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Birth Times, Death Times and Time Substitutions in Markov Chains

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SUMMARY

Given a Markov chain \((X_n)_{n \geq 0}\), random times \(\tau\) are studied which are birth times or death times in the sense that the post-\(\tau\) and pre-\(\tau\) processes are independent given the present \((X_{\tau-1}, X_\tau)\) at time \(\tau\) and the conditional post-\(\tau\) process (birth times) or the conditional pre-\(\tau\) process (death times) is again Markovian. The main result for birth times characterizes all time substitutions through homogeneous random sets with the property that all points in the set are birth times. The main result for death times is the dual of this and appears as the birth time theorem with the direction of time reversed.
1. **INTRODUCTION AND NOTATION**

An earlier paper, [7], by Jim Pitman and the author, hereafter referred to as BDC, contained a study of certain classes of birth times and death times for Markov chains in discrete time with stationary transition probabilities.

Much of the motivation for that paper came from David Williams' [14] fundamental results on path decompositions of diffusions, in particular the one-dimensional Brownian motion $BM(1)$.

The types of e.g. birth times considered in BDC were random times $\tau$ determined by the evolution of the Markov chain, with the property that the post-$\tau$ process is again Markov with a transition function possibly different from that of the given chain, and furthermore $\tau$ should have a conditional independence property similar to that in the strong Markov property, with past and future independent given the present at time $\tau$.

However, not all discrete time analogues of Williams' path decompositions are covered by BDC. For instance Williams showed that for $BM(1)$ made transient by absorption at a high level, the time $\tau$ of the ultimate minimum of the path is a birth time and a death time in the sense that given the value $x$ of the path at time $\tau$, the pre-$\tau$ and post-$\tau$ processes are both Markovian and conditionally independent. But since the transitions for the two fragments obviously depend on $x$, results of this type are not included in BDC.

One motivation behind the present paper has been the desire to fill this gap. A first discussion of the larger classes of birth and death times needed to accomplish this, appeared in the preprint Jacobsen [5], and some of the fundamentals there are repeated here. But the main results to be given below deal with a particular class of birth times (and its dual class of death times) which apart from possessing several nice properties is relevant to the theory of time substitutions in Markov processes, cf. the papers by Pittenger [11].
and Glover [3].

The main results of BDC provide characterizations of classes of birth times and death times formulated as equivalences between objects defined relative to the probabilistic structure and objects defined in terms of path-algebra. This interplay between probability theory and path-algebra is also the essence of the philosophy behind the present paper. What it amounts to technically will be discussed at the end of this introduction.

One obvious open problem left by BDC was how the results could be carried over to processes in continuous time. This has now been done by Pittenger [10] (Theorem 3.9 of BDC on birth times), Sharpe [13] (Theorem 5.2 on death times) and, most fittingly and most recently, by Pittenger and Sharpe [12] (Theorem 6.2 on times that are birth as well as death times).

Another important recent reference is the paper [2] by Getoor and Sharpe where they discuss what types of conditional independence are relevant in continuous time, and provide necessary and sufficient conditions for the different types to be valid.

The basic reference for this paper is BDC and it may be useful for the reader to have a copy available.

The notation to be used here is that of BDC with some minor changes: given a countable state space J, let \( \Omega \) denote the space of all sequences \( \omega = (\omega_0, \omega_1, \ldots) \) in \( J \) indexed by the non-negative integers \( N \), let \( (X_n, n \in N) \) be the coordinate process on \( \Omega \), i.e. \( X_n(\omega) = \omega_n \), and denote by \( (Y_n, n \in N_+) \) the sequence of transitions \( Y_n = (X_{n-1}, X_n) \) defined for \( n \in N_+ = \{1, 2, \ldots\} \). Writing \( F \) for the (uncompleted) \( \sigma \)-algebra on \( \Omega \) spanned by all \( X_n \), a probability \( P \) on \( (\Omega, F) \) is said to be Markov or Markov \( (p) \) if \( P \) makes \( (X_n) \) a Markov chain with stationary transitions \( p \). If \( \nu \) is
the \( P \)-law of \( X_o \), \( P^\mu \) may be written instead of \( P \) and, as is the custom, \( P^\mu \) if \( \mu \) is degenerate at \( x \). The following convention is adapted throughout: the same letter is used to denote a Markov probability (capital letter) and its transition function (small letter).

Adjoining a state \( \Delta \) to \( J \), write \( J_{\Delta} = J \cup \{\Delta\} \) and let \( \Omega_{\Delta} \) be the space of all sequences in \( J_{\Delta} \) that remain in \( \Delta \) once they get there. The lifetime of a sequence \( \omega \in \Omega_{\Delta} \) is \( \zeta(\omega) = \inf \{n \in \mathbb{N} : X_n(\omega) = \Delta\} \).

The space \( \Omega_{\Delta} \) and the subspace \( \Omega_0 = (\zeta < \infty) \) of paths with finite lifetime will be used mainly in Section 4 on death times. For objects pertaining to \( \Omega_{\Delta} \), the same notation will be used as for the corresponding objects on \( \Omega \).

For \( n \in \mathbb{N} \), the killing operator \( K_n : \Omega \to \Omega_{\Delta} \) and shift operator \( \theta_n : \Omega \to \Omega \) are defined by

\[
K_n(\omega_0, \omega_1, \ldots) = (\omega_0, \ldots, \omega_{n-1}, \Delta, \Delta, \ldots),
\]
\[
\theta_n(\omega_0, \omega_1, \ldots) = (\omega_n, \omega_{n+1}, \ldots).
\]

For \( n = 1 \), \( \theta \) is written instead of \( \theta_1 \).

A random time is a measurable mapping from \( \Omega \) to the extended time set \( \overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\} \). Given a random time \( \tau \), \( X_\tau \), \( Y_\tau \), \( K_\tau \), \( \theta_\tau \) are defined by local identification, e.g. \( X_\tau = X_n \) on \( (\tau = n) \). Also, \( X_\tau \), \( K_\tau \), \( \theta_\tau \) are defined only on the set \( (\tau < \infty) \) and \( Y_\tau \) on \( (0 < \tau < \infty) \). As a consequence, for instance \( (Y_\tau = (a,b)) \) will be the notation for the subset \( \{\omega : 0 < \tau(\omega) < \infty, Y_\tau(\omega) = (a,b)\} \) of \( \Omega \).

For a fixed \( n \in \mathbb{N} \) the pre-\( n \) \( \sigma \)-algebra \( F_n \) is the \( \sigma \)-algebra spanned by \( (X_0, \ldots, X_n) \). The atoms \( A_n \) are the sets of the form \( A = (X_0 = x_0, \ldots, X_n = x_n) \).

For \( \tau \) a random time, the pre-\( \tau \) \( \sigma \)-algebra \( F_\tau \) consists of the sets which
are countable unions of sets of the form (i) \((A, \tau = n)\) where \(n \in \mathbb{N}, A \in \mathcal{A}_n\) or (ii) one-point sets \((\omega)\) where \(\tau(\omega) = \infty\).

A random time \(\tau\) splits the process \((X_n)\) into two parts, the pre-\(\tau\) process, conveniently identified with and therefore labelled \(K_\tau\), given as

\[
(X_n \circ K_\tau, n \in \mathbb{N}) = (X_0, \ldots, X_{\tau-1}, \Lambda, \Lambda, \ldots),
\]

and the post-\(\tau\) process \(\theta_\tau\) given as

\[
(X_n \circ \theta_\tau, n \in \mathbb{N}) = (X_\tau, X_{\tau+1}, \ldots).
\]

As discussed earlier, the main theme in BDC and here is to provide equivalences between probabilistic and path-algebraic objects. Thus for instance two different types of definitions of random times will be used: (i) operational definitions and (ii) algebraic definitions. Definitions of type (i) give the properties of a random time relative to a Markov probability, while those of type (ii) are concerned exclusively with the properties of a random time as a function on \(\Omega\). The latter may be implicit or explicit in nature, for example a description of a random time involving a collection of parameters is explicit if the parameters may be chosen independently of each other and implicit if they are interrelated.

For an example, consider stopping (optional) times. Given a Markov probability \(P\), \(\tau\) is an operationally defined stopping time for \(P\) if conditionally on \(F_{\tau}\) within \((\tau < \infty)\), \(\theta_\tau\) is Markov with the same transitions \(p\) as \(P\). On the other hand, \(\tau\) is an algebraically defined stopping time if the following three equivalent conditions are satisfied: (a) \((\tau = n) \in F_n\) for \(n \in \mathbb{N}\); (b) \((\tau \leq n) \in F_n\) for \(n \in \mathbb{N}\); (c) \(\tau(\omega) = \inf\{n \in \mathbb{N}: \omega \in F_n\}\) for some sequence \((F_n, n \in \mathbb{N})\) of sets \(F_n \subseteq F_n\). Here one would call (a), (b) implicit and (c) an explicit definition, because in (a) the sets \((\tau = n)\) must be mutually disjoint,
in (b) the sets \( (\tau \leq n) \) must increase with \( n \), while in (c) the \( F_n \in F_n \) are arbitrary.

The characterization theorems to be given here, as those presented in BDC (or Jacobsen [6]), provide probabilistic equivalences between operationally and algebraically defined objects. For instance, and an easy consequence of the results in Section 3 of BDC, for stopping times the following is true: a random time \( \tau \) is an operationally defined stopping time for the Markov probability \( P \), iff it is \( P \)-equivalent to an algebraically defined stopping time.
2. CONDITIONAL INDEPENDENCE TIMES

In BDC two slightly different notions of conditional independence were used in the study of birth times and death times respectively: for the birth times conditional independence of the pre-$\tau$ and post-$\tau$ processes given $X_\tau$ was demanded, while for the death times it turned out that the relevant conditional independence occurs when conditioning on $X_{\tau-1}$.

In this paper we shall use the same conditional independence concept for birth times and death times as described in the following definition which replaces Definition 3.11 in BDC.

(2.1). Definition A random time $\tau$ is called a conditional independence time for the Markov probability $P$, if under $P$ the pre-$\tau$ and post-$\tau$ processes are conditionally independent given $Y_\tau$. $\Box$

Thus $\tau$ is a conditional independence time iff there is a conditional distribution of $\theta_\tau$ given $(X_0, \ldots, X_\tau)$ within $(0<\tau<\infty)$, or equivalently of $K_\tau$ given $(X_{\tau-1}, X_\tau, \ldots)$ within $(0<\tau<\infty)$, which is a function of the transition $Y_\tau$ alone.

It should be noticed that conditioning on $(X_0, \ldots, X_\tau)$ is equivalent to conditioning on $E_\tau$ and involves in particular the conditioning on the value of $\tau$. By contrast, conditioning on $(X_{\tau-1}, X_\tau, \ldots)$ does not imply knowledge of the exact value of $\tau$, wherefore in particular, as is essential, the conditional pre-$\tau$ process has a random lifetime.

It seems most natural to have a unified concept for conditional independence applying to the birth time as well as the death time theory. A second reason for using Definition 2.1 is the following: consider for a real-valued process in continuous time with, say, right-continuous, left-limit paths, the time $\tau$ where the process attains its ultimate minimum. With jumps possible,
the transition function of for instance the conditional post-\( \tau \) process given the past will in general depend on the transition \((X_{\tau}, X_{T})\) rather than on \( X_{\tau} \) alone. Translating this into the discrete time situation makes it natural to study \( K_{\tau} \) and \( \theta_{\tau} \) given the transition \( Y_{\tau} \).

The following result provides a useful characterization of conditional independence times. The proof proceeds exactly as that of Lemma 3.12 in BDC and is therefore omitted.

\textbf{(2.2). Lemma} A random time \( \tau \) is a conditional independence time for the Markov probability \( P \) iff for every \( n \in \mathbb{N}_{+} \) and every \((a, b) \in J^{2}\) there exists \( F_{n}, G_{ab} \in F \) respectively such that

\begin{equation}
(\tau = n, Y_{\tau} = (a, b)) = (F_{n}, Y_{n} = (a, b), \theta_{n} \in G_{ab}) \quad P-a.s.
\end{equation}

or equivalently iff for every \( n \in \mathbb{N}_{+} \) and every \((a, b) \in J^{2}\) there exists \( F_{n-1, ab}, G \in F \) such that

\begin{equation}
(\tau = n, Y_{\tau} = (a, b)) = (F_{n-1, ab}, Y_{n} = (a, b), \theta_{n-1} \in G) \quad P-a.s.
\end{equation}

\textbf{Remark} Conditional independence times satisfying (2.3) or (2.4) exactly are not splitting times as originally defined by Williams, see [4], equation (3.3) or [15], Section III.79. It appears most natural to generalize the definition there and call \( \tau \) a splitting time if

\begin{equation}
(\tau = n) = (F_{n}, \theta_{n-1} \in G_{n}) \quad (n \in \mathbb{N}_{+})
\end{equation}

for some \( F_{n} \in F_{n}, G_{n} \in F. \) If (2.5) holds with \( G_{n} = G \) not depending on \( n \), \( \tau \) is a stationary splitting time, cf. [11].

The definition in (2.5) is implicitly algebraic. Lemma 2.2 may now be reformulated as stating that conditional independence times are the operationally defined versions of stationary splitting times (see also the remark following
Lemma 3.12 in BDC and Lemma 2.8 in [11]. The operational definition of general splitting times demands that $K_{\tau}$ and $\theta_{\tau}$ be independent given $Y_{\tau}$ and $\tau$. \qed
3. CI-BIRTH TIMES AND THE CLASS $BTR$

The purpose of this section is to study various classes of birth times which are conditional independence times.

(3.1). Definition A random time $\tau$ is a birth time with conditional independence (in short a CI-birth time) for the Markov probability $P$ if it is a conditional independence time for $P$ and if conditionally on $Y_\tau$ within $(0 < \tau < \infty)$, the post-$\tau$ process is Markov with a stationary transition function (depending possibly on $Y_\tau$).

As defined in BDC a random time $\tau$ is a regular birth time for $P$ if there is a transition function $q$ such that conditionally on $F_\tau$, the post-$\tau$ process is Markov ($q$). Thus clearly a regular birth time is a CI-birth time. (Notice that if $\tau$ is a regular birth time for $P$, the post-$\tau$ process is itself Markov without conditioning on the past. This is of course not true in general for $\tau$ a CI-birth time).

Suppose $P$ is Markov. Recall that by Theorem 2.3 of BDC, if $D \in F$ with $P(D) > 0$, then the conditional probability $P_D = P(\cdot | D)$ is Markov iff $D = (X_0 \in H, C)$ $P$-a.s. where $H \subset J$ and $C$ is a coterminent event, i.e. $C = C_V C_\infty$ for some $V \subset J^2$ with $C_V = (Y_n \in V, n \in N_+)$ and $C_\infty \in F$ invariant for $\theta$. This result on conditioning events was used in BDC, Theorem 3.9, to give the following explicitly algebraic characterization of regular birth times:

defining $B$ to be the class of random times of the form

\[(3.2) \quad \tau_C + \rho\]

where $C$ is an arbitrary coterminent event, $\tau_C$ is the associated coterminent time

$$\tau_C = \inf\{n \in N : \theta_n \in C\},$$
and \( \rho \) is a stopping time for the family \( (F^n \cap C, n \in \mathbb{N}) \) of \( \sigma \)-algebras, it was shown that \( \tau \) is a regular birth time for \( P \) iff \( \tau \) is \( P \)-equivalent to a random time in \( B \). The proof of the theorem also showed that \( \tau \) is a regular birth time for \( P \) iff there exists \( F_n \in F^n, C \) coterminal such that

\[
(3.3) \quad (\tau = n) = (F_n, \theta_n \in C) \quad P\text{-a.s.}
\]

for all \( n \in \mathbb{N} \), cf. (3.16) of BDC. This observation amounts to an implicitly algebraic characterization of regular birth times.

The transitions \( q \) for \( \theta_\tau \) are the same as those of \( P(\cdot, C) \) and are given by

\[
(3.4) \quad q(x, y) = 1_y(x, y)p(x, y) \frac{g(y)}{g(x)},
\]

where \( g(z) = P^z(C) \).

It is easy to see that instead of using (3.2), \( B \) may be defined as follows: \( \tau \in B \) iff there is a coterminal event \( C \) and events \( F_n \in F^n, n \in \mathbb{N} \) such that

\[
(3.5) \quad \tau(\omega) = \inf\{n \in \mathbb{N}: \omega \in F_n, \theta_n \omega \in C\}.
\]

Propositions 3.6 and 3.9 provide two simple implicitly algebraic characterizations of CI-birth times. We are unable to give an explicit algebraic characterization. The main results below deal with the properties of the explicitly defined class \( BTR \), see Definition 3.14.

Proposition 3.6 is the analogue of (3.3), and is proved exactly like that using (2.3), the fact that for a random time \( \tau \) satisfying (2.3) the conditional distribution of the post-\( \tau \) process given \( F^\tau \) within \( (Y_\tau = (a, b)) \) is

\( P^b(\cdot | \mathcal{G}_{ab}) \), and the characterization of conditioning events quoted above.
(3.6) Proposition A random time $\tau$ is a CI-birth time for $P$ if and only if for every $n \in \mathbb{N}_+$ and every $(a, b) \in J^2$ there exists $F_n \in F_n$ and coterminus $C_{ab}$ respectively, such that

$$\tau = n, Y_\tau = (a, b) \Rightarrow (F_n, Y_n = (a, b), \theta_n \in C_{ab}) \quad P\text{-a.s.}$$

Notice that if $\tau$ satisfies (3.7) exactly (no exceptional sets), then $\tau$ is a stationary splitting time with the $G = C_n$ of (2.5) given by

$$G = \bigcup_{(a, b)} (Y_1 = (a, b), \theta \in C_{ab}).$$

With $\tau$ a CI-birth time for $P$ we shall write $q_{ab}$ for the transition function of the post-$\tau$ process given $Y_\tau = (a, b)$. Thus, if $\tau$ satisfies (3.7), $q_{ab}$ is the transition function for the Markov probability $Q_{ab}^b = P^b(\cdot | C_{ab})$.

The second characterization of CI-birth times is an observation due to J.W. Pitman (private communication). It follows immediately from the definitions of regular birth times and CI-birth times.

(3.9) Proposition A random time $\tau$ is a CI-birth time for the Markov probability $P$ if and only if, for every $(a, b) \in J^2$ the random time $\tau_{ab}$ defined by

$$\tau_{ab} = \begin{cases} 
\tau & \text{on } (Y_\tau = (a, b)) \\
\infty & \text{otherwise}
\end{cases}$$

is a regular birth time for $P$. 

Of course Theorem 3.9 of BDC provides an explicitly algebraic characterization of each $\tau_{ab}$. But to obtain from this an explicitly algebraic characterization of all CI-birth times requires that the $\tau_{ab}$ be chosen simultaneously in such a way that the sets $(\tau_{ab} < \infty) = (Y_\tau = (a, b))$ be disjoint, and it is not at all clear how this should be done.
Consider a CI-birth time \( \tau \) for \( P \) satisfying (3.7). Since, when ignoring some null sets, the sets on the right are mutually disjoint for \( n \), \((a,b)\) varying, it is clear that for \( P \)-almost all \( \omega \)

\[
\tau(\omega) = \inf\{n \in \mathbb{N}_+ : \omega \in F_n, \theta_n \omega \in C_{Y_n}(\omega)\},
\]

cf. (3.5). The main difficulty arising when attempting to characterize CI-birth times in an explicit algebraic fashion rests on the fact that the converse is not true: given an arbitrary collection \((C_{ab})\) of coterminal events and

\[ F_n \subseteq F_n, \text{ if } \tau \text{ is defined by (3.10), it is not in general true that (3.7) holds (exactly rather than a.s.) no matter what is the choice of the } F_n, C_{ab} \text{ appearing there. We shall now discuss systems of coterminal events for which the implication (3.10) to (3.7) holds (for all choices of } F_n).\]

For the two definitions below, let \( C = (C_{ab})_{(a,b) \in J^2} \) be a collection of coterminal events. The inclusion \( \supseteq \) is non strict, allowing for equality.

(3.11). Definition \( C \) is a \textit{transition reproducing} collection of coterminal events if \((a,b) pr (c,d)\) implies that either \( C_{ab} \supseteq (X_0 = d, C_{cd}) \) or \( (X_0 = d, C_{ab} c_{cd}) = \emptyset \). Here \((a,b) pr (c,d)\) means that there exists \( \omega \in (X_0 = b, C_{ab}) \) and \( n \in \mathbb{N}_+ \) such that \( Y_n(\omega) = (c,d) \).

Let \( p \) be a stochastic transition function on \( J \).

(3.12). Definition \( C \) is \textit{transition reproducing for} \( p \) if \((a,b) pr (p)(c,d)\) implies that either \( C_{ab} \supseteq C_{cd}^d \text{-a.s.} \) or \( C_{ab} C_{cd} = \emptyset \text{ } p^d \text{-a.s.} \). Here \((a,b) pr (p)(c,d)\) means that

\[
\sum_{n=1}^{\infty} p^b(C_{ab}, Y_n = (c,d)) > 0.
\]

Given a Markov probability \( P \) we shall call \( C \) \textit{transition reproducing for} \( P \) if \( C \) is transition reproducing for the transition function \( p \) of \( P \).
It seems plausible that if $C$ satisfies the operational definition (3.12), then each $C_{ab}$ may be replaced by a new coterminal event $C^*_{ab}$ such that $C_{ab} = C^*_{ab} P^b - a.s.$ and $C^* = (C^*_{ab})$ satisfies the algebraic definition (3.11), but this fact is not verified here.

Definition 3.11 puts some implicitly given constraints on the $C_{ab}$. There does not appear to be any explicit receipt for describing all transition reproducing $C$. Of course $C$ is transition reproducing if all $C_{ab} = C$. A more subtle example is the following

(3.13). Example Let $(C_{ab}^{'})_{(a,b) \in \mathbb{J}^2}$ be an arbitrary collection of coterminal events and define

$$C_{ab} = (Y \in V^*_ab, n \in N_+, C_{ab}^{'})$$

where

$$V^*_ab = \{(x,y) : C_{ab}^{' \rightarrow} xy \}.$$

Then we claim that $C = (C_{ab}^{'})$ is a transition reproducing collection of coterminal events. To see this, suppose $(a,b) \rho (c,d)$, and find $\omega, n$ with $\omega \in (X_0 = b, C_{ab}), Y_n(\omega) = (c,d)$. We shall show that $C_{ab} \rightarrow (X_0 = d, C_{cd})$.

Firstly, by the definition of $C_{ab}$ and because $\omega \in C_{ab}^{'}, (c,d) \in V^*_ab$ and $C_{ab}^{' \rightarrow} C_{cd}^{' \rightarrow}$. But then to show that $\omega' \in (X_0 = d, C_{cd})$ implies $\omega' \in C_{ab}$, it is enough to see that $\omega' \in (X_0 = d, C_{cd})$ implies $Y_k(\omega') \in V^*_ab$ for all $k \in N_+$, i.e. implies $C_{ab} \rightarrow C_{cd} \rightarrow C_{cd}^{'} \rightarrow Y_k(\omega')$. Since by assumption $Y_k(\omega') \in V^*_cd$, i.e. $C_{cd} \rightarrow C_{cd}^{'} \rightarrow Y_k(\omega')$, and since $C_{ab} \rightarrow C_{cd}^{'}$, the implication is evident. \[\square\]

We shall now show that transition reproducing collections of coterminal events lead to a universal class of CI-birth times.

(3.14). Definition Let $BTR$ denote the class of random times $\tau$ of the form

$$\tau(\omega) = \inf\{n \in N_+ : \omega \in F_n, \theta_n \omega \in C_{Y_n(\omega)}\}$$
where for every \( n \), \( F_n \in F \) and where \( C = (C_{ab})_{(a,b) \in J} \) is a transition reproducing collection of coterminial events.

(3.16). **Proposition** Suppose \( \tau \) belongs to the class \( BTR \). Then \( \tau \) is a CI-birth time for any Markov probability \( P \).

**Proof** We shall show that \( \tau \) satisfies (3.7) exactly (with the \( F_n \) there not the \( F_n \) from the definition of \( \tau \)). Clearly \( \tau(\omega) = n \) and \( Y_\tau(\omega) = (a,b) \) iff \( Y_n(\omega) = (a,b) \), \( \forall \omega \in C_{ab} \) and for all \( k \), \( 1 \leq k < n \) one of (i), (ii) holds:

(i) \( \omega \notin F_k \)

(ii) \( \exists \, \theta_k \omega \notin C_{Y_k(\omega)} \).

Represent each \( C_{xy} \) as \( (Y \in V_{xy}, m \in \mathbb{N}^+, C_x^\infty) \) with \( C_{xy}^\infty \) invariant. Then define

\[
V_{xy}^0 = \{(u,v) : (x,y) \text{pr}(u,v)\},
\]

recalling the meaning of \( \text{pr} \) from Definition 3.11. In particular \( V_{xy}^0 \subseteq V_{xy} \).

We claim that subject to \( \tau(\omega) = n \), \( Y_\tau(\omega) = (a,b) \), for every \( k \), \( 1 \leq k < n \), it is true that (i) or (ii) holds iff one of (i)', (ii)', (iii)' holds:

(i)' \( \omega \notin F_k \)

(ii)' \( Y_\ell(\omega) \notin V_{Y_k(\omega)}^0 \) for some \( \ell \), \( k < \ell \leq n \)

(iii)' \( Y_\ell(\omega) \in V_{Y_k(\omega)}^0 \) for all \( \ell \), \( k < \ell \leq n \) and \( (X_0 = b, C_{Y_k(\omega)}^\infty C_{ab}) = \emptyset \).

Since these three conditions involve only \( \omega_0, \ldots, \omega_n \), the proposition will then be established.

Now fix \( \omega \) with \( \tau(\omega) = n \), \( Y_\tau(\omega) = (a,b) \) and \( k \) with \( 1 \leq k < n \). Suppose
that $\omega \in F_k$. Write $(x,y) = Y_k(\omega)$. Then (i) or (ii) is equivalent to one of (i)', (ii)' or (iii)' where

$$(iii)' \quad Y_k(\omega) \in \mathcal{V}^0_{\mathcal{Y}_k(\omega)} \text{ for all } \ell, k < \ell \leq n \text{ and } \theta_n \omega \notin \mathcal{C}_{\mathcal{Y}_k(\omega)}.$$

This is clear if in (ii)' and (iii)' $V$ is written instead of $\mathcal{V}^0$. But the definition of $\mathcal{V}^0$ ensures that

$$(x_0 = y, c_{xy}^{c} \ni (x_0 = y, Y_m \notin \mathcal{V}^0_{xy} \text{ for some } m \in \mathbb{N}_+)$$

and this together with $\mathcal{V}^0_{xy} \subset \mathcal{V}_{xy}$ shows that the use of $\mathcal{V}^0$ instead of $V$ is legitimate.

The proof is completed by showing that (with the assumptions about $\omega, k$ made above), (iii) $\iff$ (iii)'. Here $\Rightarrow$ is evident because $\theta_n \omega \in (x_0 = b, C_{ab})$. Conversely, if (iii)' holds, we have $(x,y) \text{pr}(a,b)$ by the definition of $\mathcal{V}^0_{xy}$, hence since $C$ is transition reproducing, either $(x_0 = b, C_{xy}^{c} \ni \mathcal{C}_{xy}^{c} = (x_0 = b, C_{ab})$. Were the last option possible we would have $\theta_n \omega \in C_{xy}$ because $\theta_n \omega \in (x_0 = b, C_{ab}) \subset C_{xy}$ and $Y_k(\omega) \in \mathcal{V}^0_{xy} \subset \mathcal{V}_{xy}$ for $k < \ell \leq n$, and since $\omega \in F_k$ by assumption this would force $\tau(\omega) = k$ contradicting the assumption $\tau(\omega) = n$. Thus necessarily $(x_0 = b, C_{xy}^{c} \ni \mathcal{C}_{xy}^{c} = (x_0 = b, C_{ab}) = \emptyset$, and we are back to (iii)'.

**Remark** Taking $F_n = \Omega$ for all $n$, it follows in particular from the proof that if $C$ is transition reproducing, then $\inf\{n : \theta_n \in C_Y\}$ is a random time satisfying (3.7) exactly. Thus there exists $F_0^{0} \in F_n$ such that

$$(3.17) \quad (\theta_k \notin \mathcal{C}_Y, 1 \leq k < n, \theta_n \in C_Y) = (F_0^{0}, \theta_n \in C_Y) \quad (n \in \mathbb{N}_+).$$

Introducing $G$ as in (3.8), we have $\theta_m \omega \in C_Y(\omega) \text{ iff } \theta_{m-1} \omega \in G$, so (3.17) may be written

$$(\theta_k \in G^c, 0 \leq k < n-1, \theta_{n-1} \in G) = (F_0^{0}, \theta_{n-1} \in G) \quad (n \in \mathbb{N}_+).$$
Also \( \tau = \inf \{ n \in \mathbb{N} : \theta_{n-1} \in G \} \) so (with a minor modification), \( \tau \) is one of the penetration splitting times characterized by Pittenger [11], Theorem 4.4.

(3.18). Example The class of random times \( \tau \) of the form (3.15) with \( C \) as in Example 3.13, is the class \( \text{BO} \) first introduced in [5]. Specializing further one finds that with \( f : J^2 \to \mathbb{R} \) some function, the times \( \tau \) and \( \bar{\tau} \) given as the first, respectively the last time that the sequence \( (f(Y_n), n \in \mathbb{N}_*) \) attains its ultimate minimum, both belong to \( \text{BO} \).

It was shown by Millar [9], that \( \tau, \bar{\tau} \) are CI-birth times for a wide class of Markov processes in continuous time.

It is easy to give examples of CI-birth times \( \tau \) for a Markov probability \( P \) such that \( \tau \) is not \( P \)-a.s. equal to a member of \( \text{BTR} \). But apart from the explicit description (3.15), the times in \( \text{BTR} \) possess a number of nice properties, as we shall now see.

Suppose that \( \sigma > 0, \tau > 0 \) satisfy (3.7) so that for \( F_n, G_n \in F_n \) and \( C_{ab}, D_{ab} \) coterminally, the identities

\[
(\sigma = n, Y_\sigma = (a, b)) = (F_n, Y_n = (a, b), \theta_n \in C_{ab}) ,
\]

\[
(\tau = n, Y_\tau = (c, d)) = (G_n, Y_n = (c, d), \theta_n \in D_{cd})
\]

hold exactly for \( n \in \mathbb{N}_+ \), \( (a, b), (c, d) \in J^2 \). Now consider the random time \( \rho = \sigma + \tau \theta_\sigma \). Then

\[
(\rho = n, Y_\rho = (a, b), Y_\rho = (c, d)) = (H_n, cd, Y_n = (c, d), \theta_n \in C_{ab}, D_{cd})
\]

for suitable \( H_n, ab \in F_n \), and consequently for \( P \) Markov

\[
P(\theta_\rho \in \cdot | F_\rho, Y_\sigma = (a, b), Y_\rho = (c, d)) = P^d(\cdot | C_{ab}, D_{cd}) .
\]

Thus \( \rho \) will not be a CI-birth time for \( P \) unless for all \( (a, b), (c, d) \).
with \( P(Y_a = (a,b), Y_p = (c,d)) > 0 \), the inclusion \( C_{ab} \supseteq D_{cd} \) holds \( P^d \)-a.s.

However with the proof of Proposition 3.16 and Definition 3.14 in mind, the following result is not surprising and easily proved.

(3.21). Proposition If \( \sigma > 0, \tau > 0 \) belong to \( \text{BTR} \) and both satisfy (3.15) with the same family \( C \) of coterminal events, then also \( \rho \in \text{BTR} \) and satisfies (3.15) with the collection \( C \), where \( \rho = \sigma + \tau \circ \theta_\sigma \).

This result provides one explanation for the terminology 'transition reproducing' in Definition 3.12: let \( P \) be Markov and let \( q_{ab} \) be the common transition function for \( P(\theta_\kappa \in \cdot | Y_\kappa = (a,b)) \) where \( \kappa = \sigma, \tau \) or \( \rho \). Then not only are the \((q_{ab})_{(a,b) \in \mathcal{J}_2}\) the transition functions for \( \theta_\rho \) given \( Y_\rho \), but they also arise by the two stage procedure consisting in firstly considering the post-\( \sigma \) process given \( Y_\sigma \) and its distribution, and then secondly, for this new process, evaluating the transitions for the post-\( \tau \) process.

The main result of this section gives a characterization of homogeneous random sets with all points CI-birth times, in terms of collections of times in \( \text{BTR} \).

By a standard definition a process \((U_n)_{n \in \mathbb{N}_+}\) defined on \((\Omega, \mathcal{F})\) is homogeneous on \( \mathbb{N}_+ \) if

\[
(3.22) \quad U_{n+k} = U_n \circ \theta_k \quad (k, n \in \mathbb{N}_+) .
\]

(Usually (3.22) is required to hold only outside an exceptional set, but here we shall assume that it is an identity on all of \( \Omega \) for all \( k, n \). Note that we require homogeneity only for \( k, n \geq 1 \)).

A random subset \( M \) of \( \mathbb{N}_+ \) is homogeneous if \((U_n)\) is homogeneous, where \( U_n = 1_{(n \in M)} \). In terms of \( M \), (3.22) becomes

\[
(M-k) \cap N_+ = M \circ \theta_k \quad (k \in \mathbb{N}_+) ,
\]
writing $A-k=\{\ell-k: \ell \in A\}$ for $A$ a subset of $N_+$. (That $M$ is a random
subset of $N_+$ means that for every $\omega$ there is defined a subset $M(\omega)$ of $N_+$
such that $\omega \mapsto 1_{\{n \in M\}}(\omega)$ is $F$-measurable for every $n \in N_+$).

In discrete time the structure of homogeneous processes is of course very
simple: let $U=U_1$ and take $n=1$ in (3.22) to find

$$U_n = U \circ \theta_{n-1} \quad (n \in N_+).$$

Conversely, given $U$ measurable, $(U_n)$ defined this way is homogeneous.

Similarly, if $M$ is a homogeneous random set, then

(3.23) $M = \{n \in N_+: 1_{\{n \in M\}} \in G\}$

where $G = (1 \in M)$. Conversely, given $G \in F$, $M$ defined by (3.23) is homogeneous.

Given a random subset $M$ of $N_+$, let $\tau_1, \tau_2, \ldots$ denote the points of $M$
in increasing order of magnitude. Thus $\tau_1 = \inf M$, $\tau_2 = \inf \{n > \tau_1: n \in M\}$ etc.

In particular

$$(\tau_1 = 1) = (1 \in M).$$

In order that $\tau_i(\omega)$ be defined for all $\omega$, we adopt the convention that
if the cardinality $|M(\omega)|$ of $M(\omega)$ is $< i$, then $\tau_i(\omega) = \tau_{i+1}(\omega) = \ldots = \infty$.

Thus

$$(\tau_1 = \infty) = (M = \emptyset),$$

and $\tau_i < \tau_{i+1}$ on $(\tau_i < \infty)$.

In this section, when writing $M = \{\tau_1, \tau_2, \ldots\}$, we shall always assume the
$\tau_i$ to have been defined in this manner.

It is easy to see that $M = \{\tau_1, \tau_2, \ldots\}$ is homogeneous iff $\tau_i$ is of the
form
\( (3.24) \quad \tau_i = \tau_{i-1} + \tau \circ \theta \tau_{i-1} \quad (i \geq 2) \)

with \( \tau = \tau_1 \) satisfying

\( (3.25) \quad \tau = \inf \{ n \in \mathbb{N}_+ : \theta_{n-1} \in G \} \)

for some \( G \in F \).

For arbitrary \( M \), introduce \( (Z_i)_{i \geq 1} \), the process of transitions made at the timepoints in \( M \), i.e. \( Z_i = Y_{\tau_i} \). Then \( Z_i \) is defined only on \( (\tau_i < \infty) \) and the process \( (Z_i) \) may have finite lifetime. Also denote by \( G_i \) the smallest \( \sigma \)-algebra containing \( F \tau_1, \ldots, F \tau_i \). (Thus \( G_i \supseteq F \tau_i \), but in general the inclusion may be strict, also if \( M \) is homogeneous. If for every \( j < i \), \( \tau_j \) is \( F \tau_i \)-measurable, then \( G_i = F \tau_i \).

We shall show

\( (3.26) \) Theorem (a) Let \( P \) be Markov and let \( M = \{ \tau_1, \tau_2, \ldots \} \) be a homogeneous random set. Suppose each \( \tau_i \) is a CI-birth time for \( P \) with respect to \( G_i \) such that the transitions for \( \theta \tau_i \) given \( G_i \) do not depend on \( i \) i.e. for all \( i, (a,b) \),

\[ P(\theta \tau_i \in \cdot | G_i) = Q_{ab}^b \]

on \( (\tau_i < \infty, Z_i = (a,b)) \), where \( Q_{ab}^b \) is Markov with transitions \( q_{ab} \) and initial state \( b \). Then there exists a collection \( C = (C_{ab}) \) of coterminal events, transition reproducing for the transitions \( p \) of \( P \), such that

\[ M = \{ n \in \mathbb{N}_+ : \theta_{n-1} \in G' \} \quad P \text{-a.s.} \]

where \( G' = \bigcup (Y_1 = (a,b), \theta \in C_{ab}) \). Also necessarily, for every \( i, (a,b) \)

\[ F \tau_i = G_i \quad P \text{-a.s.} \]

(b) Let \( C = (C_{ab}) \) be a transition reproducing collection of coterminal events
and let $M = \{\tau_1, \tau_2, \ldots\}$ denote the homogeneous random set $\{n \in \mathbb{N}_+ : \theta_{n-1} \in G\}$ where $G = \bigcup_{(a,b)} (Y_1 = (a,b), \theta \in C_{ab})$. Then $\tau_i \in BTR$ and $F_{\tau_i} = G_i$ for all $i$ and with respect to any Markov probability $P$, each $\tau_i$ is a CI-birth time for $P$ with

$$P(\theta_{\tau_i} \in \cdot | F_{\tau_i}) = P^b(\cdot | C_{ab})$$

on $(\tau_i < \infty, Z_i = (a,b))$, in particular the transitions for $\theta_{\tau_i}$ given $F_{\tau_i}$ do not depend on $i$.

Proof (a) Since $F_{\tau_i} \subset G_i$, each $\tau_i$ is an ordinary CI-birth time for $P$, hence by Proposition 3.6 we can find $P_n^{(i)} \in F_n$ and $C_{ab}^{(i)}$ coterminal such that

$$(3.27) \quad (\tau_i = n, Z_i = (a,b)) = (P_n^{(i)}, Y_n = (a,b), \theta_n \in C_{ab}^{(i)}) \quad P-a.s.$$

for every $i,n,(a,b)$. But then on $(Z_i = (a,b))$

$$P(\theta_{\tau_i} \in \cdot | F_{\tau_i}) = P^b(\cdot | C_{ab}^{(i)}) ,$$

and since by assumption the right hand side must not depend on $i$ we deduce that for every $(a,b)$, $C_{ab}^{(i)} = C_{ab}^{(j)}$ $P-a.s.$ for all $i,j$ with $P(Z_i = (a,b)) > 0$, $P(Z_j = (a,b)) > 0$. It follows that (3.27) may be assumed to hold with $C_{ab}^{(i)} = C_{ab}^{(j)}$ coterminal not depending on $i$ provided $(a,b) \in R := \{(x,y) : \sum_i P(Z_i = (x,y)) > 0\}$ and for $(a,b) \notin R$ we may and shall take $C_{ab} = \emptyset$.

With $C = (C_{ab})$ as just found, define $G'$ as in the statement of the theorem and put $M' = \{n \in \mathbb{N}: \theta_{n-1} \in G'\}$. The definition of $M$ and (3.27) (with $C_{ab}$ replaced by $C_{ab}$) show that $M \subset M' P-a.s$. For the opposite inclusion, write $M = \{n \in \mathbb{N}_+: \theta_{n-1} \in G\}$, let $(a,b) \in R$ and find $i,n$ and an atom $A$ of $F_n$, $A \subset (Y_n = (a,b))$, such that $P(A, \tau_i = n) > 0$. With $(\tau_i = n)_A$ the section of $(\tau_i = n)$ beyond $A$ (cf. [BDC, Definition 2.7]), we find that on the $F_{\tau_i}$-atom $(A, \tau_i = n)$
\[ P(\theta_{\tau_i} \in \cdot | \theta_{\tau_i}) = P^b(\cdot | (\tau_i = n)_A), \]

while by (3.27) this also equals \( P^b(\cdot | C_{ab}) \). Consequently \((\tau_i = n)_A = C_{ab} \) \( p \)-a.s. and since by the definition of \( M \), \((\tau_i = n)_A \subset C_{ab} \) \( p \)-a.s. where \( C_{ab} = \{ \omega : \omega_0 = b, (a, b, \omega_1, \omega_2, \ldots) \in G \} \) it follows that \( C_{ab} \subset C_{ab} \) \( p \)-a.s. and

\[
P(n \in M' \setminus M) = P(\theta_n \in C_{ab}, \theta_{n-1} \notin G)
= \sum_{(a,b) \in \mathcal{E}} P(Y_n = (a,b), \theta_n \in C_{ab}, \theta_{n-1} \notin G)
= \sum_{(a,b) \in \mathcal{E}} P(Y_n = (a,b))P^b(C_{ab} \setminus G_{ab})
= 0.
\]

So we have shown \( M = M' \) \( p \)-a.s. Writing \( M' = \{ \tau_1', \tau_2', \ldots \} \), it is then clear that \( \tau_i' = \tau_i \), \( F_{\tau_i} = F_{\tau_i} \), \( G_{\tau_i}' = G_{\tau_i} \) \( p \)-a.s. for every \( i \). We complete the proof by showing that \( C \) is transition reproducing for \( p \), and that \( F_{\tau_i} = G_{\tau_i} \) \( p \)-a.s. But if \( C \) is transition reproducing it is easy to see that \( \tau_i' \) is measurable with respect to the \( p \)-completion of \( F_{\tau_i} \), or \( F_{\tau_i} \), (an argument for a similar assertion is provided in the proof of (b) below), hence so is \( \tau_i' \) and \( F_{\tau_i} = G_{\tau_i} \) \( p \)-a.s. follows.

The argument that \( C \) is transition reproducing is more lengthy and uses that \( \tau_i' \) is CI-birth with respect to \( G_{\tau_i} \) rather than just \( F_{\tau_i} \).

Let \( i < j \) and \((a,b), (c,d)\) be given such that \( P(Z_i = (a,b), Z_j = (c,d)) > 0 \). Then find \( k, n \in \mathbb{N} \), an \( F_k \)-atom \( A \subset (Y_k = (a,b)) \) and an \( F_n \)-atom \( B \subset (X_0 = b, Y_n = (c,d)) \) with \( PD > 0 \), where \( D = (A, \tau_i = k, \theta_k \in B, \tau_j = k + n) \). Since \( \tau_i \) is \( G_i \)-measurable, \( D \in G_j \) and

\[ P(\theta_{\tau_j} \in \cdot | D) = Q_{cd} \]

because \( \tau_j \) is CI-birth relative to \( G_j \). Here as below we write \( Q_{xy}^Y = P^Y(\cdot | C_{xy}) \). The conditional probability may however also be found using that
\( \tau_i \) is CI-birth. Using the homogeneity of \( M \), which implies in particular that 
\[ \tau_j = \tau_i + \tau_{j-i} \circ \theta \] 
on \( \tau_i < \infty \), we find

\[
P(\theta_i \in B, \tau_j = k + n, \theta_j \in A, \tau_i = k) 
\]
\[= P(\theta_i \in B, \tau_{j-i} \circ \theta_i = n, \theta_i \circ \theta_j \in A, \tau_i = k) 
\]
\[= Q_{ab}^b(B, \tau_{j-i} = n, \theta_j \in \cdot) 
\]
and from this it follows directly that

\[
P(\theta_j \in \cdot | D) = Q_{ab}^b(\theta_n \in \cdot | B, \tau_{j-i} = n) 
\]

Using the Markov property of \( Q_{ab}^b \) at time \( n \) shows the right hand side to equal \( Q_{cd}^d(\cdot \mid (\tau_{j-i} = n)_B) \), and comparing with (3.28) we therefore find

\[
Q_{cd}^d = Q_{ab}^d(\cdot \mid (\tau_{j-i} = n)_B) 
\]

Now not only is \( Q_{ab}^d = p^b(\cdot \mid C_{ab}) \), \( Q_{cd}^d = p^d(\cdot \mid C_{cd}) \), but we also have \( Q_{ab}^d = p^d(\cdot \mid C_{ab}) \). Thus

\[
p^d(\cdot \mid C_{cd}) = p^d(\cdot \mid C_{ab}, (\tau_{j-i} = n)_B) 
\]

i.e. \( C_{cd} = (C_{ab}, (\tau_{j-i} = n)_B) \) \( p^d \)-a.s., in particular \( C_{cd} \subset C_{ab} \) \( p^d \)-a.s.

Summarizing we have shown that if \( i < j \), then

\[
(3.29) \quad P(Z_i = (a,b), Z_j = (c,d)) > 0 \Rightarrow C_{cd} \subset C_{ab} \quad p^d \)-a.s.
\]

We must show (see Definition 3.12) that \( (a,b)_{pr}(p)(c,d) \) implies \( C_{cd} \subset C_{ab} \) \( p^d \)-a.s. or \( p^d(C_{ab}, C_{cd}) = 0 \). By assumption \( n \) may be found with

\[
(3.30) \quad p^b(C_{ab}, Y_n = (c,d)) = p^b(C_{ab})Q_{ab}^b(Y_n = (c,d)) > 0 
\]
in particular \( C_{ab} \neq \emptyset \) and \( (a,b) \in R \). If \( (c,d) \notin R, C_{cd} = \emptyset \) and there is nothing to prove, so we may also assume \( (c,d) \in R \).
Now find $i$ with $P(Z_i = (a,b)) > 0$, and consider the two possible cases:

(i) $\pi > 0$, (ii) $\pi = 0$, where

\[(3.31) \quad \pi = P(Z_1 = (a,b), Y_{\tau_1+n} = (c,d), \theta_{\tau_1+n} \in C_{cd})\]

In case (i), the identification between $M$ and $M'$ shows that $P$-a.s.

\[(Z_i = (a,b), Y_{\tau_i+n} = (c,d), \theta_{\tau_i+n} \in C_{cd}) \subset U \quad (Z_i = (a,b), Z_j = (c,d)),\quad j > i\]

and then $\pi > 0$ and (3.29) gives $C_{cd} \subset C_{ab} P^d$-a.s. as desired.

In case (ii), because $\tau_i$ is CI-birth we have

\[
0 = P(Z_i = (a,b)) Q^b_{ab}(Y_n = (c,d), \theta_n \in C_{cd})
\]

\[
= P(Z_i = (a,b)) Q^b_{ab}(Y_n = (c,d)) Q^d_{ab}(C_{cd}).
\]

The first factor is $> 0$ by assumption, the second is $> 0$ by (3.30), hence $Q^d_{ab}(C_{cd}) = 0$, i.e. $P^d(C_{cd} | C_{ab}) = 0$ or $P^d(C_{ab} | C_{cd}) = 0$ and the proof of (a) is complete.

(b) Since $\tau_1$ satisfies (3.15) with $F_n = \Omega, \tau_1 \in BTR$. Now $\tau_i = \tau_{i-1} + \tau_1 \theta_{\tau_{i-1}}$, so using Proposition 3.21 and proceeding by induction it is clear that each $\tau_i \in BTR$ and that (3.15) holds with $C_{ab}$ the given collection $C$. That each $\tau_i$ is CI-birth for any $P$ as stated, follows then from Proposition 3.16.

To show that $F_{\tau_i} = G_i$ we must check that $\tau_j$ is $F_{\tau_i}$-measurable for $j < i$. But on the $F_{\tau_i}$-atom $A = (X_0 = x_0, \ldots, X_n = x_n, \tau_i = n)$, only those time-points $k < n$ can belong to $M$ for which $x_{k-1} x_k \supset (X_0 = x_n, x_{n-1} x_n)$, and then among these exactly those succeed for which $(x_{k-1}, x_k) \in V_{x_{k-1} x_k}$ for $k < \ell < n$. Thus $M \cap \{1, \ldots, n-1\}$ and hence also $\tau_j$ is constant on $A$. □

The starting point in Theorem 3.26 is a homogeneous random set $M$, i.e. a
set with a specific algebraic structure. It is then shown essentially that the points in \( M \) are CI-birth times iff the set \( G \) characterizing \( M \) (cf. (3.23)) has a special form which is described explicitly algebraically.

Alternatively one might have taken an arbitrary random set \( M = \{ \tau_1, \tau_2, \ldots \} \), assuming as in Theorem 3.26 (a) that each \( \tau_i \) is CI-birth for \( P \). This however would allow for the possibility \( M = \{ \tau \} \) with \( \tau \) an arbitrary CI-birth time for \( P \), and as maintained above, no explicit algebraic description of such \( \tau \) appears possible.

As we shall presently see, Theorem 3.26 has connections to results in [3], [11].

Suppose that the conditions in Theorem 3.26 (b) are satisfied and let \( P \) be Markov. Because

\[
Z_1 = Z_1 \circ \theta_{\tau_{i-1}}
\]
on \( (\tau_i < \infty) \), it is easy to see that the process \( Z = (Z_i)_{i \in \mathbb{N}_+} \) (which may have finite lifetime) is Markov with transition function

\[
q((a,b),(c,d)) = P^b(Z_1 = (c,d) | C_{ab}).
\]

Thus the Markov chain \( Z \) is obtained from a time change in the original chain \( P \). Time changes of this type were discussed by Pittenger [11] - if above \( C_{ab} = C_b \) depends on \( b \) only, then \( (X_i)_{i \in \mathbb{N}_+} \) becomes a Markov chain and we have an example fitting exactly into Pittenger's theory.

More generally Pittenger considered time changes \( M = \{ \tau_1, \tau_2, \ldots \} \) (here we take \( Z \) rather than \( (X_i)_{i \in \mathbb{N}_+} \) to be the time changed chain) such that \( Z \) is Markov and each \( \tau_i \) is a conditional independence time for \( P \) (but not necessarily CI-birth). One essential algebraic condition on such \( M \) is
(3.32) \[ \tau_i = \tau_{i-1} + \sigma \theta_{\tau_{i-1}} \]

see [11], (3.5b). This leads of course to much more general time changes than those treated in Theorem 3.26, even if all \( \tau_i \) are required to be CI-birth.

For an example, let \( C \) be transition reproducing, let \( \sigma \) be given by (3.15), put \( \tau_1 = \sigma \), and define \( \tau_i \) by (3.32). Then the \( \tau_i \) satisfy (3.5a,b,c) of [11] and therefore e.g. \( Z \) is Markov with respect to any \( P \). (The set \( \Gamma \) in [11] is the set \( G \) from (3.8)). However, because of the \( F_n \) appearing in (3.15), \( M = \{ \tau_1, \tau_2, \ldots \} \)
will not in general be homogeneous. The time change corresponds to taking only
a subset of the full homogeneous set \( \{ n \geq 1 : \theta_{n-1} \in G \} \), \( G \) as in (3.8).

We shall also discuss the relation of the preceding to the time change results of Glover [3]. Translated into discrete time and the setup used here, his Theorem 1.5 states the following: let \( A = (A_n)_{n \in \mathbb{N}} \) be a raw additive functional (RAF) defined on \( \Omega \) with \( A_0 = 0 \) and suppose that for all \( n \in \mathbb{N}^+ \),
\[ \Delta A_n := A_n - A_{n-1} \]
is either 0 or 1. Define for \( i \in \mathbb{N}_+ \)
\[ \tau_i = \inf \{ n \in \mathbb{N}_+ : A_n = i \} \]
and suppose finally (condition (a) of Theorem 1.5, [3]) that for every \( i \),
(3.33) \[ A_{n \downarrow (n \leq \tau_i)} \]

is \( F_i \)-measurable for \( n \in \mathbb{N}_+ \). Then with respect to any Markov probability \( P \), each \( \tau_i \) is a conditional independence time and \( Z = (Z_i), Z_i = Y_{\tau_i} \), is a
Markov chain.

By definition, \( A \) is a RAF if \( A_{k+n} = A_k + A_n \circ \theta_k \) exactly for all \( k, n \in \mathbb{N} \).
If also \( \Delta A_n = 0 \) or 1, necessarily
\[ A_n = \sum_{k=0}^{n-1} G \circ \theta_k, \]
where \( G = (A_1 = 1) = (\Delta A_1 = 1) \). But then \( (\Delta A_n = 1) = (\theta_{n-1} \in G) \), and it is clear
that the time change determined by $A$ is the same as that induced by the homogeneous random set $M = \{ n : \theta_{n-1} \in G \}$, and that all time changes from homogeneous $M$ arise from RAF's in this manner.

The extra condition (3.33) is critical when establishing the Markov property for $Z$. A little path algebra shows it to be equivalent to the following condition, expressed in terms of $G$: for every $n \in \mathbb{N}_+$ there exists $F_n \in F_n$ such that

$$(G, \theta_{n-1} \in G) = (F_n, \theta_{n-1} \in G),$$

which except for a small modification is condition (b) of Theorem 4.4, [11]. So the time changes in [3] appear as special cases of those in [11].

Summarizing the above discussion, we have encountered three types of time changes: those in Theorem 3.26 involve homogeneous random sets with all $\tau_i$ CI-birth times; those in [3] are induced by homogeneous random sets with all $\tau_i$ conditional independence times; and finally those in [11] arise from certain subsets of homogeneous random sets with all $\tau_i$ conditional independence times.

Throughout this section we have discussed CI-birth times $\tau$ which by definition obey a strong Markov property involving conditional independence of the pre-$\tau$ and post-$\tau$ processes given $Y_{\tau}$. Other authors have studied birth times where this conditional independence occurs when conditioning not only on $Y_{\tau}$ but also on some auxiliary $F_{\tau}$-measurable variable.

Thus, in [8] Millar has introduced (for processes in continuous time) randomized coterminant times and shown that if $\tau$ is such a time, then conditionally on $F_{\tau}$ the post-$\tau$ process is Markov with a transition function depending on $X_{\tau}$ and a $F_{\tau}$-measurable variable $U$. Therefore, unless $U$ is a function of $(X_{\tau}, X_{\tau})$, $\tau$ will not be a CI-birth time.
Millar's definition of randomized coterminal times is quite complicated. Simpler examples of birth times with a conditional independence property involving an extra variable can be found in Getoor [1].

From (3.19) it follows that with $\sigma, \tau$ given by (3.19), the random time $\rho = \sigma + \tau \circ \theta_\sigma$ will be a birth time with the kind of conditional independence discussed by Millar and Getoor, provided $Y_\sigma$ is $F_\rho$-measurable. Finally it may be remarked that Millar points out that the class of randomized coterminal times is closed under the addition $(\sigma, \tau) \rightarrow \sigma + \tau \circ \theta_\sigma$. 
4. CI-DEATH TIMES AND THE CLASS DTR

This section contains the definitions and results which are the death time analogues of those in Section 3.

(4.1) Definition A random time $\tau$ is a death time with conditional independence (in short a CI-death time) for the Markov probability $P$ if it is a conditional independence time for $P$ and if conditionally on $Y_\tau$ within $(0 < \tau < \infty)$, the pre-$\tau$ process is Markov with a stationary transition function (depending possibly on $Y_\tau$).

According to a definition in Section 5 of BDC, a random time $\tau$ is a regular death time for a Markov probability $P$ if the pre-$\tau$ process is Markov ($q$) for some (substochastic) transition function $q$ with the pre-$\tau$ and post-$\tau$ processes conditionally independent given $0 < \tau < \infty$ and $X_{\tau-1}$.

Any regular death time for $P$ is a CI-death time. This statement is not as transparent as the similar one for birth times, so we shall produce an argument and at the same time introduce some notation.

Recall that if $Q$ is a Markov probability on $\Omega_\Delta$ with initial measure $\nu$ and transition function $q$, such that $(X_n)$ with positive probability, has finite lifetime, then the process reversed from the lifetime defined by the reversal transformation $R: \Omega_\Delta \to \Omega_\Delta$ given by

$$X_n \circ R = \begin{cases} X_{\xi-1-n} & \text{if } n < \xi < \infty \\ \Delta & \text{otherwise} \end{cases}$$

is again Markov with a substochastic transition function $\hat{q}$ on $\Delta$. Here

$$\hat{q}(x,y) = \xi(y)q(y,x)\xi^{-1}(x)$$

for $x,y \in \{0 < \xi < \infty\}$ with $\xi(z) = \sum_{n=0}^{\infty} Q(X_n = z)$. 

Therefore, if $\tau$ is a regular death time for the Markov probability $P$ on $\Omega$, the reversed pre-$\tau$ process $R \circ K_\tau$ is Markov with stationary transitions, hence so is $R \circ K_\tau$ given $\tau < \infty$ and for this latter process conditioning on $X_{\tau-1}$ simply amounts to freezing the initial state, so that $R \circ K_\tau$, and therefore also $R \circ R \circ K_\tau = K_\tau$, conditionally on $X_{\tau-1}$ is Markov with stationary transitions. By the conditional independence property shared by all regular death times it now follows that any regular death time is a CI-death time.

The main result, Theorem 5.2, in Section 5 of BDC states that a random time $\tau$ is a regular death time for the Markov probability $P^X$ iff $\tau$ is $P^X$-equivalent to a random time in the class $D$. A remark in BDC shows how this result may be generalized to Markov probabilities with a non-degenerate start. Since we shall work with this generalization here, we shall redefine $D$ and restate the regular death time theorem.

If $H \subseteq J, V \subseteq J^2$, let denote the modified terminal time

$$
\tau_{HV} = \begin{cases} 
0 & \text{if } X_0 \in H \\
\inf\{n \in N_+: Y_n \in V\} & \text{otherwise}
\end{cases}
$$

The class $D$ is now defined to comprise all random times $\tau$ of the form

$$
(4.3) \quad \tau = \sup\{n: 1 \leq n \leq \tau_{HV}, \emptyset_{n-1} \in F\}
$$

for some $H \subseteq J, V \subseteq J^2, F \in F$. (By the usual convention $\tau = 0$ if the set in brackets is empty; in particular $\tau = 0$ on $(X_0 \in H)$).

Then the following is true: $\tau$ is a regular death time for the Markov probability $P$ iff $\tau$ is $P$-equivalent to a random time in $D$.

As pointed out in BDC, the results on regular birth times and the corresponding results on death times are duals. This duality is prevalent also in the theory of CI-birth times and CI-death times, so the death time results
will be presented in the same order as their analogues in Section 3. Also, to keep down the length of the paper, we shall not give proofs.

In the death time theory the counterpart of a coterminial event is a sequence \( T = (T^n, n \in \mathbb{N}_+) \) of terminal events

\[
(4.4) \quad T^n = (X_0, Y_k \in V, 1 \leq k < n)
\]

where \( H \subseteq J, V \subseteq J^2 \) are sets not depending on \( n \). Of course the invariant part of the coterminial event is matched by the initial part \( (X_0 \in H) \) of each \( T^n \). Notice that \( T^n \in F_{n-1} \).

The notation from (4.4) will be used below with subscripts \( ab \) where \( (a, b) \in J^2 \). Notice that there is really a switch in notation from (4.3) to (4.4): (4.3) forbids transitions in \( V \) prior to \( \tau \) while (4.4) demands that all pre-\( n \) transitions belong to \( V \). Of course (4.3) is modelled upon the definition of \( D \) from BDC, but in the remainder of the section we shall use the notation (4.4).

The first two results are the duals of Propositions 3.6 and 3.9.

(4.5). Proposition A random time \( \tau \) is a CI-death time for \( P \) if and only if there exists \( F \in F \) and for every \( (a, b) \in J^2 \) subsets \( H_{ab} \subseteq J, V_{ab} \subseteq J^2 \) such that

\[
(4.6) \quad (\tau = n, Y_{\tau} = (a, b)) = (T^n_{ab}, Y_n = (a, b), \theta_{n-1} \in F) \quad P \text{-a.s.}
\]

for \( n \in \mathbb{N}_+ \). \( \Box \)

(4.7). Proposition A random time \( \tau \) is a CI-death time for the Markov probability \( P \) if and only if for every \( (a, b) \in J^2 \) the random time \( \tau_{ab} \) defined by
(4.8) \[ \tau_{ab}^* = \begin{cases} \tau & \text{on } (Y_\tau = (a,b)) \\ 0 & \text{otherwise} \end{cases} \]

is a regular death time for $P$. \hfill \Box

Suppose that $\tau$ is a CI-death time for $P$ so that (4.6) holds. Consider $(a,b)$ with $P(Y_\tau = (a,b)) > 0$, introduce

\[ J_{ab} = \{ x \in J : \sum_{n=0}^{\infty} P(X_n = x, n < \tau < \infty, Y_\tau = (a,b)) > 0 \} , \]

the state space for $K^*_\tau$ given $Y_\tau = (a,b)$. Straightforward calculations show that $K^*_\tau$ given $Y_\tau = (a,b)$ is Markov with transitions

\[ q_{ab}(x,y) = p(x,y) l^*_{ab} (x,y) \frac{g_{ab}(y)}{g_{ab}(x)} \quad (x,y \in J_{ab}) \]

where

\[ g_{ab}(z) = \sum_{n=1}^{\infty} P^z(Y_k \in V_{ab}, 1 \leq k < n, X_{n-1} = a) . \]

However, the symmetry between the birth time and death time theories is brought out more clearly by considering $K^*_\tau$ reversed, as was done in Theorem 2 of Jacobsen [6], and leads in a natural way to the death time analogue of Theorem 3.26.

Introduce $\xi(z) = \sum_{n=0}^{\infty} P(X_n = z)$ and the transition function in natural P-duality to $p$ (cf. [6]),

\[ \hat{p}(x,y) = \xi(y) p(y,x) \xi^{-1}(x) . \]

Without loss of generality we may assume $\xi > 0$, and then $\hat{p}(x,y)$ is defined if $\xi(x), \xi(y) < \infty$. For convenience we now assume $\xi < \infty$ (so the P-chain is transient), although it is enough that $\xi < \infty$ on the state space $U J_{ab}$ for $K^*_\tau$.

Corresponding to $(T^*_n)_{n \in \mathbb{N}^+}$ there is a natural dual coterminal event $C^*_{ab}$
which is a subset of the space \( \Omega_0 = (\zeta < \infty) \subseteq \Omega_\Delta \) of paths with finite lifetime, namely

\[
(4.9) \quad \hat{C}_{ab} = \{ Y_k \in V_{ab}, 1 \leq k < \zeta < \infty, X_{\zeta - 1} \in H_{ab} \} \\
= \{ Y_k \in V_{ab} \cup (H_{ab} \times \{ \Delta \}) \cup \{ (\Delta, \Delta) \}, k \in \mathbb{N} \},
\]

where of course

\[
\hat{V}_{ab} = \{ (x,y) \in J^2 : (y,x) \in V_{ab} \}.
\]

(About notation in the sequel: symbols with a \( \hat{\cdot} \) refer to objects pertaining to the path space \( \Omega_0 \)).

(4.10). Proposition With \( P, \hat{P}, \hat{C}_{ab} \) as above and \( \tau \) satisfying (4.6)

\[
(4.11) \quad P(\rho \circ K_\tau \in \cdot | Y_\tau = (a,b)) = \hat{P}^a(\cdot | \hat{C}_{ab}).
\]

Remark This result states that the distribution of \( K_\tau \) reversed given \( Y_\tau = (a,b) \) is the same as that of \( \theta_\tau \) given \( Y_\tau = (b,a) \) for a process \( \hat{P} \) in natural duality to \( P \) and with \( \hat{\tau} \) a CI-birth time for \( \hat{P} \) satisfying (3.7) \( \hat{P} \)-a.s. with the \( C_{ab} \) there replaced by the dual \( \hat{C}_{ab} \) of \( (T^n) \).

Proof Both chains under consideration start in \( a \), so to prove (4.11) it remains to identify their (substochastic) transition functions which by (4.2) and (3.4) are

\[
e(y)q_{ab}(y,x)e^{-1}(x) \quad \text{and} \quad 1_{\hat{V}_{ab}}(x,y)\hat{p}(x,y)\hat{g}(y)\hat{g}(x)
\]

respectively, where

\[
e(z) = \sum_{n=0}^{\infty} P(X_n = z, n < \tau < \infty | Y_\tau = (a,b)),
\]

\[
\hat{g}(z) = \hat{P}^z(\hat{C}_{ab}).
\]
It is essential to note, and fairly obvious to verify, that the two chains have \( J_{ab} \) as state space. Inserting the expressions for \( q_{ab} \) and \( \tilde{p} \) above, it is seen that we need only show that the functions \( e^{-\xi} \) and \( \hat{\xi}g \) are proportional on \( J_{ab} \). (We write \( g = g_{ab} \)). Now

\[
e(z) = \sum_{n=0}^{\infty} \sum_{n} P(X_n = z, n < t < \infty, Y_n = (a,b))
\]

\[
= \sum_{n=0}^{\infty} \sum_{k > n} P(X_n = z, t = k, Y_k = (a,b))
\]

\[
= \sum_{n=0}^{\infty} \sum_{k > n} P(X_n = z, T_{ab}^{k}, \epsilon_{k-1} \in F, Y_k = (a,b))
\]

and using the Markov property at times \( n \) and \( k-1 \) (recall that \( T_{ab}^{k} \in F_{k-1} \)) this reduces to

\[
g(z)P^a(X_1 = b, F) \sum_{n=0}^{\infty} P(X_n = z, T_{ab}^{n+1})
\]

Consequently \( e(z)g^{-1}(z) \) is proportional to

\[
(4.12) \sum_{n=0}^{\infty} P(X_n = z, T_{ab}^{n+1})
\]

On the other hand \( \hat{P}^Z(\zeta < \infty) = 1 \) and

\[
\hat{g}(z) = \sum_{n=1}^{\infty} \hat{P}^Z(Y_k \in \hat{V}_{ab}, 1 \leq k < n, X_{n-1} \in \hat{H}_{ab}, \zeta = n)
\]

\[
= \sum_{n=1}^{\infty} \sum_{u \in \hat{H}_{ab}} \hat{P}^Z(Y_k \in \hat{V}_{ab}, 1 \leq k < n, X_{n-1} = u) \hat{p}(u, \Delta)
\]

By duality

\[
\hat{P}^Z(Y_k \in \hat{V}_{ab}, 1 \leq k < n, X_{n-1} = u) = \xi(u)\hat{P}^u(Y_k \in \hat{V}_{ab}, 1 \leq k < n, X_{n-1} = z)\xi^{-1}(z)
\]

and since with \( \mu \) the initial distribution for \( \hat{P} = P^u \), \( \xi(u)\hat{p}(u, \Delta) = \mu(u) \), it follows that

\[
\xi(z)\hat{g}(z) = \sum_{n=1}^{\infty} P(X_{n-1} = z, T_{ab}^{n})
\]

which is (4.12) exactly, so the proof of (4.11) is complete.
With the preceding discussion as motivation, we shall without further comments state the analogues of the definitions and results from Definition 3.11 onwards.

Consider a collection \( T = (T^n, (a,b) \in J, n \in \mathbb{N}_+ \) of sequences of terminal events and define \( C_{ab} \) as in (4.9) above.

(4.13). **Definition** \( T \) is a transition reproducing collection of sequences of terminal events if \( (a,b) \Rightarrow (c,d) \) implies that either \( C_{ab} \Rightarrow (X_0 = c, C_{cd}) \) or \( (X_0 = c, C_{ab} \cap C_{cd}) = \emptyset \). Here \( (a,b) \Rightarrow (c,d) \) means that there exists \( \omega \in (X_0 = a, C_{ab}) \) and \( n \in \mathbb{N}_+ \) such that \( Y_n(\omega) = (d,c) \).

Let \( p \) be a substochastic transition function on \( J \) such that \( p^x(\zeta < \infty) = 1 \) for all \( x \in J \).

(4.14). **Definition** \( T \) is transition reproducing for \( p \) if \( (a,b) \Rightarrow (p)(c,d) \) implies that either \( C_{ab} \Rightarrow C_{cd} \Rightarrow C_{cd} \) or \( C_{ab} \cap C_{cd} = \emptyset \Rightarrow \hat{p}^c \) a.s. Here \( (a,b) \Rightarrow (p)(c,d) \) means that
\[
\sum_{n=1}^{\infty} \hat{p}^a(C_{ab}, Y_n = (d,c)) > 0.
\]

(4.15). **Example** Let \( (T^n, T^{\cdot n}) \) be an arbitrary collection of sequences of terminal events and define
\[
T^n_{ab} = (Y_k \in V_{ab}^*, 1 \leq k < n, T^n_{ab})
\]
where
\[
V_{ab}^* = \{ (x,y) : C_{ab} \Rightarrow C_{xy} \}.
\]
Then \( T = (T^n_{ab}) \) is a transition reproducing collection of sequences of terminal events.

(4.16) **Definition** Let \( DTR \) denote the class of random times \( \tau \) of the form
(4.17) \( \tau(\omega) = \sup \{ n \in \mathbb{N}_+ : \omega \in T_n^\infty(\omega), \theta_{n-1}^\omega \in F \} \),

where \( F \in F \) and \( T = (T_{ab}^n, (a, b) \in \mathcal{J}_2, n \in \mathbb{N}_+) \) is a transition reproducing collection of sequences of terminal events. \( \blacksquare \)

(4.18). **Proposition** Suppose \( \tau \) belongs to the class DTR. Then \( \tau \) is a CI-death time for any Markov probability \( P \).

If \( \tau \) is given by (4.17) and \( P \) is transient, then

\[ P(R \circ K_{\tau} \in \cdot | Y_{\tau} = (a, b)) = \hat{p}_t(\cdot | C_{ab}) , \]

where \( \hat{p} \) is the transition function in natural \( P \)-duality to \( p \). \( \blacksquare \)

(4.19). **Example** The class of random times \( \tau \) of the form (4.17) with \( T \) as in Example 4.15, is the class DO first introduced in [5]. The times \( \tau \) and \( \overline{\tau} \) from Example 3.18 both belong to DO. \( \blacksquare \)

Let now \( \hat{G} \) be a measurable subset of \( \Omega_0 \) and consider the random subset of \( \mathbb{N} \) given by

(4.20) \( L = \{ n \in \mathbb{N} : R \circ K_{n+1} \in \hat{G} \} \).

Sets of this form appear as the duals of homogeneous random sets. Of special interest to us is the situation where \( (T_{ab}^n) \) is a collection of sequences of terminal events, \( \hat{C}_{ab} \) is as in (4.9) and

(4.21) \( \hat{G} = \bigcup_{(a,b) \in J_2} (Y_1 = (b, a), \theta \in \hat{C}_{ab}) . \)

Then \( n \in L(\omega) \) iff \( \omega \in T_{Y_n}^n(\omega) \).

With \( \hat{G} \) given by (4.21), \( \hat{G} \subset (\zeta > 1) \) and from now on, when considering \( L \) of the form (4.20), we shall always assume the \( \hat{G} \) there to be a subset of \( (\zeta > 1) \). Then \( 0 \in L \) is impossible.
With this assumption in force, given \( L \) of the form (4.20), define random times \( (\sigma_i, i \in \mathbb{N}_+) \) on \( \Omega \) by letting \( \sigma_i = \infty \) on \( (|L| = \infty) \), \( \sigma_i = 0 \) on \( (|L| < i) \) and by writing \( L = \{\sigma_{\ell}, ..., \sigma_i\} \) on \( (|L| = \ell) \) with \( \sigma_{\ell} < ... < \sigma_i \). Also define \( W_i = (X_{\sigma_i}^\ell, X_{\sigma_i-1}^\ell) \) on \( \mathcal{A}_i := (0 < \sigma_i < \infty) \) and let \( H_i \) denote the \( \sigma \)-algebra of subsets of \( \mathcal{A}_i \) generated by \( \theta_{\sigma_i-1}^\ell \) and \( \sigma_j - \sigma_i \) for \( j < i \).

As usual, in the theorem below, \( \hat{p} \) denotes the transition function in natural \( \hat{p} \)-duality to \( p \).

(4.22) Theorem (a) Let \( P \) be Markov and transient and let \( L \) be a random set of the form (4.20) with \( \hat{G} \subseteq (\zeta > 1) \). Suppose each \( \sigma_i \) is a CI-death time for \( P \) with respect to \( H_i \) such that the transitions for \( R \circ K_{\sigma_i} \) given \( H_i \) do not depend on \( i \), i.e. for all \( i, (a,b) \),

\[
P(R \circ K_{\sigma_i} \in \cdot | A_i, H_i) = \hat{q}_{ab}^a
\]

on \( (A_i, W_i = (b,a)) \), where \( \hat{q}_{ab}^a \) is Markov on \( \hat{\Omega}_0 \) with transitions \( \hat{q}_{ab} \) not depending on \( i \) and initial state \( a \). Then there exists a collection \( T = (T^n_{ab}) \) of sequences of terminal events, transition reproducing with respect to \( \hat{p} \) such that

\[
L = \{n \in \mathbb{N}_+: R \circ K_{n+1} \in \hat{G}'\} \quad \text{P-a.s. on } (|L| < \infty)
\]

where \( \hat{G}' = \bigcup_{(a,b)} (Y_1 = (b,a), \theta \in \hat{C}_{ab}) \).

(b) Let \( T = (T^n_{ab}) \) be a transition reproducing collection of sequences of terminal events and let \( L \) denote the random set given by (4.20) with \( \hat{G} \) as in (4.21). Then \( \sigma_i \in \hat{DTR} \) for all \( i \), in particular \( \sigma_i \) is a CI-death time for any Markov probability \( P \), and if \( P \) is transient,

\[
P(R \circ K_{\sigma_i} \in \cdot | A_i, \theta_{\sigma_i-1}^\ell) = \hat{p}^a(\cdot | \hat{C}_{ab})
\]

on \( (A_i, W_i = (b,a)) \). \( \square \)
Suppose the conditions in (b) are satisfied. Rather than considering the full set $L$ (in which case nothing interesting is said when working on $|L| = \infty$), one may consider the part $L^*$ of $L$ preceding a given cooptional time. Defining $(\sigma_i^*, i \in \mathbb{N}^+_\omega)$ from $L^*$ as the $\sigma_i$ were defined from $L$, the conclusions in (b) remain valid with $\sigma_i$ replaced by $\sigma_i^*$.

As a final remark, note that under the assumptions in (b), $(W_i, i \in \mathbb{N}^+_\omega)$ is a Markov chain with respect to any Markov probability $P$ and if $P$ is transient the transitions are given by

$$P(W_{i+1} = (d,c) | W_i = (b,a)) = \hat{P}^a(\hat{Y}_\tau = (d,c) | \hat{C}_{ab})$$

where $\hat{\tau}$ is defined on $\Omega_0$ by

$$\hat{\tau} = \inf\{n \in \mathbb{N}^+_\omega : \theta_{n-1} \in \hat{G}\}.$$
5. PATH DECOMPOSITIONS WITH MARKOVIAN EXCURSIONS

We shall briefly discuss what kind of path decompositions obtain when the results from Sections 3 and 4 may be combined.

Consider a homogeneous random set $M = \{ n \in \mathbb{N}_+: \theta_{n-1} \in G \}$ with $G$ given by (3.8) and $C = (C_{ab})$ a transition reproducing collection of coterminous events.

With $M = \{ \tau_1, \tau_2, \ldots \}$ as in Section 3, we saw in Theorem 3.26 (b) that each $\tau_i$ is a CI-birth time for any Markov probability $P$. Writing $\tau = \tau_1$, the post $\tau$-process splits into the Markov chain $(Z_i) = (Y_i)$ and the sequence $(e_i, i \in \mathbb{N}_+)$ of skew excursions where

$$e_i = (X_{\tau_i}, X_{\tau_i+1}, \ldots, X_{\tau_i+1}) .$$

There may be finitely or infinitely many excursions according as $(Z_i)$ has finite lifetime or not. All excursions have finite lengths except the last one in the case where there are only finitely many excursions.

It follows immediately from the results in Section 3, that given $(Z_i, i \in \mathbb{N}_+)$, (which includes conditioning on the lifetime of the $Z_i$-chain) the excursions are, with respect to any Markovian $P$, conditionally independent with the distribution of $e_i$ not depending on $i$ but only on how $e_i$ is conditioned to start and end. More specifically, with $Q_{ab} = P^b(\cdot | C_{ab})$,

$$P(e_i \in \cdot | Z_i = (a,b), Z_{i+1} = (c,d)) = Q_{ab} (X_{\tau_i} \in \cdot | Z_1 = (c,d)) ,$$

$$P(e_i \in \cdot | Z_i = (a,b), i \text{ is the lifetime of } (Z_j)) = Q_{ab} (\cdot | \tau_1 = \infty) .$$

In general the conditional excursions will of course not be Markov. However, if $\tau = \tau_1$ is a CI-death time for each $Q_{ab}$, then certainly all the finite excursions are Markov, and if in addition $(\tau_1 = \infty)$ is $Q_{ab}$-a.s. equal to a coterminous event, then also the last infinite excursion will be
Without aiming for complete generality, we shall discuss a simple example involving Markovian excursions.

Suppose given a transitive relation $\succ$ on $J^2$ and invariant events $(C_{a,b},\infty, (a,b) \in J^2)$ compatible with $\succ$ in the sense that

$$(a,b) \succ (c,d) \Rightarrow C_{a,b,\infty} \Rightarrow C_{c,d,\infty}.$$  

Defining

$$(5.1) \quad C_{a,b} = \{(a,b) \succ Y_k, k \in N, C_{a,b,\infty}\},$$

$C_{a,b}$ is coterminal and $C = (C_{a,b})$ is transition reproducing. (This setup provides an alternative description of the class $BO$ from Example 3.18, see Proposition 3.31 in [5]).

With this choice of $C$ one finds

$$(5.2) \quad (\tau = n, Y_\tau = (a,b)) = \left[ \bigcap_{k=1}^{n-1} \bigcup_{\ell=k+1}^n (Y_k \succ Y_{\ell'}) \right] \cap (Y_n = (a,b), \theta \in C_{a,b})$$

writing $(x,y) \succ (u,v)$ if it is not true that $(x,y) \succ (u,v)$.

Suppose now in addition that the relation $\succ$ is also transitive. We claim that then

$$(5.3) \quad (\tau = n, Y_\tau = (a,b)) = (Y_k \succ (a,b), 1 \leq k < n, Y_n = (a,b), \theta \in C_{a,b}).$$

To see this, suppose $\tau(\omega) = n, Y_\tau(\omega) = (a,b)$. From (5.2) it follows (for $k = n-1$) that $Y_{n-1}(\omega) \succ (a,b)$. Suppose it has been shown that

$Y_\ell(\omega) \succ (a,b)$ for $k \leq \ell < n$. Then $Y_{k-1}(\omega) \succ (a,b)$ follows because by (5.2), $Y_{k-1}(\omega) \succ Y_\ell(\omega)$ for some $\ell, k \leq \ell < n$, by assumption $Y_\ell(\omega) \succ (a,b)$, hence $Y_{k-1}(\omega) \succ (a,b)$ since $\succ$ is transitive. So an Induction argument yields (5.3) from (5.2).
But comparing (5.3) with (4.6) it is clear that \( \tau \) is a CI-death time for any Markov probability. So by the preceding discussion, with \( C \) given by (5.1), \( \succ \) and \( \preceq \) transitive as described above, we obtain a path decomposition with the excursions \((e_i)\) being independent and Markov given \((Z_i)\).

We have here discussed path decompositions induced by certain homogeneous random sets. But it is of course also possible to obtain decompositions based on the dual homogeneous sets \( L \) considered in Theorem 4.22 (b).

Finally, there are examples of path decompositions which are perfect in the following sense: suppose \( L \) is as in Theorem 4.22 (b) with the \( \tilde{\tau} \) from (4.23) of the form (4.6) (relative to \( \Omega_0 \)), suppose \( M \) is as in Theorem 3.26 (b) with \( \tau = \tau_1 \) of the form (4.6), and suppose that the beginning \( \tau = \inf M \) of \( M \) equals the end \( \sup L \) of \( L \). Then because \( \tilde{\tau} \) and \( \tau \) are always CI-death times the given Markov chain \( P \) may be decomposed as follows: choose a random variable \( U = Z_1 = \tilde{W}_1 \) with distribution the \( P \)-distribution of \( Y_\tau \), and define \( W_1 \) by interchanging the two components of the transition \( U = \tilde{W}_1 \). Given \( U \), construct two independent Markov chains, \((Z_i)\) and \((W_i)\) with distributions matching those of the \((Z_i)\) and \((W_i)\) of Sections 3 and 4. Finally, given \( U \), \((Z_i)\) and \((W_i)\), establish two independent sequences of mutually independent Markovian excursions \((e_i)\) and \((f_i)\), where the \((e_i)\) together with \((Z_i)\) are to constitute the post-\( \tau \) process \( \theta_\tau \) as described above, while the \((f_i)\) and \((W_i)\) are to yield in a similar manner the pre-\( \tau \) process \( K_\tau \).

We leave it to the reader to check that with \( \tau = \tau_1 \) or \( \overline{\tau} \) (Examples 3.18 and 4.19), examples of such perfect path decompositions are obtained.
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REFERENCES


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