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Summary

Using properties of last-exit times, and more generally cooptional times, two necessary and sufficient conditions are established, for the existence of an invariant measure for an irreducible, transient Markov chain. The conditions are also related to the classical condition due to Harris and Veech.

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1. Introduction.

One of the main results, Theorem 2 of Harris [1], gives a sufficient condition for an irreducible, transient Markov chain to possess an invariant measure. At the time Harris wrote that he thought the condition to come in some sense close to being necessary and indeed Veech [6] 6 years later in 1963 established the necessity. The time lag is significant, since a vital part of Veech's argument exploits the Martin boundary theory as treated by Hunt [2] in 1960.

Since the appearence of Harris' and Veech's papers most work on invariant measures for transient chains in discrete time, has dealt with the case of a general state space, see e.g. Yang [7], Tweedie [5, Section 13], Shur [4].

In the present paper we return to chains with discrete state space and present in Theorem 1 a new necessary and sufficient condition for the existence of an invariant measure for an irreducible, transient Markov chain. It is then shown (Theorem 2) how one may derive, using the properties of last-exit (and more generally, cooptional) times, another necessary and sufficient condition which resembles the Harris-Veech condition. Finally the gap between these two conditions is bridged in Theorem 3.

The properties of last-exit times are also used in the proof of one half of Theorem 1. While that theorem does not require any boundary theory, the argument employed by Veech and already referred to above, is critical for the establishment of Theorems 2 and 3.

2. Results

We consider Markov chains $X = (X_n, n \geq 0)$ in discrete time on a finite or at most countably infinite state space J. Let $P = (p(x,y))$ for $x,y \in J$ denote the transition function. We shall allow P to be substochastic, i.e. for all x,y

$$p(x,y) \geq 0 , \quad \sum_{y \in J} p(x,y) \leq 1 .$$

With $P^n = (p^{(n)}(x,y))$ the n-step transitions, introduce the Green function

$$G(x,y) = \sum_{n=0}^{\infty} p^{(n)}(x,y) .$$

Throughout we shall assume that P (or X) is irreducible and transient, i.e.

$$0 < G(x,y) < \infty \qquad (x,y \in J) .$$

The reader is reminded of the basic inequalities

(1) $$G(x,y) \leq G(y,y) ,$$

(2) $$\frac{G(x,y)}{G(x,x)} \leq \frac{G(z,y)}{G(z,x)} ,$$

and, as an easy consequence of (2),

(3) $$\sup_z \frac{G(z,y)}{G(z,x)} < \infty , \qquad \sup_z \frac{G(y,z)}{G(x,z)} < \infty .$$

With P the transition function, the distribution of the chain X is specified by its initial distribution. It will be convenient for us to consider the canonical realizations of Markov chains, i.e. we introduce Ω as the space of all sequences $\omega = (\omega_0, \omega_1, \ldots)$ taking values in J, finite or infinite, and denote by P^π the measure on Ω which makes $X = (X_n)$ the

Markov chain with initial distribution π and transitions P, where $X_n(\omega) = \omega_n$. (We shall only consider P^π for π a finite measure on J. Then P^π is a finite measure defined on the smallest σ-algebra $\mathcal{F}$ of subsets of Ω, that make all X_n measurable. The measure P^π concentrates all mass on the set of infinite sequences in Ω (the chain has infinite lifetime) iff P is stochastic. If π is degenerate with unit mass at x, we write P^x instead of P^π. Finally, for $U: \Omega \to \mathbb{R}$ measurable, $P^\pi U$ will denote the P^π-expectation of U.

We have in particular, that

$$G(x,y) = P^x \sum_{n=0}^{\infty} 1_{(X_n=y)} ,$$

and more generally, if π is a probability on J,

$$\pi G(y) = P^\pi \sum_{n=0}^{\infty} 1_{(X_n=y)} ,$$

and $0 < \pi G(y) < \infty$, the finiteness following from (1).

By a <u>measure</u> on J we shall understand a function $\mu: J \to [0,\infty)$. The measure μ is finite if $\sum \mu(x) < \infty$.

<u>Definition</u>. A transition function P has an <u>invariant measure</u> if there exists a measure μ, not identically 0, such that

$$\mu P = \mu .$$

An invariant measure μ for P is <u>unique</u> if ν being any other invariant measure, there exists a constant c such that $\nu = c\mu$. □

Of course, if μ is invariant, $\mu(x) > 0$ for all x by irreducibility.

Suppose μ is invariant for P. Then

$$(4) \qquad \hat{p}(x,y) = \mu(y)p(y,x)\mu^{-1}(x)$$

defines a transition function on J which is stochastic, i.e. $\sum_y \hat{p}(x,y) = 1$. The associated Green's function satisfies

$$\hat{G}(x,y) = \mu(y)G(y,x)\mu^{-1}(x),$$

in particular $\hat{P}$ is irreducible and transient.

Because $GP = G-I$ with I the identity matrix, for any finite measure ν

$$(5) \qquad (\nu G)P = \nu G - \nu$$

with all quantities well defined and finite, i.e. $\mu = \nu G$ is <u>excessive</u> ($\mu P \leq \mu$) with ν the correction term. When searching for an invariant measure it is therefore natural that conditions (i) and (ii) in the result we shall now state, should be satisfied.

<u>Theorem 1</u>. (a) Suppose there exists a sequence $(\nu_n, n \geq 0)$ of finite measures satisfying the following 3 conditions for all $x \in J$:

(i) $\lim_{n\to\infty} \nu_n(x) = 0$,

(ii) $\lim_{n\to\infty} \nu_n G(x) = \mu(x)$,

(iii) $\lim_{n\to\infty} (\nu_n G)P(x) = \mu P(x)$, where $0 < \mu < \infty$. Then μ is invariant for P.

(b) Suppose conversely that μ is invariant for P. Then there exists a sequence (ν_n) of finite measures such that (i), (ii), (iii) are satisfied. More specifically, with $\hat{P}$ as in (4), the ν_n may be chosen as

(6) $$\nu_n(x) = \mu(x)\hat{P}^x(\sigma_n = 1)$$

where σ_n denotes the last-exit time

(7) $$\sigma_n = \sup\{k \geq 1 : X_{k-1} \in K_n\},$$

with the K_n finite subsets of J increasing to J as $n \to \infty$.

Remarks. Writing (iii) as

$$\lim_{n\to\infty} \sum_y \nu_n G(y) p(y,x) = \sum_y \mu(y) p(y,x)$$

we see that with (ii) satisfied it amounts to allowing interchanging the order of summation and taking limits. As will be seen below the condition is critical and cannot be dispensed with.

The σ_n in (7) is in particular a cooptional time. We use the convention that $\sigma_n = 0$ if the set in brackets is empty. Note that when > 0, σ_n is the last time plus 1 that the chain is in K_n. □

Proof of Theorem 1. (a) By (5), for all n, x

$$\nu_n GP(x) = \nu_n G(x) - \nu_n(x) ,$$

and for $n \to \infty$, using (i)-(iii), we obtain $\mu P(x) = \mu(x)$.

(b) With ν_n defined by (6), where μ is invariant, first note that since

$$(8) \qquad (\sigma_n = 1) = (X_0 \in K_n, X_k \in J \setminus K_n, \ k \geq 1) ,$$

$\nu_n(x) > 0$ only if $x \in K_n$, so since K_n is finite, ν_n is a finite measure. Next, also because of (8) we have

$$\nu_n(x) \leq \hat{P}^x(X_1 \in J \setminus K_n) \to 0$$

as $n \to \infty$, so (i) is satisfied.

With θ_k the shift of order k on Ω, i.e. $X_m \circ \theta_k = X_{m+k}$ for all m, we have

$$(9) \qquad (\sigma_n \circ \theta_k = 1) = (\sigma_n = k+1)$$

because σ_n is cooptional and therefore find

$$\begin{aligned}
\nu_n G(x) &= \sum_y \mu(y) \hat{P}^y(\sigma_n = 1) G(y,x) \\
&= \mu(x) \sum_y \hat{G}(x,y) \hat{P}^y(\sigma_n = 1) \\
&= \mu(x) \sum_y \sum_{k=0}^{\infty} \hat{P}^x(X_k = y, \sigma_n \circ \theta_k = 1) \\
&= \mu(x) \sum_y \sum_{k=0}^{\infty} \hat{P}(X_k = y, \sigma_n = k+1) \\
&= \mu(x) \hat{P}^x(\sigma_n > 0) .
\end{aligned}$$

Since $\hat{P}^x(\sigma_n > 0) \geq \hat{P}^x(X_0 \in K_n) \to 1$, we see that also (ii) holds.

Finally, using (6) and the expression for $\nu_n G$ just derived

$$\begin{aligned}\nu_n GP(x) &= \nu_n G(x) - \nu_n(x)\\ &= \mu(x)\hat{P}^x(\sigma_n > 1)\\ &\to \mu(x)\end{aligned}$$

as $n \to \infty$. As μ is invariant, $\mu(x) = \mu P(x)$, and thus (iii) holds. □

Remark. Referring to part (b) of the theorem, it is useful to note that if a sequence (ν_n) of finite measures has been found such that (ii) holds, with the limit μ invariant, then (i) and (iii) hold automatically. Indeed, as before

$$\nu_n GP(x) = \nu_n G(x) - \nu_n(x), \tag{10}$$

so using (ii)

$$\limsup_{n\to\infty} \nu_n(x) \le \mu(x) - \liminf_{n\to\infty} \nu_n GP(x).$$

But for any $K \subset J$ finite,

$$\begin{aligned}\liminf \nu_n GP(x) &\ge \liminf \sum_{y\in K} \nu_n G(y)p(y,x)\\ &= \sum_{y\in K} \mu(y)p(y,x),\end{aligned}$$

implying, since μ is invariant, that

$$\liminf \nu_n GP(x) \ge \mu(x)$$

and inserting this above we deduce that (i) holds. And (iii) now follows taking limits in (10) using (i), (ii) and the invariance of μ. □

<u>Example 1</u>. If J is finite, no invariant μ exists because it is impossible to have (i) and (ii) satisfied simultaneously with $\mu > 0$. This of course is trivial anyway - since P is irreducible and transient, at least one row sum must be < 1, and it follows easily that P viewed as an operator on the space of finite measures with norm $\|\mu\| = \sum_x \mu(x)$ is a genuine contraction, in particular $\|\mu P^n\| \to 0$ as $n \to \infty$ for all μ. □

In view of this example we shall assume J to be countably infinite in the remainder of the paper.

<u>Example 2</u>. Let $J = N$, the non-negative integers and assume that only the transitions $p(x,x+1) = \alpha(x)$, $p(x,x-1) = \beta(x)$ and $p(x,0) = \gamma(x)$ can be > 0. We assume of course that P is irreducible (e.g. $\alpha(x)\gamma(x) > 0$ for all x) and transient (e.g. $\Pi\ \alpha(x) > 0$). (Examples of this type were discussed by Harris [1]).

Suppose μ is invariant for P. Taking $K_n = \{0,1,\cdots,n\}$, we conclude from Theorem 1 (b) that (i)-(iii) hold with

$$\nu_n(x) = \mu(x)\hat{P}^x(\sigma_n = 1).$$

Now, $\hat{p}(x,y) > 0$ is possible only if $x = y-1, y+1$ or 0. Comparing with (8) it follows that $\nu_n(x) > 0$ only if $x = 0$ or $x = n$. But then

$$\begin{aligned}\mu(x) &= \lim_{n\to\infty} \nu_n G(x) \\ &= \lim_{n\to\infty} (\nu_n(0)G(0,x) + \nu_n(n)G(n,x)) \\ &= \lim_{n\to\infty} \nu_n(n)G(n,x)\end{aligned}$$

since $\nu_n(0) \to 0$ by (i). Consequently

(11) $$\frac{\mu(x)}{\mu(0)} = \lim_{n\to\infty} \frac{G(n,x)}{G(n,0)} ,$$

in particular we see that if an invariant measure exists, it is unique.

In the special case where $\beta \equiv 0$, $\nu_n(x) > 0$ only if $x = 0$, which is impossible if (ii) is to be satisfied with $\mu > 0$. So in this case no invariant measure exists. □

For Example 2, we just saw that the invariant measure, if it exists, is unique, and if $\mu(0) = 1$, it is given by the limit in (11). This corresponds to having (ii), and by the remark following the proof of Theorem 1 also (i) and (iii) satisfied with

$$\nu_n = \frac{1}{G(n,0)} \varepsilon_n ,$$

where, as we shall write from now on, ε_x denotes the measure degenerate at x with unit mass. As we shall see presently it is always possible to represent at least one invariant measure (assuming one to exist) using ν_n of this form.

It is a standard fact, which follows from (3) and a diagonalization argument, that given a reference state $a \in J$, an infinite sequence (x_n) of distinct states may be found such that

(12) $$\lim_{n\to\infty} \frac{G(x_n,x)}{G(x_n,a)} = \kappa(x)$$

exists simultaneously for all x, with $0 < \kappa < \infty$.

Based on this we first prove the following Corollary to Theorem 2 in [1] .

Proposition 1. Suppose that for all $y \in J$, $p(x,y) > 0$ for only finitely many x. Then P has an invariant measure.

Proof. With a, (x_n) satisfying (12), it is clear that (i) and (ii) hold, with $\nu_n = G^{-1}(x_n,a)\varepsilon_{x_n}$. But since $\nu_n GP(x) = \sum_y \nu_n G(y)p(y,x)$ involves only a finite sum, (iii) is then automatic. Hence, by Theorem 1(a), κ is invariant. □

The limit points of sequences of the form (12), determine the Martin entrance boundary for the Markov chain, see Hunt [2] . Of course in general, by Fatou's lemma, κ will be excessive but need not be invariant. If however an invariant μ exists, consider the dual chain $\hat{P}$ and observe that by Martin exit boundary theory

$$(13) \qquad \lim_{n\to\infty} \frac{\hat{G}(x,X_n)}{\hat{G}(a,X_n)} = \hat{h}(x)$$

exists $\hat{P}^a$- a.s., simultaneously for all x, with, because $\hat{P}$ is a stochastic transition function, $\hat{h}$ a $\hat{P}$-harmonic function of x, i.e. $\hat{P}\hat{h} = \hat{h}$. (see Hunt [2, Theorem 2.1]). Now choose one realization of (X_n) under $\hat{P}^a$ for which (13) holds for all x. By transience and possibly taking a subsequence, this yields an infinite sequence (x_n) of distinct states such that $\hat{G}(x,x_n) / \hat{G}(a,x_n) \to \hat{h}(x)$ for all x. Equivalently

$$\lim_{n\to\infty} \frac{G(x_n,x)}{G(x_n,a)} = \mu(x)\hat{h}(x)\mu^{-1}(a)$$

for all x, and denoting the limit by $\kappa(x)$,

$$\sum_y \kappa(y)p(y,x) = \frac{\mu(x)}{\mu(a)} \sum_y \hat{p}(x,y)\hat{h}(y) = \kappa(x)$$

so κ is invariant for P (and in general $\neq \mu$). This essentially is the argument used by Veech [6, p.860] . In our setting it has yielded the following result.

Proposition 2. Suppose P has an invariant measure. Given any $a \in J$ it is then possible to find an infinite sequence (x_n) of distinct states such that (i), (ii), (iii) of Theorem 1 are satisfied with $0 < \mu < \infty$ and

$$\nu_n = \frac{1}{G(x_n,a)} \varepsilon_{x_n} .$$ □

We shall next prove a result which in appearance is very similar to the Harris-Veech condition (HV). First we remind the readers of the contents of (HV) ([1, Theorem 2] and [6, Theorem 1]).

Let for $x \in J$, τ_x denote the hitting time $\tau_x = \inf\{n>0: X_n = x\}$ and introduce

$$A_x = (\tau_x < \infty) = \bigcup_{n=1}^{\infty} (X_n = x).$$

Then the following condition is necessary and sufficient for an invariant measure to exist: there is an infinite sequence (x_n) of distinct states such that

(HV) $\quad \lim_{K,n} P^{x_n}(X_{\tau_x - 1} \in J \setminus K \mid A_x) = 0$.

for all $x \in J$. Here $\lim_{K,n} a(n,K) = 0$ means that to every $\varepsilon > 0$, n_0 and $K_0 \subset J$ finite can be found such that $a(n,K) < \varepsilon$ for $n \geq n_0$, $K \supset K_0$, $K \subset J$ finite. If, as in (HV) above, $a(n,K)$ decreases when K increases, of course $\lim_{K,n} a(n,K) = 0$ iff to every $\varepsilon > 0$ there exists n_0 and $K_0 \subset J$ finite with $a(n,K_0) < \varepsilon$ for $n \geq n_0$.

The reader is reminded that as defined in Jacobsen [3] , a random time σ defined on Ω is modified cooptional if $(\sigma = n+1) = (\sigma \circ \theta_n = 1)$, (cf. (9)) for all $n \geq 0$. Also, the chain (X_n) when reversed from σ has a transition function as described in Theorem 2 of [3] .

Note that by irreducibility and the properties of σ, if σ is modified cooptional, and $x \in J$, either $P^y(X_{\sigma-1} = x) > 0$ for all y or $P^y(X_{\sigma-1} = x) = 0$ for all y.

<u>Theorem 2</u>. In order that P has an invariant measure, it is necessary and sufficient that there exists an infinite sequence (x_n) of distinct states and, for every x, a modified cooptional time σ_x with $P^y(X_{\sigma_x-1} = x) > 0$ for all y, such that

(14) $\quad \lim_{K,n} P^{x_n}(X_{\sigma_x - 2} \in J \setminus K, \sigma_x > 1 \mid B_x) = 0$

where $B_x = (X_{\sigma_x - 1} = x,\ 0 < \sigma_x < \infty)$.

<u>Proof</u>. The probability in (14) is the probability that the reverse of (X_n) killed at σ_x, given that it starts in x, after one transition is inside $J \setminus K$. Hence by Theorem 2 of [3] it equals

$$(15) \qquad p(n,K) = \sum_{y \in J \setminus K} G(x_n,y)p(y,x)G^{-1}(x_n,x).$$

Suppose first that P has an invariant measure. Choose $a \in J$ and find (x_n) such that (i)-(iii) of Theorem 1 are satisfied with $\nu_n = G^{-1}(x_n,a)\varepsilon_{x_n}$, cf. Proposition 2. Write the sum in (15) as

$$\frac{G(x_n,a)}{G(x_n,x)}\left\{\nu_n GP(x) - \sum_{y \in K} \frac{G(x_n,y)}{G(x_n,a)} p(y,x)\right\},$$

and with $C = \sup_z G(z,a)/G(z,x) < \infty$ (by (3)), let $n \to \infty$ and use (ii) and (iii) to obtain

$$\limsup_{n\to\infty} p(n,K) \leq C(\mu P(x) - \sum_{y \in K} \mu(y)p(y,x))$$

for every $K \subset J$ finite. Since the right hand side $\downarrow 0$ as $K \uparrow J$, it is clear that (14) holds for any modified cooptional σ_x with $P^y B_x > 0$ for all y. It remains to observe that with $\sigma_x = \sup\{n \geq 1: X_{n-1} = x\}$ this condition on B_x holds, and the first part of the proof is complete.

For the converse suppose that (x_n) and (σ_x) have been found such that (14) holds with $P^y B_x > 0$. Pick $a \in J$ and find a subsequence such that

$$\lim_{n'} \frac{G(x_{n'},x)}{G(x_{n'},a)} = \kappa(x)$$

exists simultaneously for all x, cf. (12). Then (i) and (ii) of Theorem 1 hold with $\nu_n = G^{-1}(x_n,a)\varepsilon_{x_n}$ and $\mu = \kappa$, writing $n' = n$ as we shall from now on, and it remains only to establish (iii) in order to complete the proof.

Given $\varepsilon > 0$ find n_0 and $K_0 \subset J$ finite such that $p(n,K_0) < \varepsilon$ for $n \geq n_0$. Then for $n \geq n_0$ and $K \supset K_0$ finite

$$\sum_{y \in J \setminus K} \frac{G(x_n,y)}{G(x_n,a)} p(y,x) = \frac{G(x_n,x)}{G(x_n,a)} p(n,K) < D\varepsilon$$

where $D = \sup_z G(z,x) / G(z,a)$. Using (ii) it therefore follows that

$$\limsup_{n\to\infty} \left| \nu_n GP(x) - \sum_{y\in K} \kappa(y)p(y,x) \right|$$

$$= \limsup \left| \nu_n GP(x) - \sum_{y\in K} \frac{G(x_n,y)}{G(x_n,a)} p(y,x) \right| \leq D\varepsilon .$$

Since $\nu_n GP(x) \leq \nu_n G(x) \leq D$ for all n, letting $K \uparrow J$, this implies that

$$\sum_{y\in J} \kappa(y)p(y,x) < \infty .$$

But then

$$\limsup_{n\to\infty} \left| \nu_n GP(x) - \kappa P(x) \right|$$

$$\leq D\varepsilon + \sum_{y\in J\setminus K} \kappa(y)p(y,x)$$

for $K \supset K_0$ finite. Letting $K \uparrow J$ and then $\varepsilon \downarrow 0$ the desired conclusion, $\nu_n GP(x) \to \kappa P(x)$ follows. □

In conclusion we shall establish the equivalence between (HV) and Theorem 2. More precisely, we shall show the following

Theorem 3. Let (c) be the condition that for some infinite sequence (x_n) of distinct states, (14) of Theorem 2 holds with $\sigma_x = \sup\{n \geq 1: X_{n-1} = x\}$. Then the Harris-Veech condition (HV) and the condition (c) are equivalent.

Proof. Since $A_x = (\tau_x < \infty) = (\sigma_x > 0)$, by transience $A_x = B_x$ P^y-a.s. for all y. As in (15) let $p(n,K)$ denote the conditional probability from (14), where, as we have just seen, B_x may be replaced by A_x. Also write $h(n,K)$ for the conditional probability in (HV).

Now $\sigma_x - 1$ is the last time n with $X_n = x$ and decomposing $p(n,K)$ into two terms according as $\tau_x = \sigma_x - 1$ or $\tau_x < \sigma_x - 1$, together with an application of the strong Markov property at τ_x, readily gives

(16) $$p(n,K) = h(n,K) P^x(\sigma_x = 1) + P^x(X_{\sigma_x - 2} \in J \setminus K, \sigma_x > 1) .$$

Here $P^x(\sigma_x = 1)$ is the probability that the chain, given that it starts in x, never returns to x, so by transience $P^x(\sigma_x = 1) > 0$. The last term on the right of (16) does not depend on n, and $\to 0$ as $K \uparrow J$. From these remarks, the equivalence of (HV) and (c) follows immediately. □

Although the papers [7], [5], [4] all deal with Markov chains on a general state space, the results presented here are closely connected to them. Indeed, translating the critical condition (C) of Shur [4 p.491] into the setting with discrete state space, the reader will recognize that (C) is exactly the condition that $\lim_{K,n} p(n,K) = 0$ for all x, with $p(n,K)$ given by (15). Also, (13.2) of Tweedie [5] for $r = 1$

very much resembles (but is not quite the same as) lim p(n,K) = 0. The condition (1.5) of Yang [7] , as pointed out by Yang himself, is (HV) reformulated for the general state space case.

Finally, it should be observed that there is no principal difficulties in carrying over from discrete to a general state space, the contents of Theorem 1. The reader is invited to supply the details.

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