables have an extreme value distribution.

Let $\xi_1, \xi_2, \ldots$ be independent identically distributed (i.i.d.) random variables with marginal distribution function (d.f.) F. A rather deep result in central limit theory (see[3], p. 575) is that if there are constants $a_n>0$, $b_n$ such that $|P(a_n(\Sigma_{i=1}^n \xi_i - b_n) \leq x) - \Phi(x)| = O(n^{-\theta})$, for all $\theta>0$, where $\Phi$ is the standard normal d.f., then the $\xi$'s are in fact normal. In this note the analogous (albeit less deep) problem for the maximum $M_n = \max\{\xi_1, \ldots \xi_n\}$ is studied, in particular answering a question of [2].

The basic result for maxima of i.i.d. random variables ("the Extremal Types Theorem") is that if

(1) $$P(a_n(M_n - b_n) \leq x) \to G(x), \qquad \text{as } n\to\infty,$$

at continuity points of the non-degenerate d.f. G, then G is max-stable, i.e.

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for $n=1, 2, \ldots$ there are constants $\alpha_n>0$, $\beta_n$ such that

(2) $$G(x)^n \equiv G(\alpha_n(x-\beta_n)),$$

and that $G$ then is an extreme value distribution, i.e. it is of one of only three specified forms (see e.g. [4], Chapter 1).

The uniform rate of convergence in (1) is measured by

$$\Delta_n(a_n, b_n) = \sup_x |P(a_n(M_n-b_n) \leqq x) - G(x)|.$$

Clearly $\Delta_n(a_n, b_n)$ crucially depends on the choice of the normalizing constants $a_n$, $b_n$, and interest centers on which rate is attainable if the "best" $a_n$, $b_n$ are used, i.e. on

$$\Delta_n = \inf_{a>0,\, b} \Delta_n(a, b)$$

$$= \inf_{a>0,\, b} \sup_x |P(a(M_n-b) \leqq x) - G(x)|$$

$$= \inf_{a>0,\, b} \sup_x |F(x/a + b)^n - G(x)|.$$

Further, the size of $\Delta_n$ is substantially different for different $F$'s, e.g. for the normal distribution $\Delta_n$ is of the order $1/\log n$ and for the uniform distribution of the order $1/n$, while $\Delta_n = 0$ for all $n$ if $F(x) = G(\alpha(x-\beta))$ for some $\alpha>0$, $\beta$, i.e. if $F$ is of the same type as $G$ (see [1]).

Here it will be shown that any exponential rate of decrease of $\Delta_n$ can be obtained, while a faster than exponential rate implies that $\Delta_n = 0$ for all $n$. Further we give an example showing that $\Delta_n$ can decrease arbitrarily slowly.

<u>Theorem</u> Let $\xi_1, \xi_2, \ldots$ be i.i.d. random variables with marginal d.f. $F$, let $G$ be an extreme value d.f. and let $\Delta_n$ be as defined in (3). Then

(i) $\Delta_n \geqq \Delta_1^n$, for $n = 1, 2, \ldots$, and if $\Delta_n = o(\theta^n)$, for all $\theta>0$, then $\Delta_n = 0$ for $n = 1, 2, \ldots$. In particular $F$ then is of the same type as $G$, and hence is an extreme value d.f..

(ii) For any $\theta>0$ and any (extreme value) d.f. $G$ there is a d.f. $F$ such that $0 < \Delta_n \leqq \theta^n$, for $n = 1, 2, \ldots$ .

Proof (i) We will use the elementary inequality

(4) $$|u^n - v^n| = |u - v|\,|\sum_{i=0}^{n-1} u^i v^{n-1-i}| \geqq |u - v| \max(u^{n-1}, v^{n-1}),$$

which holds for $u, v \geqq 0$. Now, for $a>0$, $b$ fixed, and writing $a' = a/\alpha_n$, $b' = b-\alpha_n\beta_n/a$, it follows from (2) and (4) that

(5) $$\begin{aligned}\Delta_n(a, b) &= \sup_x |F(x/a+b)^n - G(x)| \\ &= \sup_x |F(\alpha_n(x-\beta_n)/a+b)^n - G(\alpha_n(x-\beta_n))| \\ &= \sup_x |F(x/a'+b')^n - G(x)^n| \\ &= \sup_x \{|F(x/a'+b') - G(x)| \max(F(x/a'+b')^{n-1}, G(x)^{n-1})\}.\end{aligned}$$

By definition, for any $\Delta$ with $0 < \Delta < \Delta_1$ there is an $x'$ with $|F(x'/a'+b') - G(x')| \geqq \Delta$, so that in particular $\max\{F(x'/a'+b')^{n-1}, G(x')^{n-1}\} \geqq \Delta^{n-1}$. Then, by (5),

$$\begin{aligned}\Delta_n(a, b) &\geqq |F(x'/a'+b') - G(x')| \max(F(x'/a'+b')^{n-1}, G(x')^{n-1}) \\ &\geqq \Delta^n,\end{aligned}$$

and thus, since $\Delta < \Delta_1$ is arbitrary, $\Delta_n(a, b) \geqq \Delta_1^n$, for any $a>0$, $b$, and hence $\Delta_n \geqq \Delta_1^n$.

Next, this shows that if $\Delta_n = o(\theta^n)$, for all $\theta>0$ then $\Delta_1 = 0$, i.e. $F$ is of the same type as $G$. Since $G$ is max-stable it then follows easily that $\Delta_n = 0$, also for $n = 2, 3, \ldots$ .

(ii) We may, without loss of generality assume that $0<\theta<1$. Since $G$ is an extreme value d.f. it is continous and strictly increasing at each $x$ with $0<G(x)<1$, and hence $x_\theta = G^{-1}(\theta)$ is well defined. Let $F$ be given by

$$F(x) = \begin{cases} G(x) & \text{if } x_\theta \leqq x \\ G(x_\theta) & \text{if } x_\theta - 1 \leqq x < x_\theta \\ 0 & \text{if } x < x_\theta - 1. \end{cases}$$

If $\alpha_n$, $\beta_n$ satisfies (2) then

$$\begin{aligned} \Delta_n &\leqq \sup_x |F(x/\alpha_n + \beta_n)^n - G(x)| \\ &= \sup_x |F(x)^n - G(\alpha_n(x-\beta_n))| \\ &= \sup_x |F(x)^n - G(x)^n| \\ &= \sup_{x \leqq x_\theta} |F(x)^n - G(x)^n| \\ &\leqq \theta^n, \end{aligned}$$

since $0 \leqq F(x), G(x) \leqq \theta$, for $x \leqq x_\theta$. Since clearly $\Delta_1 > 0$, and hence $\Delta_n > 0$ for each $n$ by part (i), this proves that $0 < \Delta_n \leqq \theta^n$, for each $n$. (In passing it may be noted that $F$ is not an extreme value d.f., since it is constant and not zero or one in the interval $[x_\theta - 1, x_\theta)$.) □

Now, the promised example showing that for any sequence $\{\theta_n > 0\}$ with $\theta_n \to 0$ there is a $F$ with $\Delta_n \to 0$ but

(6) $\qquad \Delta_n \geqq \theta_n$, for all sufficiently large $n$.

Possibly after replacing $\theta_n$ by $\sup_{k \geqq n} \theta_k$, we may assume that $\theta_n$ is non-increasing, and hence, setting $\eta_n = 4e^e \theta_{e^{n+1}}$ if this quantity is less than one, and $\eta_n = 1$ otherwise, we have that

(7) $\qquad \eta_n' = \eta_{[\log n]} \geqq 4e^e \theta_n$,

for all large $n$. Let $\bar{F}(x) = 1 - e^{-x}$, $x \geqq 0$, and zero otherwise, and define

$$F(x) = \begin{cases} \bar{F}(n+\eta_n), & \text{for } x \in [n, n+\eta_n),\ n = 1, 2, \ldots \\ \bar{F}(x), & \text{otherwise.} \end{cases}$$

It is then straigthforward to check, using e.g. Theorem 1.5.1 of [4], that (1) holds with $a_n = 1$, $b_n = \log n$ and $G(x) = \exp(-e^{-x})$. Further we have the following estimate for the jump of $F^n$ at $x = x_n = [\log n]$,

$$\begin{aligned} F(x_n)^n - F(x_n-)^n &= \bar{F}([\log n] + \eta_{[\log n]})^n - \bar{F}([\log n])^n \\ &= (1-e^{-[\log n] - \eta_n'})^n - (1-e^{-[\log n]})^n \\ &= (1-e^{-[\log n]})^n\{(1+e^{-[\log n]}(1-e^{-\eta_n'})/(1-e^{-[\log n]}))^n - 1\} \\ &\geqq (1-e/n)^n\{(1+(1-e^{-\eta_n'})/n)^n - 1\} \\ &\sim e^{-e}\eta_n', \qquad \text{as } n \to \infty. \end{aligned}$$

Thus, by (7), $F(x_n)^n - F(x_n-)^n \geqq 2\theta_n$, for all large n, and (6) follows immediately, since G is continuous.

## References


[1] Davis, R.A. (1982). The rate of convergence in distribution of maxima. Statist. Neerlandica 36, 31-35.

[2] Galambos, J. (1983). Rates of convergence of maxima. Proceedings NATO Advanced Study Institute on Statistical Extremes and Applications, Vimeiro, Portugal.

[3] Ibragimov, I. (1966). On the accuracy of the Gaussian approximation to the distribution functions of sums of independent random variables. Theory Probab. Appl. 11, 559-579.

[4] Leadbetter, M.R., Lindgren, G. & Rootzén, H. (1983) Extremes and related properties of random sequences and processes. New York: Springer.

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