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SUMMARY

Given a Markov chain $(X_n)_{n\geqq 0}$, random times τ are studied which are birth times or death times in the sense that the post-τ and pre-τ processes are independent given the present $(X_{\tau-1}, X_\tau)$ at time τ and the conditional post-τ process (birth times) or the conditional pre-τ process (death times) is again Markovian. The main result for birth times characterizes all time substitutions through homogeneous random sets with the property that all points in the set are birth times. The main result for death times is the dual of this and appears as the birth time theorem with the direction of time reversed.

1. INTRODUCTION AND NOTATION

An earlier paper, [7], by Jim Pitman and the author, hereafter referred to as BDC, contained a study of certain classes of birth times and death times for Markov chains in discrete time with stationary transition probabilities.

Much of the motivation for that paper came from David Williams' [14] fundamental results on path decompositions of diffusions, in particular the one-dimensional Brownian motion BM(1).

The types of e.g. birth times considered in BDC were random times τ determined by the evolution of the Markov chain, with the property that the post-τ process is again Markov with a transition function possibly different from that of the given chain, and furthermore τ should have a conditional independence property similar to that in the strong Markov property, with past and future independent given the present at time τ.

However, not all discrete time analogues of Williams' path decompositions are covered by BDC. For instance Williams showed that for BM(1) made transient by absorption at a high level, the time τ of the ultimate minimum of the path is a birth time and a death time in the sense that given the value x of the path at time τ, the pre-τ and post-τ processes are both Markovian and conditionally independent. But since the transitions for the two fragments obviously depend on x, results of this type are not included in BDC.

One motivation behind the present paper has been the desire to fill this gap. A first discussion of the larger classes of birth and death times needed to accomplish this, appeared in the preprint Jacobsen [5], and some of the fundamentals there are repeated here. But the main results to be given below deal with a particular class of birth times (and its dual class of death times) which apart from possessing several nice properties is relevant to the theory of time substitutions in Markov processes, cf. the papers by Pittenger [11] and Glover [3].

The main results of BDC provide characterizations of classes of birth times and death times formulated as equivalences between objects defined relative to the probabilistic structure and objects defined in terms of path-algebra. This interplay between probability theory and path-algebra is also the essence of the philosophy behind the present paper. What it amounts to technically will be discussed at the end of this introduction.

One obvious open problem left by BDC was how the results could be carried over to processes in continuous time. This has now been done by Pittenger [10] (Theorem 3.9 of BDC on birth times), Sharpe [13] (Theorem 5.2 on death times) and, most fittingly and most recently, by Pittenger and Sharpe [12] (Theorem 6.2 on times that are birth as well as death times).

Another important recent reference is the paper [2] by Getoor and Sharpe where they discuss what types of conditional independence are relevant in continuous time, and provide necessary and sufficient conditions for the different types to be valid.

The basic reference for this paper is BDC and it may be useful for the reader to have a copy available.

The notation to be used here is that of BDC with some minor changes: given a countable state space J, let Ω denote the space of all sequences $\omega = (\omega_0, \omega_1, \ldots)$ in J indexed by the non-negative integers N, let $(X_n, n \in N)$ be the coordinate process on Ω, i.e. $X_n(\omega) = \omega_n$, and denote by $(Y_n, n \in N_+)$ the sequence of transitions $Y_n = (X_{n-1}, X_n)$ defined for $n \in N_+ = \{1,2,\ldots\}$. Writing $\mathcal{F}$ for the (uncompleted) σ-algebra on Ω spanned by all X_n, a probability P on $(\Omega, \mathcal{F})$ is said to be Markov or Markov (p) if P makes (X_n) a Markov chain with stationary transitions p. If μ is the P-law of X_0 , P^μ may be written instead of P and, as is the custom, P^x if μ is degenerate at x . The following convention is adapted throughout: the same letter is used to denote a Markov probability (capital letter) and its transition function (small letter).

Adjoining a state Δ to J , write $J_\Delta = J \cup \{\Delta\}$ and let Ω_Δ be the space of all sequences in J_Δ that remain in Δ once they get there. The lifetime of a sequence $\omega \in \Omega_\Delta$ is $\zeta(\omega) = \inf\{n \in N : X_n(\omega) = \Delta\}$.

The space Ω_Δ and the subspace $\Omega_0 = (\zeta < \infty)$ of paths with finite lifetime will be used mainly in Section 4 on death times. For objects pertaining to Ω_Δ, the same notation will be used as for the corresponding objects on Ω .

For $n \in N$, the killing operator $K_n : \Omega \to \Omega_\Delta$ and shift operator $\theta_n : \Omega \to \Omega$ are defined by

$$K_n(\omega_0,\omega_1,\ldots) = (\omega_0,\ldots,\omega_{n-1},\Delta,\Delta,\ldots) ,$$

$$\theta_n(\omega_0,\omega_1,\ldots) = (\omega_n,\omega_{n+1},\ldots) .$$

For $n = 1$, θ is written instead of θ_1 .

A <u>random time</u> is a measurable mapping from Ω to the extended time set $\overline{N} = N \cup \{\infty\}$. Given a random time τ, X_τ, Y_τ, K_τ, θ_τ are defined by local identification, e.g. $X_\tau = X_n$ on $(\tau = n)$. Also, X_τ, K_τ, θ_τ are defined only on the set $(\tau < \infty)$ and Y_τ on $(0 < \tau < \infty)$. As a consequence, for instance $(Y_\tau = (a,b))$ will be the notation for the subset $\{\omega : 0 < \tau(\omega) < \infty, Y_\tau(\omega) = (a,b)\}$ of Ω .

For a fixed $n \in N$ the pre-n σ-algebra $\mathcal{F}_n$ is the σ-algebra spanned by $(X_0,\ldots,X_n)$. The atoms $\mathcal{A}_n$ are the sets of the form $A = (X_0 = x_0,\ldots,X_n = x_n)$. For τ a random time, the pre-τ σ-algebra $\mathcal{F}_\tau$ consists of the sets which are countable unions of sets of the form (i) $(A, \tau = n)$ where $n \in N$, $A \in \mathcal{A}_n$ or (ii) one-point sets $\{\omega\}$ where $\tau(\omega) = \infty$.

A random time τ splits the process (X_n) into two parts, the <u>pre-τ process</u>, conveniently identified with and therefore labelled K_τ, given as

$$(X_n \circ K_\tau, n \in N) = (X_0, \ldots, X_{\tau-1}, \Delta, \Delta, \ldots) ,$$

and the <u>post-τ process</u> θ_τ given as

$$(X_n \circ \theta_\tau, n \in N) = (X_\tau, X_{\tau+1}, \ldots) .$$

As discussed earlier, the main theme in BDC and here is to provide equivalences between probabilistic and path-algebraic objects. Thus for instance two different types of definitions of random times will be used: (i) operational definitions and (ii) algebraic definitions. Definitions of type (i) give the properties of a random time relative to a Markov probability, while those of type (ii) are concerned exclusively with the properties of a random time as a function on Ω. The latter may be implicit or explicit in nature, for example a description of a random time involving a collection of parameters is explicit if the parameters may be chosen independently of each other and implicit if they are interrelated.

For an example, consider stopping (optional) times. Given a Markov probability P, τ is an operationally defined stopping time for P if conditionally on $\mathcal{F}_\tau$ within $(\tau < \infty)$, θ_τ is Markov with the same transitions p as P. On the other hand, τ is an algebraically defined stopping time if the following three equivalent conditions are satisfied: (a) $(\tau = n) \in \mathcal{F}_n$ for $n \in N$; (b) $(\tau \leqq n) \in \mathcal{F}_n$ for $n \in N$; (c) $\tau(\omega) = \inf\{n \in N : \omega \in F_n\}$ for some sequence $(F_n, n \in N)$ of sets $F_n \in \mathcal{F}_n$. Here one would call (a), (b) implicit and (c) an explicit definition, because in (a) the sets $(\tau = n)$ must be mutually disjoint, in (b) the sets $(\tau \leqq n)$ must increase with n, while in (c) the $F_n \in \mathcal{F}_n$ are arbitrary.

The characterization theorems to be given here, as those presented in BDC (or Jacobsen [6]), provide probabilistic equivalences between operationally and algebraically defined objects. For instance, and an easy consequence of the results in Section 3 of BDC, for stopping times the following is true: a random time τ is an operationally defined stopping time for the Markov probability P, iff it is P-equivalent to an algebraically defined stopping time.

2. CONDITIONAL INDEPENDENCE TIMES

In BDC two slightly different notions of conditional independence were used in the study of birth times and death times respectively: for the birth times conditional independence of the pre-τ and post-τ processes given X_τ was demanded, while for the death times it turned out that the relevant conditional independence occurs when conditioning on $X_{\tau-1}$.

In this paper we shall use the same conditional independence concept for birth times and death times as described in the following definition which replaces Definition 3.11 in BDC.

(2.1). Definition A random time τ is called a conditional independence time for the Markov probability P, if under P the pre-τ and post-τ processes are conditionally independent given Y_τ. □

Thus τ is a conditional independence time iff there is a conditional distribution of θ_τ given $(X_0,\ldots,X_\tau)$ within $(0<\tau<\infty)$, or equivalently of K_τ given $(X_{\tau-1},X_\tau,\ldots)$ within $(0<\tau<\infty)$, which is a function of the transition Y_τ alone.

It should be noticed that conditioning on $(X_0,\ldots,X_\tau)$ is equivalent to conditioning on $\mathcal{F}_\tau$ and involves in particular the conditioning on the value of τ. By contrast, conditioning on $(X_{\tau-1},X_\tau,\ldots)$ does not imply knowledge of the exact value of τ, wherefore in particular, as is essential, the conditional pre-τ process has a random lifetime.

It seems most natural to have a unified concept for conditional independence applying to the birth time as well as the death time theory. A second reason for using Definition 2.1 is the following: consider for a real-valued process in continuous time with, say, right-continuous, left-limit paths, the time τ where the process attains its ultimate minimum. With jumps possible, the transition function of for instance the conditional post-τ process given the past will in general depend on the transition $(X_{\tau-}, X_\tau)$ rather than on X_τ alone. Translating this into the discrete time situation makes it natural to study K_τ and θ_τ given the transition Y_τ.

The following result provides a useful characterization of conditional independence times. The proof proceeds exactly as that of Lemma 3.12 in BDC and is therefore omitted.

(2.2). Lemma A random time τ is a conditional independence time for the Markov probability P iff for every $n \in N_+$ and every $(a,b) \in J^2$ there exists $F_n \in \mathcal{F}_n$, $G_{ab} \in \mathcal{F}$ respectively such that

$$(2.3) \qquad (\tau = n, Y_\tau = (a,b)) = (F_n, Y_n = (a,b), \theta_n \in G_{ab}) \qquad P\text{-a.s.}$$

or equivalently iff for every $n \in N_+$ and every $(a,b) \in J^2$ there exists $F_{n-1,ab} \in \mathcal{F}_{n-1}$, $G \in \mathcal{F}$ such that

$$(2.4) \qquad (\tau = n, Y_\tau = (a,b)) = (F_{n-1,ab}, Y_n = (a,b), \theta_{n-1} \in G) \qquad P\text{-a.s.} \qquad \square$$

Remark Conditional independence times satisfying (2.3) or (2.4) exactly are not splitting times as originally defined by Williams, see [4], equation (3.3) or [15], Section III.79. It appears most natural to generalize the definition there and call τ a splitting time if

$$(2.5) \qquad (\tau = n) = (F_n, \theta_{n-1} \in G_n) \qquad (n \in N_+)$$

for some $F_n \in \mathcal{F}_n$, $G_n \in \mathcal{F}$. If (2.5) holds with $G_n = G$ not depending on n, τ is a stationary splitting time, cf. [11].

The definition in (2.5) is implicitly algebraic. Lemma 2.2 may now be reformulated as stating that conditional independence times are the operationally defined versions of stationary splitting times (see also the remark following

Lemma 3.12 in BDC and Lemma 2.8 in [11]). The operational definition of general splitting times demands that K_{τ} and θ_{τ} be independent given Y_{τ} <u>and</u> τ.

□

3. CI-BIRTH TIMES AND THE CLASS *BTR*

The purpose of this section is to study various classes of birth times which are conditional independence times.

(3.1). Definition A random time τ is a birth time with conditional independence (in short a CI-birth time) for the Markov probability P if it is a conditional independence time for P and if conditionally on Y_τ within $(0<\tau<\infty)$, the post-τ process is Markov with a stationary transition function (depending possibly on Y_τ). □

As defined in BDC a random time τ is a regular birth time for P if there is a transition function q such that conditionally on $\mathcal{F}_\tau$, the post-τ process is Markov (q). Thus clearly a regular birth time is a CI-birth time. (Notice that if τ is a regular birth time for P, the post-τ process is itself Markov without conditioning on the past. This is of course not true in general for τ a CI-birth time).

Suppose P is Markov. Recall that by Theorem 2.3 of BDC, if $D\in\mathcal{F}$ with $P(D)>0$, then the conditional probability $P_D=P(\cdot|D)$ is Markov iff $D=(X_0\in H,C)$ P-a.s. where $H\subset J$ and C is a coterminal event, i.e. $C=C_VC_\infty$ for some $V\subset J^2$ with $C_V=(Y_n\in V, n\in N_+)$ and $C_\infty\in\mathcal{F}$ invariant for θ.

This result on conditioning events was used in BDC, Theorem 3.9, to give the following explicitly algebraic characterization of regular birth times: defining $\mathcal{B}$ to be the class of random times of the form

(3.2) $\qquad \tau_C+\rho$

where C is an arbitrary coterminal event, τ_C is the associated coterminal time

$$\tau_C=\inf\{n\in N : \theta_n\in C\} ,$$

and ρ is a stopping time for the family $(\mathcal{F}_{\tau_C+n}, n \in N)$ of σ-algebras, it was shown that τ is a regular birth time for P iff τ is P-equivalent to a random time in $\mathcal{B}$. The proof of the theorem also showed that τ is a regular birth time for P iff there exists $F_n \in \mathcal{F}_n$, C coterminal such that

$$(3.3) \qquad (\tau = n) = (F_n, \theta_n \in C) \qquad\qquad P-a.s.$$

for all $n \in N$, cf. (3.16) of BDC. This observation amounts to an implicitly algebraic characterization of regular birth times.

The transitions q for θ_τ are the same as those of $P(\cdot|C)$ and are given by

$$(3.4) \qquad q(x,y) = 1_V(x,y)p(x,y)\frac{g(y)}{g(x)} \quad ,$$

where $g(z) = P^z(C)$.

It is easy to see that instead of using (3.2), $\mathcal{B}$ may be defined as follows: $\tau \in \mathcal{B}$ iff there is a coterminal event C and events $F_n \in \mathcal{F}_n$, $n \in N$ such that

$$(3.5) \qquad \tau(\omega) = \inf\{n \in N : \omega \in F_n, \theta_n \omega \in C\} \ .$$

Propositions 3.6 and 3.9 provide two simple implicitly algebraic characterizations of CI-birth times. We are unable to give an explicit algebraic characterization. The main results below deal with the properties of the explicitly defined class $\mathcal{BTR}$, see Definition 3.14.

Proposition 3.6 is the analogue of (3.3), and is proved exactly like that using (2.3), the fact that for a random time τ satisfying (2.3) the conditional distribution of the post-τ process given $\mathcal{F}_\tau$ within $(Y_\tau = (a,b))$ is $P^b(\cdot|G_{ab})$, and the characterization of conditioning events quoted above.

(3.6). <u>Proposition</u> A random time τ is a CI-birth time for P if and only if for every $n \in N_+$ and every $(a,b) \in J^2$ there exists $F_n \in \mathcal{F}_n$ and coterminal C_{ab} respectively, such that

(3.7) $$(\tau = n, Y_\tau = (a,b)) = (F_n, Y_n = (a,b), \theta_n \in C_{ab}) \qquad P-\text{a.s.} \qquad \square$$

Notice that if τ satisfies (3.7) exactly (no exceptional sets), then τ is a stationary splitting time with the $G = G_n$ of (2.5) given by

(3.8) $$G = \bigcup_{(a,b)} (Y_1 = (a,b), \theta \in C_{ab}) .$$

With τ a CI-birth time for P we shall write q_{ab} for the transition function of the post-τ process given $Y_\tau = (a,b)$. Thus, if τ satisfies (3.7), q_{ab} is the transition function for the Markov probability $Q^b_{ab} = P^b(\cdot \mid C_{ab})$.

The second characterization of CI-birth times is an observation due to J.W. Pitman (private communication). It follows immediately from the definitions of regular birth times and CI-birth times.

(3.9). <u>Proposition</u> A random time τ is a CI-birth time for the Markov probability P if and only if, for every $(a,b) \in J^2$ the random time τ_{ab} defined by

$$\tau_{ab} = \begin{cases} \tau & \text{on } (Y_\tau = (a,b)) \\ \infty & \text{otherwise} \end{cases}$$

is a regular birth time for P . $\square$

Of course Theorem 3.9 of BDC provides an explicitly algebraic characterization of each τ_{ab} . But to obtain from this an explicitly algebraic characterization of all CI-birth times requires that the τ_{ab} be chosen simultaneously in such a way that the sets $(\tau_{ab} < \infty) = (Y_\tau = (a,b))$ be disjoint, and it is not at all clear how this should be done.

Consider a CI-birth time τ for P satisfying (3.7). Since, when ignoring some null sets, the sets on the right are mutually disjoint for n, (a,b) varying, it is clear that for P-almost all ω

$$(3.10) \qquad \tau(\omega) = \inf\{n \in N_+ : \omega \in F_n, \theta_n \omega \in C_{Y_n(\omega)}\} ,$$

cf. (3.5). The main difficulty arising when attempting to characterize CI-birth times in an explicit algebraic fashion rests on the fact that the converse is not true: given an arbitrary collection (C_{ab}) of coterminal events and $F_n \in \mathcal{F}_n$, if τ is defined by (3.10), it is not in general true that (3.7) holds (exactly rather than a.s.) no matter what is the choice of the F_n, C_{ab} appearing there. We shall now discuss systems of coterminal events for which the implication (3.10) to (3.7) holds (for all choices of F_n).

For the two definitions below, let $\mathcal{C} = (C_{ab})_{(a,b)\in J^2}$ be a collection of coterminal events. The inclusion $\supset$ is non strict, allowing for equality.

(3.11). <u>Definition</u> $\mathcal{C}$ is a <u>transition reproducing</u> collection of coterminal events if $(a,b)\mathrm{pr}(c,d)$ implies that either $C_{ab} \supset (X_0 = d, C_{cd})$ or $(X_0 = d, C_{ab}C_{cd}) = \emptyset$. Here $(a,b)\mathrm{pr}(c,d)$ means that there exists $\omega \in (X_0 = b, C_{ab})$ and $n \in N_+$ such that $Y_n(\omega) = (c,d)$. □

Let p be a stochastic transition function on J.

(3.12). <u>Definition</u> $\mathcal{C}$ is <u>transition reproducing for p</u> if $(a,b)\mathrm{pr}(p)(c,d)$ implies that either $C_{ab} \supset C_{cd}$ P^d - a.s. or $C_{ab}C_{cd} = \emptyset$ P^d - a.s. Here $(a,b)\mathrm{pr}(p)(c,d)$ means that

$$\sum_{n=1}^{\infty} P^b(C_{ab}, Y_n = (c,d)) > 0 . \qquad \square$$

Given a Markov probability P we shall call $\mathcal{C}$ transition reproducing for P if $\mathcal{C}$ is transition reproducing for the transition function p of P.

It seems plausible that if $\mathcal{C}$ satisfies the operational definition (3.12), then each C_{ab} may be replaced by a new coterminal event C^*_{ab} such that $C_{ab} = C^*_{ab}$ P^b - a.s. and $\mathcal{C}^* = (C^*_{ab})$ satisfies the algebraic definition (3.11), but this fact is not verified here.

Definition 3.11 puts some implicitly given constraints on the C_{ab}. There does not appear to be any explicit receipt for describing all transition reproducing $\mathcal{C}$. Of course $\mathcal{C}$ is transition reproducing if all $C_{ab} = C$. A more subtle example is the following

(3.13). <u>Example</u> Let $(C'_{ab})_{(a,b)\in J^2}$ be an arbitrary collection of coterminal events and define

$$C_{ab} = (Y_n \in V^*_{ab}, n \in N_+, C'_{ab})$$

where

$$V^*_{ab} = \{(x,y) : C'_{ab} \supset C'_{xy}\} .$$

Then we claim that $\mathcal{C} = (C_{ab})$ is a transition reproducing collection of coterminal events. To see this, suppose $(a,b)\mathrm{pr}(c,d)$, and find ω, n with $\omega \in (X_0 = b, C_{ab}), Y_n(\omega) = (c,d)$. We shall show that $C_{ab} \supset (X_0 = d, C_{cd})$.

Firstly, by the definition of C_{ab} and because $\omega \in C_{ab}$, $(c,d) \in V^*_{ab}$ and $C'_{ab} \supset C'_{cd}$. But then to show that $\omega' \in (X_0 = d, C_{cd})$ implies $\omega' \in C_{ab}$, it is enough to see that $\omega' \in (X_0 = d, C_{cd})$ implies $Y_k(\omega') \in V^*_{ab}$ for all $k \in N_+$, i.e. implies $C'_{ab} \supset C'_{Y_k(\omega')}$. Since by assumption $Y_k(\omega') \in V^*_{cd}$, i.e. $C'_{cd} \supset C'_{Y_k(\omega')}$, and since $C'_{ab} \supset C'_{cd}$, the implication is evident. □

We shall now show that transition reproducing collections of coterminal events lead to a universal class of CI-birth times.

(3.14). <u>Definition</u> Let $\mathcal{BTR}$ denote the class of random times τ of the form

$$(3.15) \qquad \tau(\omega) = \inf\{n \in N_+ : \omega \in F_n, \theta_n\omega \in C_{Y_n(\omega)}\}$$

where for every n, $F_n \in \mathcal{F}_n$ and where $\mathcal{C} = (C_{ab})_{(a,b)\in J^2}$ is a transition reproducing collection of coterminal events. □

(3.16). Proposition Suppose τ belongs to the class *BTR*. Then τ is a CI-birth time for any Markov probability P.

Proof We shall show that τ satisfies (3.7) exactly (with the F_n there not the F_n from the definition of τ). Clearly $\tau(\omega) = n$ and $Y_\tau(\omega) = (a,b)$ iff $Y_n(\omega) = (a,b)$, $\theta_n\omega \in C_{ab}$ and for all k, $1 \leqq k < n$ one of (i), (ii) holds:

(i) $\omega \notin F_k$

(ii) $\theta_k\omega \notin C_{Y_k(\omega)}$.

Represent each C_{xy} as $(Y_m \in V_{xy}, m \in N_+, C^\infty_{xy})$ with C^∞_{xy} invariant. Then define

$$V^0_{xy} = \{(u,v) : (x,y)\,pr\,(u,v)\} ,$$

recalling the meaning of pr from Definition 3.11. In particular $V^0_{xy} \subset V_{xy}$.

We claim that subject to $\tau(\omega) = n$, $Y_\tau(\omega) = (a,b)$, for every k, $1 \leqq k < n$, it is true that (i) or (ii) holds iff one of (i)', (ii)', (iii)' holds:

(i)' $\omega \notin F_k$

(ii)' $Y_\ell(\omega) \notin V^0_{Y_k(\omega)}$ for some ℓ, $k < \ell \leqq n$

(iii)' $Y_\ell(\omega) \in V^0_{Y_k(\omega)}$ for all ℓ, $k < \ell \leqq n$ and $(X_0 = b, C_{Y_k(\omega)}\ C_{ab}) = \emptyset$.

Since these three conditions involve only $\omega_0, \ldots, \omega_n$, the proposition will then be established.

Now fix ω with $\tau(\omega) = n$, $Y_\tau(\omega) = (a,b)$ and k with $1 \leqq k < n$. Suppose that $\omega \in F_k$. Write $(x,y) = Y_k(\omega)$. Then (i) or (ii) is equivalent to one of (i)', (ii)' or (iii)" where

(iii)" $\quad Y_\ell(\omega) \in V^0_{Y_k(\omega)}$ for all ℓ, $k < \ell \leqq n$ and $\theta_n \omega \notin C_{Y_k(\omega)}$.

This is clear if in (ii)' and (iii)" V is written instead of V^0. But the definition of V^0 ensures that

$$(X_0 = y, C^c_{xy}) \supset (X_0 = y,\ Y_m \notin V^0_{xy} \text{ for some } m \in N_+)$$

and this together with $V^0_{xy} \subset V_{xy}$ shows that the use of V^0 instead of V is legitimate.

The proof is completed by showing that (with the assumptions about ω, k made above), (iii)'$\Leftrightarrow$(iii)". Here $\Rightarrow$ is evident because $\theta_n \omega \in (X_0 = b, C_{ab})$. Conversely, if (iii)" holds, we have $(x,y)\mathrm{pr}(a,b)$ by the definition of V^0_{xy}, hence since $\mathcal{C}$ is transition reproducing, either $(X_0 = b, C_{xy} C_{ab}) = \emptyset$ or $C_{xy} \supset (X_0 = b, C_{ab})$. Were the last option possible we would have $\theta_k \omega \in C_{xy}$ because $\theta_n \omega \in (X_0 = b, C_{ab}) \subset C_{xy}$ and $Y_\ell(\omega) \in V^0_{xy} \subset V_{xy}$ for $k < \ell \leqq n$, and since $\omega \in F_k$ by assumption this would force $\tau(\omega) = k$ contradicting the assumption $\tau(\omega) = n$. Thus necessarily $(X_0 = b, C_{xy} C_{ab}) = \emptyset$, and we are back to (iii)'. □

<u>Remark</u> Taking $F_n = \Omega$ for all n, it follows in particular from the proof that if $\mathcal{C}$ is transition reproducing, then $\inf\{n : \theta_n \in C_{Y_n}\}$ is a random time satisfying (3.7) exactly. Thus there exists $F^0_n \in \mathcal{F}_n$ such that

$$(3.17) \qquad (\theta_k \notin C_{Y_k}, 1 \leqq k < n, \theta_n \in C_{Y_n}) = (F^0_n, \theta_n \in C_{Y_n}) \qquad (n \in N_+).$$

Introducing G as in (3.8), we have $\theta_m \omega \in C_{Y_m(\omega)}$ iff $\theta_{m-1} \omega \in G$, so (3.17) may be written

$$(\theta_k \in G^c,\ 0 \leqq k < n-1, \theta_{n-1} \in G) = (F^0_n, \theta_{n-1} \in G) \qquad (n \in N_+).$$

Also $\tau = \inf\{n \in N_+ : \theta_{n-1} \in G\}$ so (with a minor modification), τ is one of the penetration splitting times characterized by Pittenger [11], Theorem 4.4.

□

(3.18). Example The class of random times τ of the form (3.15) with $\mathcal{C}$ as in Example 3.13, is the class *BO* first introduced in [5]. Specializing further one finds that with $f: J^2 \to R$ some function, the times $\underline{\tau}$ and $\overline{\tau}$ given as the first, respectively the last time that the sequence $(f(Y_n), n \in N_+)$ attains its ultimate minimum, both belong to *BO*.

It was shown by Millar [9], that $\underline{\tau}, \overline{\tau}$ are CI-birth times for a wide class of Markov processes in continuous time. □

It is easy to give examples of CI-birth times τ for a Markov probability P such that τ is not P - a.s. equal to a member of *BTR*. But apart from the explicit description (3.15), the times in *BTR* possess a number of nice properties, as we shall now see.

Suppose that $\sigma > 0, \tau > 0$ satisfy (3.7) so that for $F_n, G_n \in \mathcal{F}_n$ and C_{ab}, D_{ab} coterminal, the identities

(3.19)
$$(\sigma = n, Y_\sigma = (a,b)) = (F_n, Y_n = (a,b), \theta_n \in C_{ab}) ,$$
$$(\tau = n, Y_\tau = (c,d)) = (G_n, Y_n = (c,d), \theta_n \in D_{cd})$$

hold exactly for $n \in N_+$, (a,b), $(c,d) \in J^2$. Now consider the random time $\rho = \sigma + \tau \circ \theta_\sigma$. Then

$$(\rho = n, Y_\sigma = (a,b), Y_\rho = (c,d)) = (H_{n,cd}, Y_n = (c,d), \theta_n \in C_{ab}D_{cd})$$

for suitable $H_{n,ab} \in \mathcal{F}_n$, and consequently for P Markov

(3.20) $$P(\theta_\rho \in \cdot \,|\, \mathcal{F}_\rho, Y_\sigma = (a,b), Y_\rho = (c,d)) = P^d(\cdot \,|\, C_{ab}D_{cd}) .$$

Thus ρ will not be a CI-birth time for P unless for all $(a,b),(c,d)$

with $P(Y_\sigma = (a,b), Y_\rho = (c,d)) > 0$, the inclusion $C_{ab} \supset D_{cd}$ holds P^d-a.s. However with the proof of Proposition 3.16 and Definition 3.14 in mind, the following result is not surprising and easily proved.

(3.21). <u>Proposition</u> If $\sigma > 0, \tau > 0$ belong to *BTR* and both satisfy (3.15) with the same family C of coterminal events, then also $\rho \in BTR$ and satisfies (3.15) with the collection C, where $\rho = \sigma + \tau \circ \theta_\sigma$. □

This result provides one explanation for the terminology 'transition reproducing' in Definition 3.12: let P be Markov and let q_{ab} be the common transition function for $P(\theta_\kappa \in \cdot \mid Y_\kappa = (a,b))$ where $\kappa = \sigma, \tau$ or ρ. Then not only are the $(q_{ab})_{(a,b)\in J^2}$ the transition functions for θ_ρ given Y_ρ, but they also arise by the two stage procedure consisting in firstly considering the post-σ process given Y_σ and its distribution, and then secondly, for this new process, evaluating the transitions for the post-τ process.

The main result of this section gives a characterization of homogeneous random sets with all points CI-birth times, in terms of collections of times in *BTR*.

By a standard definition a process $(U_n)_{n \in N_+}$ defined on (Ω, F) is <u>homogeneous</u> on N_+ if

(3.22) $$U_{n+k} = U_n \circ \theta_k \qquad (k,n \in N_+) .$$

(Usually (3.22) is required to hold only outside an exceptional set, but here we shall assume that it is an identity on all of Ω for all k,n. Note that we require homogeneity only for $k,n \geq 1$) .

A random subset M of N_+ is homogeneous if (U_n) is homogeneous, where $U_n = 1_{(n \in M)}$. In terms of M, (3.22) becomes

$$(M - k) \cap N_+ = M \circ \theta_k \qquad (k \in N_+) ,$$

writing $A - k = \{\ell - k : \ell \in A\}$ for A a subset of N_+. (That M is a random subset of N_+ means that for every ω there is defined a subset $M(\omega)$ of N_+ such that $\omega \to 1_{(n \in M)}(\omega)$ is $\mathcal{F}$-measurable for every $n \in N_+$).

In discrete time the structure of homogeneous processes is of course very simple: let $U = U_1$ and take $n = 1$ in (3.22) to find

$$U_n = U \circ \theta_{n-1} \qquad (n \in N_+) .$$

Conversely, given U measurable, (U_n) defined this way is homogeneous.

Similarly, if M is a homogeneous random set, then

(3.23) $$M = \{n \in N_+ : \theta_{n-1} \in G\}$$

where $G = (1 \in M)$. Conversely, given $G \in \mathcal{F}$, M defined by (3.23) is homogeneous.

Given a random subset M of N_+, let $\tau_1, \tau_2, \ldots$ denote the points of M in increasing order of magnitude. Thus $\tau_1 = \inf M$, $\tau_2 = \inf\{n > \tau_1 : n \in M\}$ etc. In particular

$$(\tau_1 = 1) = (1 \in M) .$$

In order that $\tau_i(\omega)$ be defined for all ω, we adopt the convention that if the cardinality $|M(\omega)|$ of $M(\omega)$ is $< i$, then $\tau_i(\omega) = \tau_{i+1}(\omega) = \ldots = \infty$. Thus

$$(\tau_1 = \infty) = (M = \emptyset) ,$$

and $\tau_i < \tau_{i+1}$ on $(\tau_i < \infty)$.

In this section, when writing $M = \{\tau_1, \tau_2, \ldots\}$, we shall always assume the τ_i to have been defined in this manner.

It is easy to see that $M = \{\tau_1, \tau_2, \ldots\}$ is homogeneous iff τ_i is of the form

(3.24) $\tau_i = \tau_{i-1} + \tau \circ \theta_{\tau_{i-1}}$ $(i \geqq 2)$

with $\tau = \tau_1$ satisfying

(3.25) $\tau = \inf\{n \in N_+ : \theta_{n-1} \in G\}$

for some $G \in \mathcal{F}$.

For arbitrary M, introduce $(Z_i)_{i \geqq 1}$, the process of transitions made at the timepoints in M, i.e. $Z_i = Y_{\tau_i}$. Then Z_i is defined only on $(\tau_i < \infty)$ and the process (Z_i) may have finite lifetime. Also denote by $\mathcal{G}_i$ the smallest σ-algebra containing $\mathcal{F}_{\tau_1}, \ldots, \mathcal{F}_{\tau_i}$. (Thus $\mathcal{G}_i \supset \mathcal{F}_{\tau_i}$, but in general the inclusion may be strict, also if M is homogeneous. If for every $j < i$, τ_j is $\mathcal{F}_{\tau_i}$-measurable, then $\mathcal{G}_i = \mathcal{F}_{\tau_i}$).

We shall show

(3.26). <u>Theorem</u> (a) Let P be Markov and let $M = \{\tau_1, \tau_2, \ldots\}$ be a homogeneous random set. Suppose each τ_i is a CI-birth time for P with respect to $\mathcal{G}_i$ such that the transitions for θ_{τ_i} given $\mathcal{G}_i$ do not depend on i, i.e. for all i, (a,b),

$$P(\theta_{\tau_i} \in \cdot \mid \mathcal{G}_i) = Q^b_{ab}$$

on $(\tau_i < \infty, Z_i = (a,b))$, where Q^b_{ab} is Markov with transitions q_{ab} and initial state b. Then there exists a collection $\mathcal{C} = (C_{ab})$ of coterminal events, transition reproducing for the transitions p of P, such that

$$M = \{n \in N_+ : \theta_{n-1} \in G'\} \qquad P\text{-a.s.}$$

where $G' = \bigcup_{(a,b)} (Y_1 = (a,b), \theta \in C_{ab})$. Also necessarily, for every i, $\mathcal{F}_{\tau_i} = \mathcal{G}_i$ P-a.s.

(b) Let $\mathcal{C} = (C_{ab})$ be a transition reproducing collection of coterminal events and let $M=\{\tau_1,\tau_2,\dots\}$ denote the homogeneous random set $\{n\in N_+ : \theta_{n-1}\in G\}$ where $G=\bigcup_{(a,b)}(Y_1=(a,b),\theta\in C_{ab})$. Then $\tau_i\in BTR$ and $F_{\tau_i}=G_i$ for all i and with respect to any Markov probability P, each τ_i is a CI-birth time for P with

$$P(\theta_{\tau_i}\in\cdot\,|F_{\tau_i})=P^b(\cdot\,|C_{ab})$$

on $(\tau_i<\infty, Z_i=(a,b))$, in particular the transitions for θ_{τ_i} given F_{τ_i} do not depend on i.

Proof (a) Since $F_{\tau_i}\subset G_i$, each τ_i is an ordinary CI-birth time for P, hence by Proposition 3.6 we can find $F_n^{(i)}\in F_n$ and $C_{ab}^{(i)}$ coterminal such that

$$(3.27)\qquad (\tau_i=n, Z_i=(a,b))=(F_n^{(i)},Y_n=(a,b),\theta_n\in C_{ab}^{(i)})\qquad P\text{-a.s.}$$

for every $i,n,(a,b)$. But then on $(Z_i=(a,b))$

$$P(\theta_{\tau_i}\in\cdot\,|F_{\tau_i})=P^b(\cdot\,|C_{ab}^{(i)}),$$

and since by assumption the right hand side must not depend on i we deduce that for every (a,b), $C_{ab}^{(i)}=C_{ab}^{(j)}$ P^b-a.s. for all i,j with $P(Z_i=(a,b))>0$, $P(Z_j=(a,b))>0$. It follows that (3.27) may be assumed to hold with $C_{ab}^{(i)}=C_{ab}$ coterminal not depending on i provided $(a,b)\in R:=\{(x,y):\sum_i P(Z_i=(x,y))>0\}$ and for $(a,b)\notin R$ we may and shall take $C_{ab}=\emptyset$.

With $C=(C_{ab})$ as just found, define G' as in the statement of the theorem and put $M'=\{n\in N_+ : \theta_{n-1}\in G'\}$. The definition of M and (3.27) (with $C_{ab}^{(i)}$ replaced by C_{ab}) show that $M\subset M'$ P-a.s. For the opposite inclusion, write $M=\{n\in N_+ : \theta_{n-1}\in G\}$, let $(a,b)\in R$ and find i,n and an atom A of F_n, $A\subset(Y_n=(a,b))$, such that $P(A,\tau_i=n)>0$. With $(\tau_i=n)_A$ the section of $(\tau_i=n)$ beyond A (cf. BDC, Definition 2.7), we find that on the F_{τ_i}-atom $(A,\tau_i=n)$

$$P(\theta_{\tau_i} \in \cdot \,|\, \mathcal{F}_{\tau_i}) = P^b(\cdot \,|\, (\tau_i = n)_A) \ ,$$

while by (3.27) this also equals $P^b(\cdot|C_{ab})$. Consequently $(\tau_i = n)_A = C_{ab}$ P^b-a.s. and since by the definition of M, $(\tau_i = n)_A \subset G_{ab}$ P^b-a.s. where $G_{ab} = \{\omega : \omega_0 = b, (a,b,\omega_1,\omega_2,\ldots) \in G\}$ it follows that $C_{ab} \subset G_{ab}$ P^b-a.s. and

$$\begin{aligned} P(n \in M' \setminus M) &= P(\theta_n \in C_{Y_n}, \theta_{n-1} \notin G) \\ &= \sum_{(a,b)\in R} P(Y_n = (a,b), \theta_n \in C_{ab}, \theta_{n-1} \notin G) \\ &= \sum_{(a,b)\in R} P(Y_n = (a,b)) P^b(C_{ab} \setminus G_{ab}) \\ &= 0 \ . \end{aligned}$$

So we have shown $M = M'$ P-a.s. Writing $M' = \{\tau_1', \tau_2', \ldots\}$, it is then clear that $\tau_i' = \tau_i, \mathcal{F}_{\tau_i} = \mathcal{F}_{\tau_i'}, \mathcal{G}_i' = \mathcal{G}_i$ P-a.s. for every i. We complete the proof by showing that $\mathcal{C}$ is transition reproducing for p, and that $\mathcal{F}_{\tau_i} = \mathcal{G}_i$ P-a.s. But if $\mathcal{C}$ is transition reproducing it is easy to see that τ_{i-1}' is measurable with respect to the P-completion of $\mathcal{F}_{\tau_i'}$ or $\mathcal{F}_{\tau_i}$, (an argument for a similar assertion is provided in the proof of (b) below), hence so is τ_{i-1} and $\mathcal{F}_{\tau_i} = \mathcal{G}_i$ P-a.s. follows.

The argument that $\mathcal{C}$ is transition reproducing is more lengthy and uses that τ_i is CI-birth with respect to $\mathcal{G}_i$ rather than just $\mathcal{F}_{\tau_i}$.

Let $i < j$ and $(a,b),(c,d)$ be given such that $P(Z_i = (a,b), Z_j = (c,d)) > 0$. Then find $k,n \in N_+$, an $\mathcal{F}_k$-atom $A \subset (Y_k = (a,b))$ and an $\mathcal{F}_n$-atom $B \subset (X_0 = b, Y_n = (c,d))$ with $PD > 0$, where $D = (A, \tau_i = k, \theta_k \in B, \tau_j = k+n)$. Since τ_i is $\mathcal{G}_j$-measurable, $D \in \mathcal{G}_j$ and

$$(3.28) \qquad P(\theta_{\tau_j} \in \cdot \,|\, D) = Q^d_{cd}$$

because τ_j is CI-birth relative to $\mathcal{G}_j$. Here as below we write $Q^y_{xy} = P^y(\cdot|C_{xy})$. The conditional probability may however also be found using that

τ_i is CI-birth. Using the homogeneity of M, which implies in particular that $\tau_j = \tau_i + \tau_{j-i} \circ \theta_{\tau_i}$ on $(\tau_i < \infty)$, we find

$$P(\theta_{\tau_i} \in B, \tau_j = k+n, \theta_{\tau_j} \in \cdot \,|A, \tau_i = k)$$

$$= P(\theta_{\tau_i} \in B, \tau_{j-i} \circ \theta_{\tau_i} = n, \theta_{\tau_{j-i}} \circ \theta_{\tau_i} \in \cdot \,|A, \tau_i = k)$$

$$= Q^b_{ab}(B, \tau_{j-i} = n, \theta_{\tau_{j-i}} \in \cdot)$$

and from this it follows directly that

$$P(\theta_{\tau_j} \in \cdot \,|D) = Q^b_{ab}(\theta_n \in \cdot \,|B, \tau_{j-i} = n) .$$

Using the Markov property of Q^b_{ab} at time n shows the right hand side to equal $Q^d_{ab}(\cdot \,|(\tau_{j-i} = n)_B)$, and comparing with (3.28) we therefore find

$$Q^d_{cd} = Q^d_{ab}(\cdot \,|(\tau_{j-i} = n)_B) .$$

Now not only is $Q^b_{ab} = P^b(\cdot \,|C_{ab})$, $Q^d_{cd} = P^d(\cdot \,|C_{cd})$, but we also have $Q^d_{ab} = P^d(\cdot \,|C_{ab})$. Thus

$$P^d(\cdot \,|C_{cd}) = P^d(\cdot \,|C_{ab}, (\tau_{j-i} = n)_B) ,$$

i.e. $C_{cd} = (C_{ab}, (\tau_{j-i} = n)_B)$ P^d-a.s., in particular $C_{cd} \subset C_{ab}$ P^d-a.s.

Summarizing we have shown that if $i < j$, then

(3.29) $$P(Z_i = (a,b), Z_j = (c,d)) > 0 \Rightarrow C_{cd} \subset C_{ab} \quad P^d\text{-a.s.}$$

We must show (see Definition 3.12) that $(a,b)\mathrm{pr}(p)(c,d)$ implies $C_{cd} \subset C_{ab}$ P^d-a.s. or $P^d(C_{ab}\, C_{cd}) = 0$. By assumption n may be found with

(3.30) $$P^b(C_{ab}, Y_n = (c,d)) = P^b(C_{ab}) Q^b_{ab}(Y_n = (c,d)) > 0 ,$$

in particular $C_{ab} \neq \emptyset$ and $(a,b) \in R$. If $(c,d) \notin R$, $C_{cd} = \emptyset$ and there is nothing to prove, so we may also assume $(c,d) \in R$.

Now find i with $P(Z_i = (a,b)) > 0$, and consider the two possible cases: (i) $\pi > 0$, (ii) $\pi = 0$, where

(3.31) $$\pi = P(Z_i = (a,b), Y_{\tau_i+n} = (c,d), \theta_{\tau_i+n} \in C_{cd})$$

In case (i), the identification between M and M' shows that P-a.s.

$$(Z_i = (a,b), Y_{\tau_i+n} = (c,d), \theta_{\tau_i+n} \in C_{cd}) \subset \bigcup_{j>i} (Z_i = (a,b), Z_j = (c,d)),$$

and then $\pi > 0$ and (3.29) gives $C_{cd} \subset C_{ab}$ P^d-a.s. as desired.

In case (ii), because τ_i is CI-birth we have

$$0 = P(Z_i = (a,b)) Q^b_{ab}(Y_n = (c,d), \theta_n \in C_{cd})$$

$$= P(Z_i = (a,b)) Q^b_{ab}(Y_n = (c,d)) Q^d_{ab}(C_{cd}) .$$

The first factor is > 0 by assumption, the second is > 0 by (3.30), hence $Q^d_{ab}(C_{cd}) = 0$, i.e. $P^d(C_{cd} | C_{ab}) = 0$ or $P^d(C_{ab} C_{cd}) = 0$ and the proof of (a) is complete.

(b) Since τ_1 satisfies (3.15) with $F_n = \Omega$, $\tau_1 \in BTR$. Now $\tau_i = \tau_{i-1} + \tau_1 \circ \theta_{\tau_{i-1}}$, so using Proposition 3.21 and proceeding by induction it is clear that each $\tau_i \in BTR$ and that (3.15) holds with (C_{ab}) the given collection $\mathcal{C}$. That each τ_i is CI-birth for any P as stated, follows then from Proposition 3.16.

To show that $\mathcal{F}_{\tau_i} = \mathcal{G}_i$ we must check that τ_j is $\mathcal{F}_{\tau_i}$-measurable for $j < i$. But on the $\mathcal{F}_{\tau_i}$-atom $A = (X_0 = x_0, \ldots, X_n = x_n, \tau_i = n)$, only those time-points $k < n$ can belong to M for which $C_{x_{k-1}x_k} \supset (X_0 = x_n, C_{x_{n-1}x_n})$, and then among these exactly those succeed for which $(x_{\ell-1}, x_\ell) \in V_{x_{k-1}x_k}$ for $k < \ell < n$. Thus $M \cap \{1, \ldots, n-1\}$ and hence also τ_j is constant on A. □

The starting point in Theorem 3.26 is a homogeneous random set M, i.e. a o

set with a specific algebraic structure. It is then shown essentially that the points in M are CI-birth times iff the set G characterizing M (cf. (3.23)) has a special form which is described explicitly algebraically.

Alternatively one might have taken an arbitrary random set $M=\{\tau_1,\tau_2,\ldots\}$, assuming as in Theorem 3.26 (a) that each τ_i is CI-birth for P. This however would allow for the possibility $M=\{\tau\}$ with τ an arbitrary CI-birth time for P, and as maintained above, no explicit algebraic description of such τ appears possible.

As we shall presently see, Theorem 3.26 has connections to results in [3], [11].

Suppose that the conditions in Theorem 3.26 (b) are satisfied and let P be Markov. Because

$$Z_i = Z_1 \circ \theta_{\tau_{i-1}}$$

on $(\tau_i < \infty)$, it is easy to see that the process $Z=(Z_i)_{i\in N_+}$ (which may have finite lifetime) is Markov with transition function

$$q((a,b),(c,d)) = P^b(Z_1 = (c,d) \mid C_{ab}) .$$

Thus the Markov chain Z is obtained from a <u>time change</u> in the original chain P. Time changes of this type were discussed by Pittenger [11] - if above $C_{ab}=C_b$ depends on b only, then $(X_{\tau_i})_{i\in N_+}$ becomes a Markov chain and we have an example fitting exactly into Pittenger's theory.

More generally Pittenger considered time changes $M=\{\tau_1,\tau_2,\ldots\}$ (here we take Z rather than (X_{τ_i}) to be the time changed chain) such that Z is Markov and each τ_i is a conditional independence time for P (but not necessarily CI-birth). One essential algebraic condition on such M is

(3.32) $\tau_i = \tau_{i-1} + \sigma \circ \theta_{\tau_{i-1}}$,

see [11], (3.5b). This leads of course to much more general time changes than those treated in Theorem 3.26, even if all τ_i are required to be CI-birth. For an example, let C be transition reproducing, let σ be given by (3.15), put $\tau_1 = \sigma$, and define τ_i by (3.32). Then the τ_i satisfy (3.5a,b,c) of [11] and therefore e.g. Z is Markov with respect to any P . (The set Γ in [11] is the set G from (3.8)). However, because of the $\mathbf{F}_n$ appearing in (3.15), $M = \{\tau_1, \tau_2, ...\}$ will not in general be homogeneous. The time change corresponds to taking only a subset of the full homogeneous set $\{n \geqq 1 : \theta_{n-1} \in G\}$, G as in (3.8).

We shall also discuss the relation of the preceding to the time change results of Glover [3]. Translated into discrete time and the setup used here, his Theorem 1.5 states the following: let $A = (A_n)_{n \in N}$ be a raw additive functional (RAF) defined on Ω with $A_0 = 0$ and suppose that for all $n \in N_+$, $\Delta A_n := A_n - A_{n-1}$ is either 0 or 1. Define for $i \in N_+$

$$\tau_i = \inf\{n \in N_+ : A_n = i\} ,$$

and suppose finally (condition (a) of Theorem 1.5, [3]) that for every i ,

(3.33) $A_n 1_{(n \leqq \tau_i)}$

is F_{τ_i}-measurable for $n \in N_+$. Then with respect to any Markov probability P , each τ_i is a conditional independence time and $Z = (Z_i)$, $Z_i = Y_{\tau_i}$, is a Markov chain.

By definition, A is a RAF if $A_{k+n} = A_k + A_n \circ \theta_k$ exactly for all $k,n \in N$. If also $\Delta A_n = 0$ or 1 , necessarily

$$A_n = \sum_{k=0}^{n-1} 1_G \circ \theta_k ,$$

where $G = (A_1 = 1) = (\Delta A_1 = 1)$. But then $(\Delta A_n = 1) = (\theta_{n-1} \in G)$, and it is clear that the time change determined by A is the same as that induced by the homogeneous random set $M = \{n : \theta_{n-1} \in G\}$, and that all time changes from homogeneous M arise from RAF's in this manner.

The extra condition (3.33) is critical when establishing the Markov property for Z. A little path algebra shows it to be equivalent to the following condition, expressed in terms of G: for every $n \in N_+$ there exists $F_n \in \mathcal{F}_n$ such that

$$(G, \theta_{n-1} \in G) = (F_n, \theta_{n-1} \in G) \quad ,$$

which except for a small modification is condition (b) of Theorem 4.4, [11]. So the time changes in [3] appear as special cases of those in [11].

Summarizing the above discussion, we have encountered three types of time changes: those in Theorem 3.26 involve homogeneous random sets with all τ_i CI-birth times; those in [3] are induced by homogeneous random sets with all τ_i conditional independence times; and finally those in [11] arise from certain subsets of homogeneous random sets with all τ_i conditional independence times.

Throughout this section we have discussed CI-birth times τ which by definition obey a strong Markov property involving conditional independence of the pre-τ and post-τ processes given Y_τ. Other authors have studied birth times where this conditional independence occurs when conditioning not only on Y_τ but also on some auxiliary $\mathcal{F}_\tau$-measurable variable.

Thus, in [8] Millar has introduced (for processes in continuous time) randomized coterminal times and shown that if τ is such a time, then conditionally on $\mathcal{F}_\tau$ the post-τ process is Markov with a transition function depending on X_τ and a $\mathcal{F}_\tau$-measurable variable U. Therefore, unless U is a function of $(X_{\tau-}, X_\tau)$, τ will not be a CI-birth time.

Millar's definition of randomized coterminal times is quite complicated. Simpler examples of birth times with a conditional independence property involving an extra variable can be found in Getoor [1].

From (3.19) it follows that with σ, τ given by (3.19), the random time $\rho = \sigma + \tau \circ \theta_\sigma$ will be a birth time with the kind of conditional independence discussed by Millar and Getoor, provided Y_σ is $\mathcal{F}_\rho$-measurable. Finally it may be remarked that Millar points out that the class of randomized coterminal times is closed under the addition $(\sigma,\tau) \to \sigma + \tau \circ \theta_\sigma$.

4. CI-DEATH TIMES AND THE CLASS *DTR*

This section contains the definitions and results which are the death time analogues of those in Section 3.

(4.1). Definition A random time τ is a death time with conditional independence (in short a CI-death time) for the Markov probability P if it is a conditional independence time for P and if conditionally on Y_τ within $(0<\tau<\infty)$, the pre-τ process is Markov with a stationary transition function (depending possibly on Y_τ).

According to a definition in Section 5 of BDC, a random time τ is a regular death time for a Markov probability P if the pre-τ process is Markov (q) for some (substochastic) transition function q with the pre-τ and post-τ processes conditionally independent given $0<\tau<\infty$ and $X_{\tau-1}$.

Any regular death time for P is a CI-death time. This statement is not as transparent as the similar one for birth times, so we shall produce an argument and at the same time introduce some notation.

Recall that if Q is a Markov probability on Ω_Δ with initial measure ν and transition function q, such that (X_n) with positive probability, has finite lifetime, then the process reversed from the lifetime defined by the reversal transformation $R:\Omega_\Delta \to \Omega_\Delta$ given by

$$X_n \circ R = \begin{cases} X_{\zeta-1-n} & \text{if } n<\zeta<\infty \\ \Delta & \text{otherwise} \end{cases}$$

is again Markov with a substochastic transition function $\hat{q}$ on J. Here

$$(4.2) \qquad \hat{q}(x,y) = \xi(y)q(y,x)\xi^{-1}(x)$$

for $x,y \in \{0<\xi<\infty\}$ with $\xi(z) = \sum_{n=0}^{\infty} Q(X_n = z)$.

Therefore, if τ is a regular death time for the Markov probability P on Ω, the reversed pre-τ process $R \circ K_\tau$ is Markov with stationary transitions, hence so is $R \circ K_\tau$ given $\tau < \infty$ and for this latter process conditioning on $X_{\tau-1}$ simply amounts to freezing the initial state, so that $R \circ K_\tau$, and therefore also $R \circ R \circ K_\tau = K_\tau$, conditionally on $X_{\tau-1}$ is Markov with stationary transitions. By the conditional independence property shared by all regular death times it now follows that any regular death time is a CI-death time.

The main result, Theorem 5.2, in Section 5 of BDC states that a random time τ is a regular death time for the Markov probability P^x iff τ is P^x-equivalent to a random time in the class $\mathcal{D}$. A remark in BDC shows how this result may be generalized to Markov probabilities with a non-degenerate start. Since we shall work with this generalization here, we shall redefine $\mathcal{D}$ and restate the regular death time theorem.

If $H \subset J$, $V \subset J^2$, let τ_{HV} denote the modified terminal time

$$\tau_{HV} = \begin{cases} 0 & \text{if } X_0 \in H \\ \inf\{n \in N_+ : Y_n \in V\} & \text{otherwise} \end{cases} .$$

The class $\mathcal{D}$ is now defined to comprise all random times τ of the form

$$\tau = \sup\{n : 1 \leqq n \leqq \tau_{HV},\ \theta_{n-1} \in F\} \tag{4.3}$$

for some $H \subset J$, $V \subset J^2$, $F \in \mathcal{F}$. (By the usual convention $\tau = 0$ if the set in brackets is empty; in particular $\tau = 0$ on $(X_0 \in H)$).

Then the following is true: τ is a regular death time for the Markov probability P iff τ is P-equivalent to a random time in $\mathcal{D}$.

As pointed out in BDC, the results on regular birth times and the corresponding results on death times are duals. This duality is prevalent also in the theory of CI-birth times and CI-death times, so the death time results will be presented in the same order as their analogues in Section 3. Also, to keep down the length of the paper, we shall not give proofs.

In the death time theory the counterpart of a coterminal event is a sequence $T = (T^n, n \in N_+)$ of terminal events

(4.4) $\quad T^n = (X_0 \in H, Y_k \in V,\ 1 \leqq k < n)$

where $H \subset J$, $V \subset J^2$ are sets not depending on n. Of course the invariant part of the coterminal event is matched by the initial part $(X_0 \in H)$ of each T^n. Notice that $T^n \in \mathcal{F}_{n-1}$.

The notation from (4.4) will be used below with subscripts ab where $(a,b) \in J^2$. Notice that there is really a switch in notation from (4.3) to (4.4): (4.3) forbids transitions in V prior to τ while (4.4) demands that all pre-n transitions belong to V. Of course (4.3) is modelled upon the definition of $\mathcal{D}$ from BDC, but in the remainder of the section we shall use the notation (4.4).

The first two results are the duals of Propositions 3.6 and 3.9.

(4.5). <u>Proposition</u> A random time τ is a CI-death time for P if and only if there exists $F \in \mathcal{F}$ and for every $(a,b) \in J^2$ subsets $H_{ab} \subset J$, $V_{ab} \subset J^2$ such that

(4.6) $\quad (\tau = n, Y_\tau = (a,b)) = (T^n_{ab}, Y_n = (a,b), \theta_{n-1} \in F) \qquad P\text{-a.s.}$

for $n \in N_+$. □

(4.7). <u>Proposition</u> A random time τ is a CI-death time for the Markov probability P if and only if for every $(a,b) \in J^2$ the random time τ_{ab} defined by

(4.8) $$\tau_{ab} = \begin{cases} \tau & \text{on } (Y_\tau = (a,b)) \\ 0 & \text{otherwise} \end{cases}$$

is a regular death time for P . □

Suppose that τ is a CI-death time for P so that (4.6) holds. Consider (a,b) with $P(Y_\tau = (a,b)) > 0$, introduce

$$J_{ab} = \{x \in J : \sum_{n=0}^{\infty} P(X_n = x, n < \tau < \infty, Y_\tau = (a,b)) > 0\} ,$$

the state space for K_τ given $Y_\tau = (a,b)$. Straightforward calculations show that K_τ given $Y_\tau = (a,b)$ is Markov with transitions

$$q_{ab}(x,y) = p(x,y) 1_{V_{ab}}(x,y) \frac{g_{ab}(y)}{g_{ab}(x)} \qquad (x,y \in J_{ab})$$

where

$$g_{ab}(z) = \sum_{n=1}^{\infty} P^z(Y_k \in V_{ab}, 1 \leqq k < n, X_{n-1} = a) .$$

However, the symmetry between the birth time and death time theories is brought out more clearly by considering K_τ reversed, as was done in Theorem 2 of Jacobsen [6], and leads in a natural way to the death time analogue of Theorem 3.26.

Introduce $\xi(z) = \sum_{n=0}^{\infty} P(X_n = z)$ and the transition function in <u>natural P-duality</u> to p (cf. [6]),

$$\hat{p}(x,y) = \xi(y) p(y,x) \xi^{-1}(x) .$$

Without loss of generality we may assume $\xi > 0$, and then $\hat{p}(x,y)$ is defined if $\xi(x), \xi(y) < \infty$. For convenience we now assume $\xi < \infty$ (so the P-chain is transient), although it is enough that $\xi < \infty$ on the state space $\cup J_{ab}$ for K_τ .

Corresponding to $(T^n_{ab})_{n \in N_+}$ there is a natural dual coterminal event $\hat{C}_{ab}$

which is a subset of the space $\Omega_0 = (\zeta < \infty) \subset \Omega_\Delta$ of paths with finite lifetime, namely

$$(4.9) \qquad \hat{C}_{ab} = (Y_k \in \hat{V}_{ab}, 1 \leq k < \zeta < \infty, X_{\zeta-1} \in H_{ab})$$

$$= (Y_k \in \hat{V}_{ab} \cup (H_{ab} \times \{\Delta\}) \cup \{(\Delta,\Delta)\}, k \in N_+) ,$$

where of course

$$\hat{V}_{ab} = \{(x,y) \in J^2 : (y,x) \in V_{ab}\} .$$

(About notation in the sequel: symbols with a ^ refer to objects pertaining to the path space Ω_0) .

(4.10). Proposition With $P, \hat{P}, \hat{C}_{ab}$ as above and τ satisfying (4.6)

$$(4.11) \qquad P(R \circ K_\tau \in \cdot \,|\, Y_\tau = (a,b)) = \hat{P}^a(\cdot \,|\, \hat{C}_{ab}) .$$

Remark This result states that the distribution of K_τ reversed given Y_τ = (a,b) is the same as that of $\theta_{\hat{\tau}}$ given $Y_{\hat{\tau}} = (b,a)$ for a process $\hat{P}$ in natural duality to P and with $\hat{\tau}$ a CI-birth time for $\hat{P}$ satisfying (3.7) $\hat{P}$ - a.s. with the C_{ab} there replaced by the dual $\hat{C}_{ab}$ of (T^n_{ab}) .

Proof Both chains under consideration start in a , so to prove (4.11) it remains to identify their (substochastic) transition functions which by (4.2) and (3.4) are

$$e(y) q_{ab}(y,x) e^{-1}(x) \quad \text{and} \quad 1_{\hat{V}_{ab}}(x,y) \hat{p}(x,y) \frac{\hat{g}(y)}{\hat{g}(x)}$$

respectively, where

$$e(z) = \sum_{n=0}^{\infty} P(X_n = z, n < \tau < \infty \,|\, Y_\tau = (a,b)) ,$$

$$\hat{g}(z) = \hat{P}^z(\hat{C}_{ab}) .$$

It is essential to note, and fairly obvious to verify, that the two chains have J_{ab} as state space. Inserting the expressions for q_{ab} and $\hat{p}$ above, it is seen that we need only show that the functions eg^{-1} and $\xi\hat{g}$ are proportional on J_{ab}. (We write $g = g_{ab}$). Now

$$e(z) \propto \sum_{n=0}^{\infty} P(X_n = z, n < \tau < \infty, Y_\tau = (a,b))$$

$$= \sum_{n=0}^{\infty} \sum_{k>n} P(X_n = z, \tau = k, Y_k = (a,b))$$

$$= \sum_{n=0}^{\infty} \sum_{k>n} P(X_n = z, T^k_{ab}, \theta_{k-1} \in F, Y_k = (a,b))$$

and using the Markov property at times n and $k-1$ (recall that $T^k_{ab} \in \mathcal{F}_{k-1}$) this reduces to

$$g(z)P^a(X_1 = b, F) \sum_{n=0}^{\infty} P(X_n = z, T^{n+1}_{ab}) .$$

Consequently $e(z)g^{-1}(z)$ is proportional to

$$\sum_{n=0}^{\infty} P(X_n = z, T^{n+1}_{ab}) . \tag{4.12}$$

On the other hand $\hat{P}^z(\zeta < \infty) = 1$ and

$$\hat{g}(z) = \sum_{n=1}^{\infty} \hat{P}^z(Y_k \in \hat{V}_{ab}, 1 \leqq k < n, X_{n-1} \in H_{ab}, \zeta = n)$$

$$= \sum_{n=1}^{\infty} \sum_{u \in H_{ab}} \hat{P}^z(Y_k \in \hat{V}_{ab}, 1 \leqq k < n, X_{n-1} = u)\hat{p}(u,\Delta) .$$

By duality

$$\hat{P}^z(Y_k \in \hat{V}_{ab}, 1 \leqq k < n, X_{n-1} = u) = \xi(u)P^u(Y_k \in V_{ab}, 1 \leqq k < n, X_{n-1} = z)\xi^{-1}(z)$$

and since with μ the initial distribution for $P = P^\mu$, $\xi(u)\hat{p}(u,\Delta) = \mu(u)$, it follows that

$$\xi(z)\hat{g}(z) = \sum_{n=1}^{\infty} P(X_{n-1} = z, T^n_{ab})$$

which is (4.12) exactly, so the proof of (4.11) is complete. □

With the preceding discussion as motivation, we shall without further comments state the analogues of the definitions and results from Definition 3.11 onwards.

Consider a collection $\mathcal{T} = (T^n_{ab}, (a,b) \in J^2, n \in N_+$, of sequences of terminal events and define $\hat{C}_{ab}$ as in (4.9) above.

(4.13). Definition $\mathcal{T}$ is a transition reproducing collection of sequences of terminal events if $(a,b)\hat{pr}(c,d)$ implies that either $\hat{C}_{ab} \supset (X_0 = c, \hat{C}_{cd})$ or $(X_0 = c, \hat{C}_{ab}\hat{C}_{cd}) = \emptyset$. Here $(a,b)\hat{pr}(c,d)$ means that there exists $\omega \in (X_0 = a, \hat{C}_{ab})$ and $n \in N_+$ such that $\hat{Y}_n(\omega) = (d,c)$. □

Let $\hat{p}$ be a substochastic transition function on J such that $\hat{P}^x(\zeta<\infty)=1$ for all $x \in J$.

(4.14). Definition $\mathcal{T}$ is transition reproducing for $\hat{p}$ if $(a,b)\hat{pr}(\hat{p})(c,d)$ implies that either $\hat{C}_{ab} \supset \hat{C}_{cd}$ $\hat{P}^c$ - a.s. or $\hat{C}_{ab}\hat{C}_{cd} = \emptyset$ $\hat{P}^c$ - a.s. Here $(a,b)\hat{pr}(\hat{p})(c,d)$ means that

$$\sum_{n=1}^{\infty} \hat{P}^a(\hat{C}_{ab}, Y_n = (d,c)) > 0 \,. \qquad \square$$

(4.15). Example Let (T'^n_{ab}) be an arbitrary collection of sequences of terminal events and define

$$T^n_{ab} = (Y_k \in V^*_{ab}, 1 \leqq k < n, T'^n_{ab})$$

where

$$V^*_{ab} = \{(x,y) : \hat{C}'_{ab} \supset \hat{C}'_{xy}\} \,.$$

Then $\mathcal{T} = (T^n_{ab})$ is a transition reproducing collection of sequences of terminal events. □

(4.16) Definition Let $\mathcal{DTR}$ denote the class of random times τ of the form

(4.17) $\tau(\omega) = \sup\{n \in N_+ : \omega \in T^n_{Y_n(\omega)}, \theta_{n-1}\omega \in F\}$,

where $F \in \mathcal{F}$ and $\mathcal{T} = (T^n_{ab}, (a,b) \in J^2, n \in N_+)$ is a transition reproducing collection of sequences of terminal events. □

(4.18). Proposition Suppose τ belongs to the class *DTR*. Then τ is a CI-death time for any Markov probability P .

If τ is given by (4.17) and P is transient, then

$$P(R \circ K_\tau \in \cdot \,|\, Y_\tau = (a,b)) = \hat{P}^a(\cdot \,|\, \hat{C}_{ab}) ,$$

where $\hat{p}$ is the transition function in natural P-duality to p . □

(4.19). Example The class of random times τ of the form (4.17) with $\mathcal{T}$ as in Example 4.15, is the class *DO* first introduced in [5]. The times $\underline{\tau}$ and $\overline{\tau}$ from Example 3.18 both belong to *DO*. □

Let now $\hat{G}$ be a measurable subset of Ω_0 and consider the random subset of N given by

(4.20) $L = \{n \in N : R \circ K_{n+1} \in \hat{G}\}$.

Sets of this form appear as the duals of homogeneous random sets. Of special interest to us is the situation where (T^n_{ab}) is a collection of sequences of terminal events, $\hat{C}_{ab}$ is as in (4.9) and

(4.21) $\hat{G} = \bigcup_{(a,b)\in J^2} (Y_1 = (b,a), \theta \in \hat{C}_{ab})$.

Then $n \in L(\omega)$ iff $\omega \in T^n_{Y_n(\omega)}$.

With $\hat{G}$ given by (4.21), $\hat{G} \subset (\zeta > 1)$ and from now on, when considering L of the form (4.20), we shall always assume the $\hat{G}$ there to be a subset of $(\zeta > 1)$. Then $0 \in L$ is impossible.

With this assumption in force, given L of the form (4.20), define random times $(\sigma_i, i \in N_+)$ on Ω by letting $\sigma_i = \infty$ on $(|L| = \infty)$, $\sigma_i = 0$ on $(|L| < i)$ and by writing $L = \{\sigma_\ell, \ldots, \sigma_1\}$ on $(|L| = \ell)$ with $\sigma_\ell < \ldots < \sigma_1$. Also define $W_i = (X_{\sigma_i}, X_{\sigma_i - 1})$ on $A_i := (0 < \sigma_i < \infty)$ and let $\mathcal{H}_i$ denote the σ-algebra of subsets of A_i generated by $\theta_{\sigma_i - 1}$ and $\sigma_j - \sigma_i$ for $j < i$.

As usual, in the theorem below, $\hat{p}$ denotes the transition function in natural P-duality to p.

(4.22) <u>Theorem</u> (a) Let P be Markov and transient and let L be a random set of the form (4.20) with $\hat{G} \subset (\zeta > 1)$. Suppose each σ_i is a CI-death time for P with respect to $\mathcal{H}_i$ such that the transitions for $R \circ K_{\sigma_i}$ given $\mathcal{H}_i$ do not depend on i, i.e. for all i, (a,b),

$$P(R \circ K_{\sigma_i} \in \cdot \mid A_i, \mathcal{H}_i) = \hat{Q}^a_{ab}$$

on $(A_i, W_i = (b,a))$, where $\hat{Q}^a_{ab}$ is Markov on Ω_0 with transitions $\hat{q}_{ab}$ not depending on i and initial state a. Then there exists a collection $\mathcal{T} = (T^n_{ab})$ of sequences of terminal events, transition reproducing with respect to $\hat{p}$ such that

$$L = \{n \in N_+ : R \circ K_{n+1} \in \hat{G}'\} \qquad P\text{-a.s. on } (|L| < \infty),$$

where $\hat{G}' = \bigcup_{(a,b)} (Y_1 = (b,a), \theta \in \hat{C}_{ab})$.

(b) Let $\mathcal{T} = (T^n_{ab})$ be a transition reproducing collection of sequences of terminal events and let L denote the random set given by (4.20) with $\hat{G}$ as in (4.21). Then $\sigma_i \in \mathit{DTR}$ for all i, in particular σ_i is a CI-death time for any Markov probability P, and if P is transient,

$$P(R \circ K_{\sigma_i} \in \cdot \mid A_i, \theta_{\sigma_i - 1}) = \hat{P}^a(\cdot \mid \hat{C}_{ab})$$

on $(A_i, W_i = (b,a))$. □

Suppose the conditions in (b) are satisfied. Rather than considering the full set L (in which case nothing interesting is said when working on $(|L| = \infty)$) one may consider the part L^* of L preceding a given cooptional time. Defining $(\sigma_i^*, i \in N_+)$ from L^* as the σ_i were defined from L, the conclusions in (b) remain valid with σ_i replaced by σ_i^*.

As a final remark, note that under the assumptions in (b), $(W_i, i \in N_+)$ is a Markov chain with respect to any Markov probability P and if P is transient the transitions are given by

$$P(W_{i+1} = (d,c) \mid W_i = (b,a)) = \hat{P}^a(Y_{\hat{\tau}} = (d,c) \mid \hat{C}_{ab})$$

where $\hat{\tau}$ is defined on Ω_0 by

(4.23) $\quad \hat{\tau} = \inf\{n \in N_+ : \theta_{n-1} \in \hat{G}\}$.

5. PATH DECOMPOSITIONS WITH MARKOVIAN EXCURSIONS

We shall briefly discuss what kind of path decompositions obtain when the results from Sections 3 and 4 may be combined.

Consider a homogeneous random set $M=\{n\in N_+ : \theta_{n-1}\in G\}$ with G given by (3.8) and $\mathcal{C}=(C_{ab})$ a transition reproducing collection of coterminal events.

With $M=\{\tau_1,\tau_2,\ldots\}$ as in Section 3, we saw in Theorem 3.26 (b) that each τ_i is a CI-birth time for any Markov probability P. Writing $\tau=\tau_1$, the post τ-process splits into the Markov chain $(Z_i)=(Y_{\tau_i})$ and the sequence $(e_i, i\in N_+)$ of skew excursions where

$$e_i=(X_{\tau_i},X_{\tau_i+1},\ldots,X_{\tau_{i+1}-1}) .$$

There may be finitely or infinitely many excursions according as (Z_i) has finite lifetime or not. All excursions have finite lengths except the last one in the case where there are only finitely many excursions.

It follows immediately from the results in Section 3, that given $(Z_i, i\in N_+)$, (which includes conditioning on the lifetime of the Z_i-chain) the excursions are, with respect to any Markovian P, conditionally independent with the distribution of e_i not depending on i but only on how e_i is conditioned to start and end. More specifically, with $Q^b_{ab}=P^b(\cdot\,|\,C_{ab})$,

$$P(e_i\in\cdot\,|\,Z_i=(a,b),Z_{i+1}=(c,d))=Q^b_{ab}(K_{\tau_1}\in\cdot\,|\,Z_1=(c,d)) ,$$

$$P(e_i\in\cdot\,|\,Z_i=(a,b),\ i \text{ is the lifetime of } (Z_j)) = Q^b_{ab}(\cdot\,|\,\tau_1=\infty) .$$

In general the conditional excursions will of course not be Markov. However, if $\tau=\tau_1$ is a CI-death time for each Q^b_{ab}, then certainly all the finite excursions are Markov, and if in addition $(\tau_1=\infty)$ is Q^b_{ab}-a.s. equal to a coterminal event, then also the last infinite excursion will be

Markov.

Without aiming for complete generality, we shall discuss a simple example involving Markovian excursions.

Suppose given a transitive relation $\succ$ on J^2 and invariant events $(C^*_{ab,\infty}, (a,b) \in J^2)$ compatible with $\succ$ in the sense that

$$(a,b) \succ (c,d) \Rightarrow C^*_{ab,\infty} \supset C^*_{cd,\infty} .$$

Defining

$$(5.1) \qquad C_{ab} = ((a,b) \succ Y_k, k \in N_+, C^*_{ab,\infty}) ,$$

C_{ab} is coterminal and $\mathcal{C} = (C_{ab})$ is transition reproducing. (This setup provides an alternative description of the class *BO* from Example 3.18, see Proposition 3.31 in [5]).

With this choice of $\mathcal{C}$ one finds

$$(5.2) \qquad (\tau = n, Y_\tau = (a,b)) = [\bigcap_{k=1}^{n-1} \bigcup_{\ell=k+1}^{n} (Y_k \not\succ Y_\ell)] \cap (Y_n = (a,b), \theta_n \in C_{ab})$$

writing $(x,y) \not\succ (u,v)$ if it is not true that $(x,y) \succ (u,v)$.

Suppose now in addition that the relation $\not\succ$ is also transitive. We claim that then

$$(5.3) \qquad (\tau = n, Y_\tau = (a,b)) = (Y_k \not\succ (a,b), 1 \leqq k < n, Y_n = (a,b), \theta_n \in C_{ab}) .$$

To see this, suppose $\tau(\omega) = n$, $Y_\tau(\omega) = (a,b)$. From (5.2) it follows (for $k = n-1$) that $Y_{n-1}(\omega) \not\succ (a,b)$. Suppose it has been shown that $Y_\ell(\omega) \not\succ (a,b)$ for $k \leqq \ell < n$. Then $Y_{k-1}(\omega) \not\succ (a,b)$ follows because by (5.2) , $Y_{k-1}(\omega) \not\succ Y_\ell(\omega)$ for some $\ell, k \leqq \ell \leqq n$, by assumption $Y_\ell(\omega) \not\succ (a,b)$, hence $Y_{k-1}(\omega) \not\succ (a,b)$ since $\not\succ$ is transitive. So an Induction argument yields (5.3) from (5.2).

But comparing (5.3) with (4.6) it is clear that τ is a CI-death time for any Markov probability. So by the preceding discussion, with C given by (5.1), $\succ$ and $\not\succ$ transitive as described above, we obtain a path decomposition with the excursions (e_i) being independent and Markov given (Z_i).

We have here discussed path decompositions induced by certain homogeneous random sets. But it is of course also possible to obtain decompositions based on the dual homogeneous sets L considered in Theorem 4.22 (b).

Finally, there are examples of path decompositions which are perfect in the following sense: suppose L is as in Theorem 4.22 (b) with the $\hat{\tau}$ from (4.23) of the form (4.6) (relative to Ω_0), suppose M is as in Theorem 3.26 (b) with $\tau = \tau_1$ of the form (4.6), and suppose that the beginning $\tau = \inf M$ of M equals the end sup L of L. Then because $\hat{\tau}$ and τ are always CI-death times the given Markov chain P may be decomposed as follows: choose a random variable $U = Z_1 = \tilde{W}_1$ with distribution the P-distribution of Y_τ, and define W_1 by interchanging the two components of the transition $U = \tilde{W}_1$. Given U, construct two independent Markov chains, (Z_i) and (W_i) with distributions matching those of the (Z_i) and (W_i) of Sections 3 and 4. Finally, given U, (Z_i) and (W_i), establish two independent sequences of mutually independent Markovian excursions (e_i) and (f_i), where the (e_i) together with (Z_i) are to constitute the post-τ process θ_τ as described above, while the (f_i) and (W_i) are to yield in a similar manner the pre-τ process K_τ.

We leave it to the reader to check that with $\tau = \underline{\tau}$ or $\overline{\tau}$ (Examples 3.18 and 4.19), examples of such perfect path decompositions are obtained.

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REFERENCES

1. Getoor, R.K.: Splitting Times and Shift Functionals. Z. Wahrsch. Verw. Gebiete 47, 69-81 (1979).

2. Getoor, R.K., Sharpe, M.J.: Markov Properties of Markov Processes. Z. Wahrsch. Verw. Gebiete 55, 313-330 (1981).

3. Glover, J.: Raw Time Changes of Markov Processes. Ann. Probab. 9, 90-102 (1981).

4. Jacobsen, M.: Splitting Times for Markov Processes and a Generalized Markov Property for Diffusions. Z. Wahrsch. Verw. Gebiete 30, 27-43 (1974).

5. Jacobsen, M.: Markov Chains: Birth and Death Times with Conditional Independence. Institute of Mathematical Statistics, University of Copenhagen, Preprint 7 (1979).

6. Jacobsen, M.: Two Operational Characterizations of Cooptional Times. To appear in Ann. Probab.

7. Jacobsen, M., Pitman, J.W.: Birth, Death and Conditioning of Markov Chains. Ann. Probab. 5, 430-450 (1977).

8. Millar, P.W.: Random Times and Decomposition Theorems. Probability (Proc. Sympos. Pure Math., Vol. XXXI, Univ. Illinois, Urbana, Ill., 1976), pp. 91-103. Amer. Math. Soc., Providence, R.I. (1977).

9. Millar, P.W.: A Path Decomposition for Markov Processes. Ann. Probab. 6, 345-348 (1978).

10. Pittenger, A.O.: Regular Birth Times for Markov Processes. Ann. Probab. 9, 769-780 (1981).

11. Pittenger, A.O.: Time Changes of Markov Chains. Stochastic Process. Appl. 13, 189-199 (1982).

12. Pittenger, A.O., Sharpe, M.J.: Regular Birth and Death Times. Z. Wahrsch. Verw. Gebiete 58, 231-246 (1981).

13. Sharpe, M.J.: Killing Times for Markov Processes. Z. Wahrsch. Verw. Gebiete 58, 223-230 (1981).

14. Williams, D.: Path Decomposition and Continuity of Local Time for One-dimensional Diffusions, Proc. London Math. Soc. 28, 738-768 (1974).

15. Williams, D.: Diffusions, Markov Processes, and Martingales. Volume 1: Foundations. Chichester, Wiley 1979.

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