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CONJUGATE DISTRIBUTIONS AND VARIANCE REDUCTION IN RUIN PROBABILITY SIMULATION

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A general method is developed for giving simulation estimates of the probability $\psi(u,T)$ of ruin before time $T$. When the probability law $P$ governing the given risk reserve process is imbedded in an exponential family $(P_\theta)$, one can write $\psi(u,T) = E_\theta R_\theta$ for certain random variables $R_\theta$ given by the fundamental identity of sequential analysis. Using this to simulate from $P_\theta$ rather than $P$, it is possible not only to overcome the difficulties connected with the case $T=\infty$, but also to obtain a considerable variance reduction. It is shown that the solution of the Lundberg equation determines the asymptotically optimal value of $\theta$ in heavy traffic when $T=\infty$, and some results guidelining the choice of $\theta$ when $T<\infty$ are also given. The potential of the method in complex models is illustrated by two examples.

Risk reserve process; ruin probability; simulation; conjugate distributions; importance sampling; heavy traffic; fundamental identity of sequential analysis; Lundberg equation.

## 1. INTRODUCTION

Let $U(t)$ be the risk reserve at time $t$ of a risk business (evolving in a manner unspecified for a while), and assume that the initial reserve is $u = U(0)$. In the center of classical risk theory lies then the wish to give reasonable precise values of the probabilities

$$(1.1) \qquad \psi(u,T) = P(\inf_{0 \leq t < T} U(t) < 0) \; ,$$

$$(1.2) \qquad \psi(u) = \psi(u,\infty) = P(\inf_{0 \leq t < \infty} U(t) < 0)$$

of ruin before time $T$, resp. of ultimate ruin.

As is well-known, closed forms of even $\psi(u)$ can only be found in models which are much to simple to be of any practical relevance. For $\psi(u,T)$, such closed forms hardly exist at all. It is therefore necessary to develop substitutes for theoretical solutions, and tentatively three main types occur: Numerical methods), examples of which can be found in [21], [23], [28], [29], [ 2 ]; Approximations), one of the classical topics in risk theory with surveys in [7] and (more recently) in [24], [2]; and finally Simulation), the simplest example of which would be crude simulation of $\psi(u,T)$, i.e. to perform $N$ independent runs of the risk process in the time interval $[0,T)$ and estimate $\psi(u,T)$ by the fraction of runs where ruin has occured.

Without embarking into a comparison of the merits of these approaches, it would occur to the author that, as in many other applied probability problem, simulation is by no means always preferable, but appealing by its simplicity and insensivity to the complexity of the model. Nevertheless, the particular aspects of ruin simulation seem to have received extremely little attention compared to related fields like queueing problems (e.g. [9],[14],[6] Ch. 6) and also, the literature (e.g. [20],[4] pp. 91-97, 134-136) does not go deep into the methodology of the subject. Of course, crude simulation is too simple a topic to deserve much attention from the theoretician. However, the method is not immediately applicable in infinite time problems and has some further disadvantages. Thus each run may require much computer time and in applications, the ruin probabilities are typically small so that the relative error on the estimates becomes large.

The purpose of the present paper is to present a general method which overcomes these difficulties. The idea is quite simple and comes from one of the classical tools in risk theory, conjugate distributions. These may be thought of as arising from an imbedding of the probability law $P = P_{\theta_0}$ governing the given risk process in an exponential family $(P_\theta)$, cf. [2], and a given $\psi(u,T)$ can then by means of the fundamental identity of sequential analysis be expressed as $\psi(u,T) = E_\theta R_\theta$ for certain random variables $R_\theta$ (thus $R_{\theta_0}$ is simply the indicator of ruin before $T$). The point is that for suitable choice of $\theta$ the $R_\theta$ can be simulated in finitely many steps even when $T = \infty$, and that their variances are small compared to $R_{\theta_0}$. Thus the approach relates to two topics discussed in the statistically orientated literature on simulation, viz. that of simulating infinite time problems (e.g. [8]) and that of variance reduction techniques (e.g. [12] Ch.5, [16] Ch.III, [19] Ch.4). From this last point of view there is some relation to importance sampling, a concept which is intuitively appealing but which it so far mainly seems to have been possible to implement in idealized textbook situations rather than in real world problems.

The paper is organised as follows. In Section 2, we present the classical Poisson model, which serves as our illustration in most cases, and the above mentioned exponential family $(P_\theta)$ and responses $R_\theta$. The simulation method is presented in Section 3 and offered some preliminary discussion. We start by a simple example and some computer simulations and numerical studies are presented which naturally raise the question of the optimal choice of $\theta$. This is resolved in Section 5, where it is shown that in heavy traffic conditions the solution $\gamma$ of the Lundberg equation determines the asymptotically optimal value $\theta_1 = \gamma + \theta_0$ of $\theta$ if $T = \infty$. Also some approximations are derived which are of relevance for the case $T < \infty$. The asymptotic considerations are based on approximations by Brownian motions with drift and their first passage time distributions (inverse Gaussians) in the same way as in [25], [2], and the relevant preliminaries are given in Section 4. Finally it is argued in Section 6 that the approach can be applied in substantially more complex models than the classical Poisson one. Two examples are considered, time-dependent intensities for the arrival of claims and state-dependent premiums. More generally, it is our belief that the method has a scope also outside risk theory in cases where one is faced with the evaluation of first passage time probabilities. Examples are emptiness problems in dams and some special queueing problems.

## 2. THE CLASSICAL POISSON MODEL, EXPONENTIAL FAMILIES AND THE FUNDAMENTAL IDENTITY OF SEQUENTIAL ANALYSIS

Assume that claims arrive according to a Poisson process $\{N(t)\}_{t \geq 0}$ with intensity $\alpha$, that the claim sizes $Y_1, Y_2, \ldots$ are independent of $\{N(t)\}$ and i.i.d. with common moment generating function $\phi(s) = Ee^{sY}$, and that premiums come in at rate $p$ per unit time. That is, the interclaim times $Z_1, Z_2$ are i.i.d. with $P(Z > z) = e^{-\alpha z}$, we can write

$$U(t) = u - X(t) \quad \text{where} \quad X(t) = \sum_{n=1}^{N(t)} Y_n - pt$$

and the ruin probabilities are given in terms of the first passage time $\tau = \tau(u) = \inf\{t \geq 0 : X(t) > u\}$ by

$$\psi(u,T) = P(\tau(u) \leq T), \quad \psi(u) = P(\tau(u) < \infty) .$$

We shall refer to this case as the classical Poisson model. It plays a predominant role in the literature and it should be noted that both analytical solutions in particular for $\psi(u)$ and approximations have been extensively developed. However, as is also done to a large extent in the literature on variance reduction techniques, it seems reasonable to us to first exploit and present the basic ideas in a simple case. We shall therefore not hesitate to even frequently to specialize to the Poisson/Exponential (P/E) case $P(Y > y) = e^{-\beta y}$. Let $\eta = (p - \alpha EY)/\alpha EY$ denote the safety loading. We consider only the case $\eta > 0$ which is equivalent to $\psi(u) < 1$ for all $u$.

As is well-known (e.g. [17], [18], [22]), this situation has a queueing analogue as well. If for convenience the time scale is chosen such that $p = 1$, then $\psi(u) = P(V > u)$ where $V$ is the virtual waiting time in a stationary M/G/1 queue with arrival intensity $\alpha$ and service times distributed as $Y$. Thus $\alpha EY$ is the traffic intensity of the queue which is $< 1$ in view of $\eta > 0$.

We define the basic exponential family $(P_\theta)$ of risk processes exactly as in [2]. We cite only the most basic facts and formulas and refer to [2] for a more complete discussion and references.

We first note that since $\{X(t)\}_{t \geqq 0}$ has stationary independent increments, the cumulant generating function of $X(t)$ is given by

$$Ee^{sX(t)} = e^{t\kappa(s)}$$ where

$$\kappa(s) = \log Ee^{sX(1)} = \alpha(\phi(s) - 1) - ps \ .$$

The arrival intensity for $P_\theta$ is denoted by $\alpha_\theta$, the m.g.f. of $Y$ by $\phi_\theta(s)$ and the c.g.f. of $X(1)$ by $\kappa_\theta(s)$. These quantities are defined in (2.2), (2.3) below and make the distributions of the interclaim time, $Y$ and the $X(t)$ conjugates to the given ones (i.e. obtained by multiplying an exponential function to the density and normalizing). The premium rate is $p$ for all $\theta$. The given process corresponds to $P = P_{\theta_0}$ where $\theta_0 < 0$ is given by

(2.1) $$\kappa'(-\theta_0) = \alpha\phi'(-\theta_0) - p = 0$$

(the formulas simplify somewhat if instead one defines the origin by $P = P_0$ but the present choice is more convenient for the asymptotic considerations of Sections 4-5). For any $\theta$ such that $\phi(\theta - \theta_0) < \infty$, we define

(2.2) $$\alpha_\theta = \alpha\phi(\theta - \theta_0), \quad \phi_\theta(s) = \frac{\phi(s + \theta - \theta_0)}{\phi(\theta - \theta_0)} \ .$$ Then

(2.3) $$\kappa_{\theta''}(s) = \kappa_{\theta'}(s + \theta'' - \theta') - \kappa_{\theta'}(\theta'' - \theta')$$

(the parameter set $\{\theta : \phi(\theta - \theta_0) < \infty\}$ contains always the negative halfline, but will typically be bounded to the right except if the tail of $Y$ decreases very rapidly). It is seen that $\kappa'_\theta(0) \gtreqless 0$ (or equivalently $E_\theta X(t) \gtreqless 0$)

exactly when $\theta \gtreqless 0$. This is well-known to imply that in particular ruin occurs a.s. when $\theta \geqq 0$. I.e., $P_\theta(\tau(u) < \infty) = 1$ $\theta \geqq 0$. Besides $\theta_0$, a very important quantity in risk theory is the solution $\gamma > 0$ of the Lundberg equation $\kappa(\gamma) = 0$, cf. e.g. [7], and we let $\theta_1 = \gamma + \theta_0$.

<u>Lemma 2.1</u> (Fundamental Identity of Sequential Analysis). <u>For any stopping time</u> $\tau^*$ <u>w.r.t.</u> $\{X(t)\}_{t \geqq 0}$ <u>and any random variable</u> $Y$ <u>measurable w.r.t. the usual stopping time</u> $\sigma$<u>-algebra</u> $F_{\tau^*}$ <u>and satisfying</u> $Y = 0$ <u>on</u> $\{\tau^* = \infty\}$, <u>it holds for all</u> $\theta', \theta''$ <u>that</u>

$$(2.4) \qquad E_{\theta'} Y = E_{\theta''}[Y \exp\{(\theta' - \theta'')X(\tau^*) - \tau^* \kappa_{\theta''}(\theta' - \theta'')\}] .$$

Letting $\theta' = \theta_0, \theta'' = \theta, \tau^* = \tau(u), Y = I\{\tau(u) < T\}$ we get

<u>Corollary 2.2</u> <u>Let</u> $T \leqq \infty$, $\tau = \tau(u)$. <u>Then</u> $\psi(u,T) = E_\theta R_\theta$ <u>where</u>

$$(2.5) \qquad R_\theta = \exp\{(\theta_0 - \theta)X(\tau) - \tau\kappa_\theta(\theta_0 - \theta)\} I(\tau < T) .$$

Note that, as one would expect, $R_{\theta_0} = I(\tau < T)$. If $T = \infty$ and $\theta \geqq 0$, then $\tau < \infty$ a.s. so that $I(\tau < T)$ in (2.5) is vacuous. A further simplification occurs if $\theta = \theta_1$. Then by (2.3)

$$\kappa_{\theta_1}(\theta_0 - \theta_1) = \kappa(0) - \kappa(\theta_1 - \theta_0) = 0 - 0 \qquad \text{so that}$$

$$(2.6) \qquad R_{\theta_1} = e^{-\gamma X(\tau)} I(\tau < T) \quad (= e^{-\gamma X(\tau)} \text{ if } T = \infty) .$$

A further application of (2.3) yields

$$(2.7) \qquad R_{\theta_1(1+\Delta)} = e^{-(\gamma+\theta_1\Delta)X(\tau)+\tau\kappa_{\theta_1}(\theta_1\Delta)} I(\tau < T)$$

which is the form of (2.5) which we shall most often be using in the following.

<u>Corollary 2.3</u> <u>If</u> $\theta \geqq 0, \theta + \lambda \geqq 0, \kappa(\theta + \lambda - \theta_0) < \infty$, <u>then</u>

(2.8) $$E_\theta \exp\{\lambda X(\tau) - \tau\kappa_\theta(\lambda)\} = 1 .$$

This follows by letting $Y = I(\tau < \infty), \theta' = \theta + \lambda$ and $\theta'' = \theta$ in (2.4). Alternatively, this can be proved by a martingale argument, see e.g. [5] Prop. 5.34 for the discrete time case.

When working with the above expressions, it is also frequently convenient to write $X(\tau) = X(\tau(u)) = u + B(u)$ where $B(u)$ is the overshot. As can be seen from [1],[2],[25], one can most often think of $u$ as the dominating part of $X(\tau)$ and of $B(u)$ as a small disturbance which is furthermore asymptotically independent of $\tau(u)$.

We conclude this section by stating the relevant formulas for the P/E case $P(Y > y) = e^{-\beta y}$ which is used in most of our computer illustrations. Here

(2.9) $$\theta_0 = -\beta(1 - \sqrt{\alpha/\beta p}), \quad \gamma = \theta_1 - \theta_0 = \beta - \alpha/p$$

(2.10) $$\alpha_{\theta_1(1+\Delta)} = \frac{\alpha\beta}{\alpha/p - \theta_1\Delta} ,$$

(2.11) $$P_{\theta_1(1+\Delta)}(Y > y) = \exp\{-\beta_{\theta_1(1+\Delta)}y\}, \quad \beta_{\theta_1(1+\Delta)} = \alpha/p - \theta_1\Delta$$

Furthermore, the requirement $\phi(\theta_1(1+\Delta) - \theta_0) < \infty$ is equivalent to $\theta_1\Delta < \alpha/p$. The following lemma is an easy consequence of the lack of memory of the exponential distribution:

<u>Lemma 2.4</u> <u>Consider the</u> P/E <u>case and let</u> $\theta \geq 0$. <u>Then</u> $B(u)$ <u>is independent of</u> $\tau(u)$ <u>and distributed as</u> $Y$, $P_\theta(B(u) > b) = e^{-\beta_\theta b}$.

Combining this with Corollary 2.3 and substituting $s = \kappa_\theta(\lambda)$, one can compute the Laplace transform of $\tau$. We quote the following formula from [2], which assumes the normalization $\beta = p = 1$:

$$(2.12) \qquad E_{\theta_1} e^{-s\tau(u)} = e^{-\lambda u}(1 - \frac{\lambda}{\rho}) \ ,$$

$$\lambda = \lambda(s) = [\alpha - 1 - s + \sqrt{(1 - \alpha + s)^2 + 4\alpha s}]/2$$

The form of the density or cumulative distribution function is much more complicated and involves Bessel functions.

## 3. SIMULATING FROM $P_\theta$

The idea to be exploited in the rest of the paper amounts in its simplest form to apply the identity $\psi(u) = E_{\theta_1} R_{\theta_1} = E_{\theta_1} e^{-X(\tau(u))}$ to simulate $X(\tau(u))$ (or equivalently $B(u)$) from $P_{\theta_1}$ rather than simulating the event of ruin from $P = P_{\theta_0}$ itself. At least one benefit is immediately obtained: Since $R_{\theta_1}$ can be simulated in finitely many steps, the difficulties connected with the case $T = \infty$ are overcomed and it is furthermore possible to apply standard statistical techniques to estimate $\psi(u)$ and the error on the observate. The lines are standard for this type of experiment: We create $N$ i.i.d. replicates $R_\theta(1),\ldots,R_\theta(N)$ of our response $R_\theta$ and estimate $\psi(u,T) = E_\theta R_\theta$ and $V_\theta R_\theta$ by

$$\overline{R}_\theta = \frac{1}{N}\sum_{n=1}^{N} R_\theta(n), \text{ resp. } s^2 = \frac{1}{N-1}\sum_{n=1}^{N}\{R_\theta(n)^2 - \overline{R}_\theta\}$$

Then as $N \to \infty$, $\overline{R}_\theta$ is strongly consistent for $\psi(u,T)$ and $s^2$ strongly for $\mathrm{Var}_\theta R_\theta$. Furthermore $\overline{R}$ is asymptotically normal

$$N^{\frac{1}{2}}(\overline{R}_\theta - \psi(u,T)) \to N(0, \mathrm{Var}_\theta R_\theta) \quad \text{in } P_\theta\text{-distribution}$$

a fact which it is customary in the literature to state in form of asymptotic $(1-\alpha)$ confidence bands of the form

$$(3.1) \qquad [\overline{R}_\theta - \Phi^{-1}(1-\frac{\alpha}{2})\frac{s}{N^{\frac{1}{2}}}, \overline{R}_\theta + \Phi^{-1}(1-\frac{\alpha}{2})\frac{s}{N^{\frac{1}{2}}}] .$$

Now of course the difficulties connected with crude simulation in the case $T = \infty$ are not unsurmountable. E.g., one could use bounds or approximations to stop each individual run when either the risk reserve or the time has become sufficiently large (we comment on a further alternative, regenerative simulation, later on). However, we shall show in this paper that the method of simulating from $P_\theta$ also creates a considerable reduction of the variance obtained from crude simulation. We start in this section by some largely empirical illustrations and examples, and follow up later on with theoretical analysis.

We first give a simple example. Consider the P/E case with $\beta = 1/EY = 1, p = 1$, $\alpha = 0.85$ and $\psi(u) = \alpha e^{-(1-\alpha)u} = 5\%$, i.e. $u = 18.9$ (this set of parameters could be argued to be typical and will be used repeatedly in the paper). In crude simulation, it follows from properties of the binomial distribution that the variance on the estimate of $\psi(u)$ based on $N_c$ runs is

$$\frac{0.05(1-0.05)}{N_c} = \frac{0.0475}{N_c} .$$

If instead we create $N$ i.i.d. replicates of $R_{\theta_1}$, the variance on $\overline{R}_{\theta_1}$ becomes

$$\frac{\mathrm{Var}_{\theta_1} R_{\theta_1}}{N} = \frac{e^{-2\gamma u}\mathrm{Var}_{\theta_1} e^{-\gamma B(u)}}{N} =$$

$$\frac{e^{-2\gamma u}}{N}[E_{\theta_1} e^{-2\gamma Y} - (E_{\theta_1} e^{-\gamma Y})^2] = \frac{e^{-2\gamma u}}{N}[\frac{\alpha}{\alpha+2\gamma} - (\frac{\alpha}{\alpha+\gamma})^2] =$$

$$\frac{0.00346}{N}[0.7391 - 0.7225] = \frac{0.0000575}{N}$$

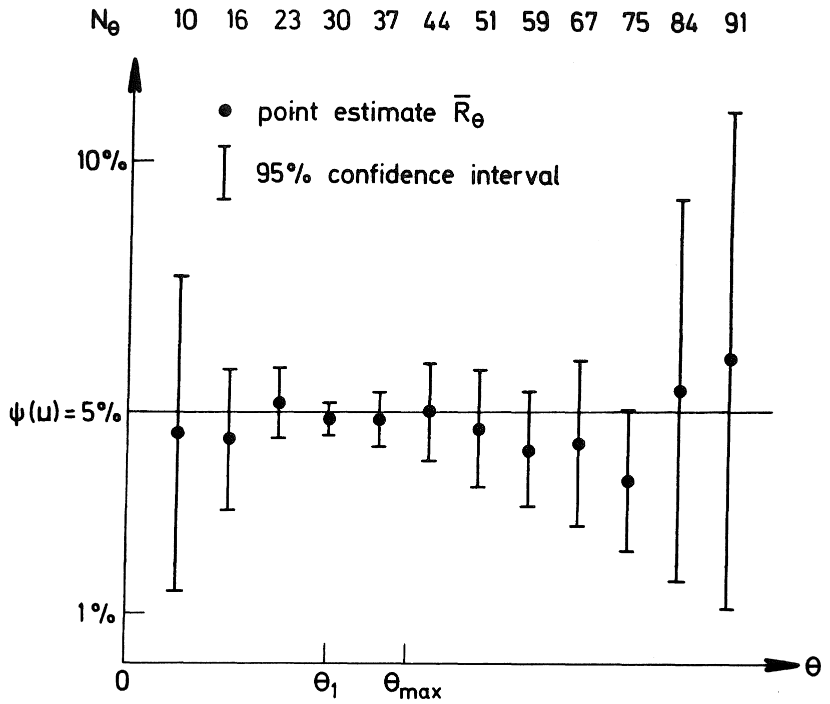
(using material of Section 2 for the calculations). Needless to say that this is a dramatic reduction. Some further reflection seems to indicate that even more is gained. In fact, in crude simulation the runs with ruin will have about the same length as in the $P_{\theta_1}$-process ([1], slightly adapted) whereas the ones without ruin will be longer since we must wait until our stopping criterion is met. Thus if $N = N_c$, $P_{\theta_1}$-simulation should require less computer time than $P_{\theta_0}$-(crude) simulation.

These last considerations are in the spirit of a common point of view in the literature, that not only the variance VarR on the response is of importance but also the computer time IR needed to create one replicate. More precisely, it seems reasonable to take IR VarR as a measure of performance of a particular method, since the variance obtained within T units computer time will then asymptotically be $\mathrm{VarR}/(T/\mathrm{IR}) = T\,\mathrm{IR}\,\mathrm{VarR}$. See e.g. [19] p.119.

In the present case, the time needed to create $R_\theta$ is a random variable so that $I_\theta R$ should denote the expected value. Inspection of the way $I_\theta R$ depends on $\theta$ suggests that it might be worthwhile to look into simulation from also $P_\theta$'s with $\theta \neq \theta_1$. Indeed, it is easy to see that when $\theta' < \theta''$, then the $P_{\theta''}$-distribution of the whole process $\{X(t)\}_{t \geq 0}$ is stochastically larger than its $P_{\theta'}$-distribution (in the sense of the usual ordering in $D[0,\infty)$). Hence ruin occurs earlier so that $I_{\theta''} R_{\theta''} < I_{\theta'} R_{\theta'}$ (whereas it seems less clear what happens to the $\mathrm{Var}_\theta R_\theta$).

To illustrate these phenomena, we return to the P/E example $\alpha = 0.85$, $\psi(u) = 5\%$. Computer simulations were performed for $\theta_1$, and for larger as well as smaller $\theta$. For each value of $\theta$, the computer time allowed was the same, one second CPU time. That is, the number $N_\theta$ of runs was finalized the first time the CPU time at the end of a run exceeded one second. The estimates and asymptotic 95% confidence bands are depicted in Figure 1.

Figure 1 Simulation results from one second CPU time simulation from $P_{\theta_1(1+\Delta)}$. P/E case, $\beta = p = 1$, $\psi(u) = 5\%$, $\Delta = -0.75(0.25)2.00$




It should be noted that the estimates and confidence bands are computed exactly as above, i.e. as if $N_\theta$ were deterministic. That this is immaterial for the asymptotic considerations follows by Anscombe's theorem and standard results from renewal theory (related remarks are given in [8] p. 54).

The simulations were carried out in Pascal at the Regional Computing Center, University of Copenhagen, on their Univac 1181 Machine. Uniform random numbers were produced by N.A.G. routine SO5CAF, initialized by SO5CBF(I) with $I = 17$ for any single simulation estimate reported in the paper. The algorithm (extremely simple) is based on considering the times of claims only as follows:

1) Put $S = SS = 0$, $N = 0$, and initialize the random number generator;

2) Put $X = T = 0$, $N = N + 1$ ;

3) Generate a claim size $Y$ and an interarrival time $Z$ according to $P_\theta$ ; Put $X = X + Y - pZ$, $T = T + Z$;

4) If $X < u$, return to 3). Otherwise let $R = \exp\{(\theta_0 - \theta)X - T\kappa_\theta(\theta_0 - \theta)\}$, $S = S + R$, $SS = SS + R^2$;

5) If less than one second CPU time has elapsed since 1), return to 2).

We shall now give some preliminary discussion of some of the phenomena underlying Fig. 1. First, it is seen as expected that $N_\theta$ increases with $\theta$. Next we note that the parameter set has the form $(-\infty, \bar{\theta})$, where $\bar{\theta} - \theta_0$ is the first singularity of $\phi$, i.e. $\bar{\theta} = \theta_0 + \beta = \theta_0 + 1 \cong 1.07$, and we are interested in the range $(0, \bar{\theta})$. The values of $\theta = \theta_1(1 + \Delta)$ used for illustration are at most $3\theta_1 \cong 0.22$ and thus well inside. However, this only guarantees consistency and asymptotic normality will in fact only hold in a much more restricted range. To see this, we need the relation

$$(3.2) \qquad \mathrm{Var}_{\theta_1(1+\Delta)} R_{\theta_1(1+\Delta)} = E_{\theta_1} R^2_{\theta_1(1+\Delta)} - \psi(u)^2 =$$

$$E_{\theta_1} \exp\{-(2\gamma + \theta_1\Delta)X(\tau) + \tau\kappa_{\theta_1}(\theta_1\Delta)\} - \psi(u)^2$$

which easily follows from (2.7), (2.4). It can then be derived from [27] and the fundamental identity that (3.2) is only finite when

$$\kappa_{\theta_1}(\theta_1\Delta) \leqq -\kappa(\theta_0) \quad (= (1 - \alpha^{\frac{1}{2}})^2),$$

which amounts to $\Delta < 0.4$ (on Fig. 2, $\theta_{max}$ denotes the corresponding upper bound on $\theta$). Thus only the first five confidence bands are meaningful and we shall hereafter only be concerned with the range $(0, \theta_{max})$.

It is seen that in the remaining five cases the width of the confidence band is minimal when $\theta = \theta_1$. This was found also in other simulations with different sets of parameters and naturally raises the question of a possible theoretical investigation supporting the optimality of $\theta_1$. This would be interesting not only with the simulation application in mind but also for further elucidating the role of $\theta_1$ (or equivalently $\gamma = \theta_1 - \theta_0$) which is well-known to be one of the very fundamental quantities in risk theory, and it is therefore a question which shall occupy us in some detail. We let $T = \infty$ for quite a while.

As a first step, it is necessary to put the somewhat unprecise definition of $I_\theta R_\theta$ into a form more suitable for theoretical analysis. A look at the algorithm above seems to suggest that the main time consuming factor when generating a single $R_\theta$ is the repetitions of step 3), the number of which is

$$\underline{n} = \inf\{n \geqq 1 : \sum_{k=1}^{n} \{Y_k - pZ_k\} > u\} .$$

The time needed for each step is of course machine- and programming language dependent but does not significantly vary with $\theta$, and in the following we shall therefore insert $I_\theta R_\theta = E_\theta \underline{n}$. That is, we are concerned with minimizing

(3.3) $$f(\Delta) = E_{\theta_1(1+\Delta)}\underline{n}\, \mathrm{Var}_{\theta_1(1+\Delta)} R_{\theta_1(1+\Delta)}$$

subject to $\theta_1 \Delta < \theta_{max}$.

The two following formulas, which are easy consequences of Wald's identity and $\tau = Z_1 + \ldots + Z_{\underline{n}}$, will be useful in the following:

(3.4) $$E_\theta \tau = E_\theta Z \cdot E_\theta \underline{n} ,$$

(3.5) $$E_\theta \tau = E_\theta X(\tau)/\kappa_\theta'(0) = (u + E_\theta B(u))/\kappa_\theta'(0) .$$

We first consider the P/E case where (3.2) can be computed explicitly. In fact, if $\beta = p = 1$ then

$$\gamma = 1-\alpha,\ \alpha_{\theta_1} = 1,\ \beta_{\theta_1}(1+\Delta) = \alpha - \theta_1\Delta,\ \kappa_{\theta_1}(\lambda) = \frac{\alpha}{\alpha-\lambda} - 1 - \lambda$$

and using Lemma 2.4, it follows from (3.4),(3.5) that

$$E_{\theta_1}(1+\Delta)\underline{n} = [u + \frac{1}{\alpha-\theta_1\Delta}]\frac{\alpha(\alpha-\theta_1\Delta)}{\alpha-(\alpha-\theta_1\Delta)^2} ,$$

whereas (3.2) can be evaluated using (2.12) Numerical tabulations showed $f(\Delta)$ to be convex with the minimum attained at a point $\Delta_{min}$ which is in general $\neq 0$ and is tabulated in Table 1 for some selected parameter values.

Table 1 $\Delta_{min}$ and variance reduction $f(\Delta_{min})/f(0)$ for classical P/E model $(\beta = p = 1)$

| $\alpha$ | $\psi(u)$ (%) | $\Delta_{min}$ | $f(\Delta_{min})/f(0)$ |
|---|---|---|---|
| 0.25 | 1 | - 0.0202 | 0.997 |
| 0.25 | 5 | - 0.0441 | 0.989 |
| 0.25 | 20 | - 0.1142 | 0.958 |
| 0.55 | 5 | 0.0021 | 1.000 |
| 0.55 | 20 | - 0.2036 | 1.000 |
| 0.55 | 50 | - 0.0332 | 0.997 |
| 0.85 | 5 | 0.0004 | 1.000 |
| 0.85 | 20 | 0.0007 | 1.000 |
| 0.85 | 50 | 0.0012 | 1.000 |

It is seen that many values of $\Delta_{min}$ are so close to zero and the variance reduction to small that it is strongly suggested that in some asymptotical sense $\theta_1$ is optimal. To show this is one of the topics of the next sections.

## 4. DIFFUSION APPROXIMATIONS IN HEAVY TRAFFIC

Various approximation procedures are reviewed in [2]. For the present purpose, the relevant ones seem to be normal approximations $(u\to\infty)$ and heavy traffic approximations (which all require $p \cong \alpha EY$ in some way). We shall here exploit the last point of view, since [2] indicates that it can provide better approximations and since it has a natural implementation within the framework of imbedding in an exponential family.

We shall consider the same limiting procedure as in [25],[2] Sect. 5. That is, we think of $P_0$ (i.e. of $p,\alpha_0,\phi_0$) as the fixed parameter and consider the limit

(4.1) $$\theta_0 \uparrow 0, u \uparrow \infty \text{ in such a way that } \theta_0 u \to -\xi$$

for some $\xi > 0$ (note that in [2] we write $\theta_0 u \to \xi$ with $\xi < 0$). As explained in [2] (the argument is essentially the same as in [10],[11],[25]), it holds subject to (4.1) that

(4.2) $$\frac{\tau(u)\alpha_0 E_0 Y^2}{u^2} \to \tau_{-\xi} \text{ in } P_{\theta_0}\text{-distribution}$$

where $\tau_\xi$ is the time of first passage of Brownian motion with unit variance and drift $\xi$ to level 1 (thus $\tau_\xi$ is defective when $\xi < 0$). The distribution of $\tau_\xi$ is the so-called <u>inverse Gaussian distribution</u> and has density, cumulative d.f., resp. moment generating function

(4.3) $$g(t;\xi) = \frac{1}{\sqrt{2\pi}} t^{-3/2} \exp\{\xi - \tfrac{1}{2}(\tfrac{1}{t} + \xi^2 t)\}, \quad t > 0 ,$$

(4.4) $$G(t;\xi) = P(\tau_\xi < t) = 1 - \Phi(t^{-\frac{1}{2}} - \xi t^{\frac{1}{2}}) + e^{2\xi}\,\Phi(-t^{-\frac{1}{2}} - \xi t^{\frac{1}{2}}), \quad t > 0 ,$$

(4.5) $$Ee^{\lambda\tau_\xi} = \begin{cases} \infty & \lambda > \xi^2/2 \\ \exp\{\xi - \sqrt{\xi^2 - 2\lambda}\} & \lambda \leqq \xi^2/2 \end{cases}$$

See [26] Ch.7 or [15] Ch.15 for more detail.

We quote some consequences of (4.1),(4.2) from the above references. First

$$\psi(u) = P_{\theta_0}(\tau < \infty) \cong P(\tau_{-\xi} < \infty) = G(\infty, -\xi) = e^{-2\xi} , \tag{4.6}$$

$$\psi(u, Tu^2/\alpha_0 E_0 Y^2) = P_{\theta_0}(\tau < Tu^2/\alpha_0 E_0 Y^2) \cong \tag{4.7}$$

$$P(\tau_{-\xi} < T) = G(T; -\xi) .$$

Next, since $\theta_0 < 0, \theta_1 > 0$ are connected by $\kappa_0(\theta_0) = \kappa_0(\theta_1)$ , it follows by Taylor expansion that (4.1) is equivalent to

$$\theta_1 u \to \xi . \tag{4.8}$$

From this relations similar to (4.7) for the $P_{\theta_1(1+\Delta)}$-distribution of $\tau$ follow by replacing $-\xi$ by $\xi(1+\Delta)$ . Furthermore:

_Lemma 4.1_ _Subject to_ (4.8), _it holds that_ $B(u) \to B(\infty)$ _in_ $P_{\theta_1}$_-distribution. Here_ $B(\infty)$ _has the limiting_ $P_0$_-distribution of_ $B(u)$ _as_ $u \to \infty$, _viz._

$$E_0 e^{\lambda B(\infty)} = \alpha_0 \{E_0 Y + \lambda \frac{E_0 Y^2}{2} + \lambda^2 \frac{E_0 Y^3}{6} + \lambda^3 \frac{E_0 Y^4}{24} + \ldots\} . \tag{4.9}$$

_Furthermore_ $E_{\theta_1} e^{\lambda B(u)} \to E_0 e^{\lambda B(\infty)}$ _in a neighbourhood of zero._

Indeed, the first statement is contained in [25], the formula (4.9) in the proof of Lemma 5.1 of [2] whereas the last statement as well as some further estimates to be used in the sequel requires some uniform integrability arguments. As example of how to carry out these, we give the proof of the following Lemma:

_Lemma 4.2_ _Subject to_ (4.8), _it holds that_

$$E_{\theta_1}\exp\{\lambda\tau\rho_0 E_0 Y^2/u^2\}\begin{cases} = \infty \text{ ultimately} & \lambda > \xi^2/2 \\ \to \exp\{\xi - \sqrt{\xi^2 - 2\lambda}\} & \lambda < \xi^2/2 \end{cases}$$

(we have not investigated the case $\lambda = \xi^2/2$).

Proof  Since $\tau \to \tau_\xi$ in $P_{\theta_1}$-distribution, the result is trivial for $\lambda \leqq 0$, and for $0 < \lambda < \xi^2/2$ it suffices by standard uniform integrability arguments to show

(4.10) $$\overline{\lim}\, E_{\theta_1}\exp\{\lambda'\tau\alpha_0 E_0 Y^2/u^2\} < \infty$$

for some $\lambda' > \lambda$. Choose $\lambda' < \xi^2/2$ and let $\varepsilon, p, q$ satisfy $0 < \varepsilon < 1, p > 1$, $1/p + 1/q = 1$, $\lambda' < (2\varepsilon - \varepsilon^2)\xi^2/2p < \xi^2/2$. Since

(4.11) $$-\kappa_{\theta_1}(-\theta_1\varepsilon) = \kappa_0(\theta_1) - \kappa_0(\theta_1(1-\varepsilon)) \cong$$

$$\frac{\kappa_0''(0)}{2}\{\theta_1^2 - \theta_1^2(1-\varepsilon)^2\} \cong \alpha_0 E_0 Y^2/u^2(2\varepsilon - \varepsilon^2)\xi^2/2$$

we can then bound (4.10) by

(4.12) $$\overline{\lim}\, E_{\theta_1} e^{-\tau\kappa_{\theta_1}(-\theta_1\varepsilon)/p} = \overline{\lim}\, E_{\theta_1}(A_{\theta_1}B_{\theta_1}),$$

$$A_{\theta_1} = e^{-\theta_1\varepsilon X(\tau)/p - \tau\kappa_{\theta_1}(-\theta_1\varepsilon)/p}, B_{\theta_1} = e^{\theta_1\varepsilon X(\tau)/p}.$$

Here $E_{\theta_1}A_{\theta_1}^p = 1$ by (2.8), whereas

$$E_{\theta_1}B_{\theta_1}^q = e^{\theta_1\varepsilon uq/p}E_{\theta_1}e^{\theta_1\varepsilon B(u)q/p} \cong e^{\xi\varepsilon q/p}\cdot 1 .$$

Hence (4.12) is finite by Hölder's inequality.

Suppose finally $\lambda > \xi^2/2$ and let

$$\beta = \frac{\lambda\alpha_0 E_0 Y^2}{u^2}\ ,\quad \beta_0 = -\kappa(\theta_0) = \kappa_0(-\theta_0) \cong \frac{\alpha_0 E_0 Y^2}{u^2} \cdot \frac{\xi^2}{2}\ .$$

Then $\beta > \beta_0$ ultimately and as remarked in Section 3, $E_{\theta_1} e^{\beta\tau} = \infty$ whenever $\beta > \beta_0$. □

## 5. ASYMPTOTIC CRITERIA FOR CHOOSING $\theta$

We can now easily obtain the limiting behaviour of (3.3):

Proposition 5.1 *Subject to* (4.1),(4.8), *it holds that* $f(\Delta)=\infty$ *ultimately if* $\Delta > \sqrt{2}-1$ (i.e. $\Delta^2+2\Delta>1$). *If* $-1<\Delta<\sqrt{2}-1$, *then*

$$(5.1) \qquad f(\Delta) \cong \frac{u^2 e^{-4\xi}}{E_0 Y^2 \xi} \frac{1}{1+\Delta} \{e^{-\Delta\xi+\xi-\xi(1-\Delta^2-2\Delta)^{\frac{1}{2}}} - 1\} .$$

*More precisely it holds for* $\Delta = 0$ *that*

$$(5.2) \qquad f(0) \to 4\xi e^{-4\xi} \alpha_0 \{\frac{E_0 Y^3}{3E_0 Y^2} - \frac{\alpha_0 E_0 Y^2}{4}\} .$$

Clearly, this result contains the asymptotic optimality of $\theta_1$ since $\{\ldots\}$ vanishes for $\Delta = 0$ and is necessarily always $\geqq 0$ as limit of non-negative quantities (this is also easily proved directly). Proposition 5.1 contains, however, some further information: Since $f(\Delta)/f(0) \cong cu^2$ for $\Delta \neq 0$, the difference between $\theta_1$ and $\theta_1(1+\Delta)$ becomes more and more marked as the traffic increases.

Proof of Proposition 5.1 We first note that

$$(5.3) \qquad E_{\theta_1(1+\Delta)} \underline{n} = \frac{E_{\theta_1(1+\Delta)}\tau}{E_{\theta_1(1+\Delta)}Z} \cong \frac{u^2/\alpha_0 E_0 Y^2 E\tau_{\xi(1+\Delta)}}{E_0 Z}$$

$$= \frac{u^2}{E_0 Y^2 \xi(1+\Delta)}$$

(the estimate for $E\tau$ requires some uniform integrability argument along the lines of Lemma 4.2. We omit the details). In the remaining factor (3.2), $\psi(u)^2 \cong e^{-4\xi}$ by (4.6). Furthermore

(5.4) $$(2\gamma + \theta_1\Delta)X(\tau) \cong \theta_1(4+\Delta)(u + B(\infty)) \cong (4+\Delta)\xi \quad ,$$

(5.5) $$\kappa_{\theta_1}(\theta_1\Delta) \cong \alpha_0 E_0 Y^2/u^2(\Delta^2 + 2\Delta)\xi^2/2, \quad \text{cf. (4.11)} .$$

Therefore

$$\exp\{-(2\gamma + \theta_1\Delta)X(\tau) + \tau\kappa_{\theta_1}(\theta_1\Delta)\} \to$$

$$\exp\{-(4+\Delta)\xi + (\Delta^2 + 2\Delta)\xi^2/2 \cdot \tau_\xi\}$$

in distribution. By Lemma 4.2 and standard results on weak convergence, the expectations converge as well with limit

$$\exp\{-(4+\Delta)\xi + \xi - \xi\sqrt{1 - \Delta^2 - 2\Delta}\}$$

given by (4.5). Combining the above estimates, (5.1) follows.

For (5.2), we need to estimate $\mathrm{Var}_{\theta_1} R_{\theta_1} = \mathrm{Var}_{\theta_1} e^{-\gamma X(\tau)}$ more precisely. However,

$$|e^{-\gamma B(u)} - 1 + \gamma B(u)| \leq \gamma^2 B(u)^2 e^{\gamma B(u)} .$$

Hence by Lemma 4.1

$$\mathrm{Var}_{\theta_1} R_{\theta_1} \cong e^{-2\gamma u}\gamma^2 \,\mathrm{Var}_{\theta_1} B(u) \cong e^{-4\xi} \frac{4\xi^2}{u^2} \mathrm{Var}_0 B(\infty)$$

But according to (4.9),

$$\mathrm{Var}_0 B(\infty) = \alpha_0 \frac{E_0 Y^3}{3} - (\alpha_0 E_0 Y^2/2)^2 .$$

Combining with (5.3), (5.2) follows. Finally the assertion for $\Delta^2 + 2\Delta > 1$ is an easy consequence of (5.4), (5.5) and Lemma 4.2. □

We shall make two further comments on Proposition 5.1.

First, the discussion of [25], [2] suggests that the approximations given in Section 4 and underlying Prop. 5.1 are not terribly accurate until the term of next order $(0(u^{-1}))$ are added. Presumably such refinements in the asymptotic form of $f(\Delta)$ could be made and thereby provide a better approximation to $\Delta_{min}$ than just $\Delta_{min} = 0$. We have not carried this out since Table 1 suggests that the resulting variance reduction would be small.

Next, we make some illustrations relating to a question which the reader familiar with queues may have posed a long while ago. Indeed, the relation to the virtual waiting time $V(t)$ in the M/G/1 queue given in Section 2 immediately suggests to apply regenerative simulation, cf. [8],[9],[19] Ch. 6. This amounts to simulating a two-dimensional response variable $R = (R^{(1)}, R^{(2)})$, with components

$$R^{(1)} = \inf\{t > 0 : V(t) = 0, V(s) > 0 \quad \text{for some } s < t\},$$

$$R^{(2)} = \int_0^{R^{(1)}} I(V(t) > u)\,dt$$

and estimate $\psi(u)$ by $\overline{R}^{(2)}/\overline{R}^{(1)}$ (asymptotic expressions for the variance and the confidence intervals can be found in the above references). To compare these two approaches, we performed $P_{\theta_1}$-simulation and regenerative simulation in each one second CPU time for various sets of parameters and obtained the following results:

Table 2 Empirical variance on simulation estimates of $\psi(u)$ obtained within one second CPU time. $\beta = p = 1$

| $\alpha$ | $\psi(u)$ (%) | Regenerative | $P_{\theta_1}$ | Ratio |
|---|---|---|---|---|
| 0.50 | 5 | $1.6_{10}-4$ | $1.8_{10}-6$ | 91.8 |
| 0.50 | 50 | $2.0_{10}-4$ | $6.2_{10}-5$ | 3.2 |
| 0.85 | 5 | $9.8_{10}-4$ | $2.2_{10}-6$ | 436 |
| 0.85 | 50 | $2.8_{10}-3$ | $3.2_{10}-5$ | 86.4 |
| 0.85 | 85 | $5.0_{10}-4$ | $3.4_{10}-5$ | 14.6 |

This indicates that in a broad range of parameters $P_{\theta_1}$-simulation is widely superior. We have not looked into theoretical support for this, but conjecture that as the traffic increases, then the measure of performance similar to IR VarR for the regenerative method tends to infinity so that by (5.2) the difference becomes more marked.

We now turn to the finite time problem $T < \infty$.

Whereas $R_\theta$ has been found already in Section 2, we need to redefine $I_\theta R_\theta$. Since simulation goes on until the risk reserve becomes negative or time $T$ has passed, the appropriate choice appears to be $I_\theta R_\theta = E_\theta \underline{n} \wedge \underline{n}_T$ where

$$\underline{n}_T = \inf\{n \geqq 1 : \sum_{k=1}^{n} Z_k > T\} .$$

We then have the following extension of Proposition 5.1:

Proposition 5.2 Suppose that $T\alpha_0 E_0 Y^2/u^2 \to T_0 \in (0,\infty)$ subject to (4.1),(4.8). Then for all $\Delta > -1$,

$$(5.6) \qquad I_{\theta_1(1+\Delta)} R_{\theta_1(1+\Delta)} \simeq \frac{u^2}{E_0 Y^2} E\tau_{\xi(1+\Delta)} \wedge T_0 \qquad \text{where}$$

(5.7) $$E\tau_\xi \wedge T_0 = \frac{1}{\xi}\{G(T_0;\xi) - 2e^{2\xi}\Phi(-T_0^{-\frac{1}{2}} - \xi T_0^{\frac{1}{2}})\}$$

$$+ T_0\{1 - G(T_0;\xi)\} ,$$

(5.8) $$\mathrm{Var}_{\theta_1(1+\Delta)} R_{\theta_1(1+\Delta)} \cong$$

$$e^{-4\xi}[e^{-\Delta\xi} E e^{(\Delta^2+2\Delta)\xi^2/2\tau_\xi} I(\tau_\xi < T_0) - G(T_0;\xi)^2]$$

It should be noted that it is no longer required that $\Delta^2 + 2\Delta < 1$. This is simply because $\tau_\xi$ when restricted to $\{\tau_\xi < T_0\}$ is bounded and hence has exponential moments of all order. A slight simplification in (5.8) occurs, however, if $\Delta^2 + 2\Delta < 1$ in view of the formula

(5.9) $$E e^{\beta\tau_\xi} I(\tau_\xi < t_0) = e^{\xi - \sqrt{\xi^2 - 2\beta}} G(T_0; \sqrt{\xi^2 - 2\beta}) , \quad \beta \leqq \frac{\xi^2}{2} ,$$

which is immediate by an exponential family argument. We have not been able to find closed expressions if $\beta > \xi^2/2$.

<u>Proof of Proposition 5.2</u> If $C(T)$ is the waiting time until the next claim following $T$, then

(5.8) $$\tau \wedge T = Z_1 + \ldots + Z_{\underline{n}\wedge\underline{n}_T} - C(T) I(\underline{n}_T \leqq \underline{n}) .$$

Clearly, $P_\theta(C(T) > c) = e^{-\alpha_\theta c}$. Therefore the last term in (5.8) vanishes in the limit and we get

$$E_{\theta_1(1+\Delta)} \underline{n} \wedge \underline{n}_T \cong E_{\theta_1(1+\Delta)} \tau \wedge T / E_{\theta_1(1+\Delta)} Z \cong$$

$$\frac{u^2}{\alpha_0 E_0 Y^2} E\tau_{\xi(1+\Delta)} \wedge T_0 \cdot \alpha_0$$

proving (5.4). Obviously (5.7) is equivalent to

(5.9) $$E(\tau_\xi | \tau_\xi < T_0) = \frac{1}{\xi}\{1 - \frac{2e^{2\xi}\phi(-T_0^{\frac{1}{2}} - \xi T_0^{\frac{1}{2}})}{G(T_0;\xi)}\} .$$

Now the class of distributions of $\tau_\xi$ given $\{\tau_\xi < T_0\}$ form an exponential family with canonical parameter $\mu = -\xi^2/2$ and densities

$$\frac{e^\xi}{G(T_0;\xi)} e^{\mu t} \quad (0 < t < T_0) \text{ w.r.t. } \frac{1}{\sqrt{2\pi}} t^{-3/2} e^{-\frac{1}{2}t} dt .$$

Hence ([3] Th.8.1)

$$E(\tau_\xi | \tau_\xi < T_0) = \frac{d}{d\mu} \log \frac{G(T_0;\xi(\mu))}{e^{\xi(\mu)}} = \frac{\frac{\partial G}{\partial \xi}\frac{\partial \xi}{\partial \mu}}{G(T_0;\xi)} - \frac{\partial \xi}{\partial \mu} .$$

Using (4.4), it is easily verified that

$$\partial G/\partial \xi = 2e^{2\xi}\Phi(-T_0^{-\frac{1}{2}} - \xi T_0^{\frac{1}{2}})$$

and since $\xi = \sqrt{-2\mu}$ , we have $\partial \xi/\partial \mu = -1/\xi$ and (5.5) follows.

In (5.6), we get from (4.7) that

$$(E_{\theta_1(1+\Delta)} R_{\theta_1(1+\Delta)})^2 = \psi(u)^2 \cong G(T_0; -\xi)^2 = e^{-4\xi} G(T_0;\xi)^2 .$$

and (5.6) now follows immediately from (5.4), (5.5) since

$$E_{\theta_1(1+\Delta)} R^2_{\theta_1(1+\Delta)} =$$

$$E_{\theta_1} \exp\{-(2\gamma + \theta_1\Delta)X(\tau) + \tau\kappa_{\theta_1}(\theta_1\Delta)\} I(\tau < T) \cong$$

$$\exp\{-(4+\Delta)\xi\} E e^{\tau_\xi(\Delta^2+2\Delta)\xi^2/2} I(\tau_\xi < T_0) . \qquad \square$$

In the same way as for $T = \infty$, we are concerned with finding the value $\Delta_{min}$ of $\Delta$ for which

$$g(\Delta) = \lim I_{\theta_1(1+\Delta)} R_{\theta_1(1+\Delta)} \mathrm{Var}_{\theta_1(1+\Delta)} R_{\theta_1(1+\Delta)} / u^2$$

is minimized.

As the first main consequence of Proposition 5.2, it is immediately observed that it is no longer true that $\Delta_{min} = 0$. That is, one can do better than to apply $P_{\theta_1}$-simulation.

To find closed forms for $\Delta_{min}$ does not look easy. A tabulation of $g(\Delta)$ (using (5.9) when possible and otherwise numerical integration) seemed to indicate that indeed a well-defined minimum of $g(\Delta)$ exists. Some values of $\xi$ and $T_0$ which we consider typical were selected, and $\Delta_{min}$ computed numerically, cf. Table 4.

Table 3 $\Delta_{min}$ as function of selected values of $\xi, T_0$

| $T_0$ \ $\xi$ | 0.5 | 1 | 2 | 5 |
|---|---|---|---|---|
| 0.5 | 2.717 | 1.395 | 0.636 | 0.148 |
| 1 | 2.067 | 0.927 | 0.332 | 0.038 |
| 2 | 1.640 | 0.614 | 0.148 | 0.005 |
| 5 | 1.306 | 0.356 | 0.033 | 0.000 |

It is seen that for T small is $\Delta_{min}$ not only significantly different from zero but also larger than the value $\sqrt{2}-1$ which is critical when $T=\infty$. As expected, $\Delta_{min}$ approaches zero as $T_0 \to \infty$ with $\xi$ fixed.

For a comparison of simulations with $\theta = \theta_1$ or $\theta = \theta_1(1+\Delta_{min})$ it is straightforward to compute $g(\Delta_{min})/g(0)$ and we obtain the following table of the asymptotic variance reduction:

Table 4 Asymptotic variance reduction $g(\Delta_{min})/g(0)$

| $T_0\xi$ \ $\xi$ | 0.5 | 1 | 2 | 5 |
|---|---|---|---|---|
| 0.5 | 0.15 | 0.21 | 0.35 | 0.69 |
| 1 | 0.15 | 0.27 | 0.49 | 0.87 |
| 2 | 0.14 | 0.32 | 0.66 | 0.98 |
| 5 | 0.08 | 0.40 | 0.87 | 1.00 |

It is also of interest to compare the two parameters to crude $(\theta = \theta_0)$ simulation. Here

$$\mathrm{Var}_{\theta_0} R_{\theta_0} = \psi(u,T) - \psi(u,T)^2 \underset{=}{\sim} G(T_0; -\xi)(1 - G(T_0; -\xi)) \quad ,$$

and it follows exactly as above that

$$I_{\theta_0} R_{\theta_0} = E_{\theta_0} \underline{n} \wedge \underline{n}_T \underset{=}{\sim} \frac{u^2}{E_0 Y^2} E\tau_{-\xi} \wedge T_0 \ .$$

Now

$$E\tau_{-\xi} \wedge T_0 =$$

$$E(\tau_{-\xi} \wedge T_0 | \tau_{-\xi} < \infty) P(\tau_{-\xi} < \infty) + T_0 P(\tau_{-\xi} = \infty) =$$

$$E\tau_{\xi} \wedge T_0 e^{-2\xi} + T_0(1 - e^{-2\xi}) \quad .$$

Combining with (5.7), one can thus compute

$$g_c = \lim I_{\theta_0} R_{\theta_0} \mathrm{Var}_{\theta_0} R_{\theta_0} / u^2$$

and the following tables give the corresponding asymptotic variance reductions $g(0)/g_c$, resp. $g(\Delta_{min})/g_c$ when passing from $\theta = \theta_0$ to $\theta = \theta_1$, resp. $\theta = \theta_1(1 + \Delta_{min})$ :

Table 5    Asymptotic variance reduction $g(0)/g_c$

| $T_0\xi$ \ $\xi$ | 0.5 | 1 | 2 | 5 |
|---|---|---|---|---|
| 0.5 | $4.4_{10}-2$ | $4.1_{10}-2$ | $2.3_{10}-2$ | $3.5_{10}-3$ |
| 1 | $1.9_{10}-2$ | $1.6_{10}-2$ | $6.4_{10}-3$ | $3.4_{10}-4$ |
| 2 | $3.0_{10}-3$ | $2.6_{10}-3$ | $6.4_{10}-4$ | $6.3_{10}-6$ |
| 5 | $8.6_{10}-6$ | $7.8_{10}-6$ | $7.2_{10}-7$ | $9.0_{10}-11$ |

Table 6    Asymptotic variance reduction $g(\Delta_{min})/g_c$

| $T_0\xi$ \ $\xi$ | 0.5 | 1 | 2 | 5 |
|---|---|---|---|---|
| 0.5 | $6.5_{10}-3$ | $8.6_{10}-3$ | $8.0_{10}-3$ | $2.4_{10}-3$ |
| 1 | $2.9_{10}-3$ | $4.4_{10}-3$ | $3.2_{10}-3$ | $2.9_{10}-4$ |
| 2 | $4.3_{10}-4$ | $8.3_{10}-4$ | $4.3_{10}-4$ | $6.1_{10}-6$ |
| 5 | $6.9_{10}-7$ | $3.1_{10}-6$ | $6.2_{10}-7$ | $9.0_{10}-11$ |

It is seen that $\theta_1$ is much preferable to $\theta_0$. In some cases the further variance reduction by passing on to $\theta_1(1+\Delta_{min})$ is considerable, in others not.

As an illustration of how results of the above type may be applied in practice and of the accuracy of the approximations, we shall give a final example.

Consider again the P/E case with $\beta = p = 1$, $\alpha = 0.85$ and let $u = 15$, $T = 100$. Here $\gamma = 0.15$, $\theta_1 = \alpha^{\frac{1}{2}} - \alpha = 0.072$ so according to (4.8) we let $\xi = \theta_1 u = 1.079$. Finally let $T_0 = T\alpha_0 E_0 Y^2/u^2 = 2T/\alpha^{\frac{1}{2}}u^2 = 0.964$.

It was found numerically that $\Delta_{min} = 0.8408$. Simulations were performed with $\theta = \theta_0$, $\theta = \theta_1$ and $\theta = \theta_1(1+\Delta_{min})$, each $\theta$ allowed 5 seconds CPU time and the results can be summarised as follows

Table 7 Simulation estimates and empirical and asymptotic variances obtained by 5 seconds CPU time $P_\theta$-simulation. P/E model, $\beta = p = 1$, $u = 15$, $T = 100$

| | $\overline{R}$ | $s^2/N$ | $s^2/N$ in % of $\theta_0$-value | asymptotical (%) |
|---|---|---|---|---|
| crude $\theta = \theta_0$ | 0.068 | $3.1_{10}-4$ | 100 | $g_c/g_c = 100$ |
| Lundberg $\theta = \theta_1$ | 0.064 | $6.4_{10}-6$ | 2.0 | $g(0)/g_c = 1.4$ |
| optimal $\theta = \theta_1(1+\Delta_{min})$ | 0.059 | $3.1_{10}-6$ | 1.0 | $g(\Delta_{min})/g_c = 0.4$ |

As a comparison, approximation (5.8) of [2] gave $\psi(u,T) = 0.062$ where according to the numerical evidence of [2] all figures should be correct.

It is seen that our theoretical results are supported qualitatively from Table 7: $\theta_1$ is much preferable to $\theta_0$ and $\theta_1(1+\Delta_{min})$ even somewhat better. The quantitative agreement of the empirical and theoretical results (i.e., of the two last columns) occurs also reasonable at least when absolute (rather than relative) deviations are considered.

## 6. EXAMPLES OF MORE COMPLEX MODELS

It is often argued in the literature that sophisticated variance reduction methods become increasingly hard to apply in practice when the model is made more realistic and thereby more complex (possible exceptions are extremely simple ideas like antithetic variables or common random numbers). We therefore considered it worthwhile to point out that the method developed here is in this respect quite insensitive, and to substantiate this claim by some worked out examples.

Consider first the case where the rate $\alpha$ of arrival of claims varies with time, $\alpha = \alpha(t)$ . A typical case would be seasonal fluctuations so assume that $\alpha$ is periodic, say $\alpha(t+r) = \alpha(t)$ and let

$$\overline{\alpha} = \lim_{T\to\infty} \frac{1}{T}\int_0^T \alpha(t)\,dt = \frac{1}{r}\int_0^r \alpha(t)\,dt$$

be the average arrival rate. Assume further that the premium rate $p$ and the distribution of claims $Y$ are constant with time and that $\overline{\alpha}EY < p$ . It is then easy to see that $\psi(u) < 1$ for all $u$ and for simplicity, we shall only consider simulation of $\psi(u)$ and not $\psi(u,T)$ .

The main idea is to get an input process with stationary independent increments by passing to operational time $T_{op}$ , which is connected to physical time $T$ by

$$\frac{dT_{op}}{dT} = \alpha(T), \quad T_{op} = \int_0^T \alpha(t)\,dt = \overline{\alpha}T + \delta(T)$$

where $\delta(T) = \int_0^T (\alpha(t) - \overline{\alpha})\,dt$ has period $r$ . Similarly

$$T = \frac{1}{\overline{\alpha}} T_{op} + \varepsilon(T_{op}), \quad \varepsilon(T_{op}) = \frac{1}{\overline{\alpha}}\,\delta(T)$$

where $\varepsilon$ has period $r^{-1}$ . In the operational time scale, the input process

$I(T_{op})$ is compound Poisson with unit rate and jumps distributed as $Y$. The premiums $q(T_{op})$ received before $T_{op}$ are given by

$$q(T_{op}) = pT = \frac{p}{\bar{\alpha}} T_{op} + p\varepsilon(T_{op}) \ .$$

Hence if we let $X(T_{op}) = I(T_{op}) - p/\bar{\alpha} \cdot T_{op}$, then our ruin probability is

$$\psi(u) = P\left( \sup_{0 \leqq T_{op} < \infty} \{I(T_{op}) - q(T_{op})\} > u\right)$$

$$= P(X(T_{op}) > u + p\varepsilon(T_{op}) \text{ for some } T_{op})$$

$$= P(\tau(u) < \infty)$$

where

$$\tau(u) = \inf\{T_{op} \geqq 0 : X(T_{op}) > u + p\varepsilon(T_{op})\} \ .$$

Thus if we compute the solution $\gamma$ of the Lundberg equation for the $X$ process, it holds according to Lemma 2.1 that

$$\psi(u) = E_{\theta_1} R_{\theta_1} \, , \; R_{\theta_1} = e^{-\gamma X(\tau(u))} \ .$$

In summary, this suggests the following procedure

1) Pass from the given periodic model to a classical Poisson model $X$ by considering operational time and the average premium rate.

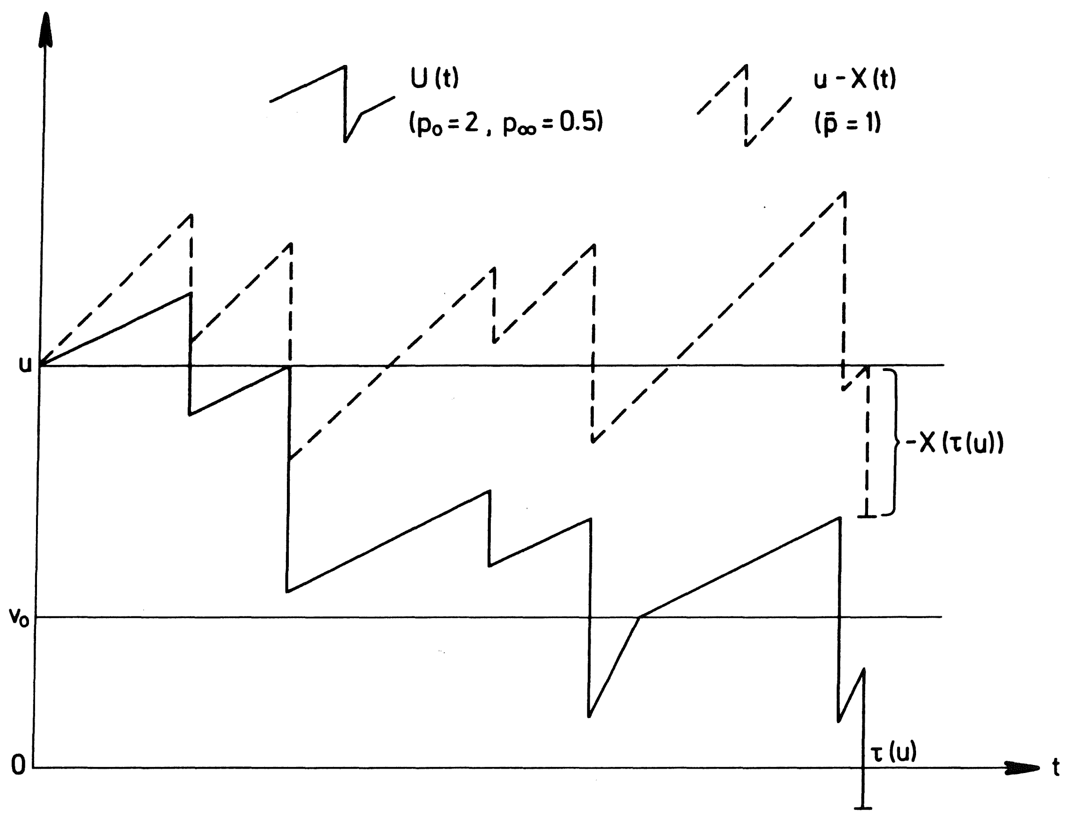
2) Solve the Lundberg equation for $X$.

3) Simulate $X$ from $P_{\theta_1}$ until at time $\tau(u)$ the periodic boundary $u + p\varepsilon(T(op))$ is hit and observe the response $R_{\theta_1}$. Replicate the experiment a suitable number of times.

The preceding part of the paper strongly suggests that this yields reasonable tight confidence intervals. Of course, some arbitrariness is inherent by the choice of which premium rate to subtract from $I(T_{op})$ before solving the Lundberg equation. The present choice $p/\overline{\alpha}$ appears sensible by long-run considerations if $\tau(u) >> r^{-1}$ , whereas otherwise modifications may be required.

In our second example, we assume that the premium rate $p$ is a function of the current risk reserve $v = U(t)$ , say $p = p(v)$ . This appears sensible since an insurance company would want to take some action, typically by increasing the premiums, if the risk reserve approaches exhaustion. An example of the paths of the risk reserve process $\{U(t)\}_{t \geqq 0}$ is given in Figure 2 for the case where $p$ has only two values $p_0 > p_\infty$ , $p(v) = p_0$ when $v \leqq v_0$ and $p(v) = p_\infty$ when $v > v_0$

Figure 2



The assumptions on the input process $I(t)$ are as for the classical Poisson model. As a reasonable set of general conditions on $p = p(v)$, assume that $p$ is non-increasing with limits $p_0 < \infty$, $p_\infty > \alpha EY$ as $v \downarrow 0$, resp. $v \uparrow \infty$. Clearly, the risk reserve process satisfies the storage equation

$$V(T) = u - I(T) + \int_0^T p(V(t))dt$$

as in [13], and the ruin probability is

$$\psi(u) = P(\tau(u) < \infty), \tau(u) = \inf\{t \geqq 0 : V(t) < 0\} .$$

Since $p(v) \geqq p(\infty) > \alpha EY$, $P(\tau(u) < \infty) < 1$ for all n and crude simulation does not apply (neither is it immediately clear how to apply regenerative simulation in this case). Instead we note that $\tau(u)$ is a stopping time not only w.r.t. $\{V(t)\}$ but also w.r.t. $\{I(t)\}$ and hence w.r.t. $\{X(t)\}_{t\geqq 0}$ where $X(t) = I(t) - \overline{p}t$ with some arbitrary choice of $\overline{p}$. This suggests the following procedure.

1) Choose $\overline{p} \in [p_\infty, p_0]$ and consider the classical Poisson model $X(t) = I(t) - \overline{p}t$

2) Solve the Lundberg equation for $X$

3) Simulate $I$ from $P_{\theta_1}$ and keep track of both $V$ and $X$. Stop when at time $t = \tau(u)$ $V(t) < 0$ and observe the response $R_{\theta_1} = e^{-\gamma X(\tau(u))}$ (see Fig. 2 for an illustration). Replicate the experiment a suitable number of times.

Again, an arbitrariness is inherent in the choice of $p$. It is less obvious whether here in fact $P_{\theta_1}(\tau(u) < \infty) = 1$ no matter $\overline{p}$ so we shall sketch a proof of this. Since $\alpha_{\theta_1} E_{\theta_1} Y > \overline{p} \geqq p_\infty$, we can find $p^* < \alpha_{\theta_1} E_{\theta_1} Y$ and $v_0$ such that $p(v) \leqq p^*$ when $v \geqq v_0$. Thus when $V$ comes above $v_0$, it increases less than the classical Poisson model $p^*t - I(t)$ which drifts to $-\infty$ $P_{\theta_1}$-a.s. Hence $[0, v_o]$ is a $P_{\theta_1}$-recurrent set for the Markov process

$\{V(t)\}$ and since it is readily seen that for some $T < \infty$

$$\inf_{v \leqq v_0} P_{\theta_1}(\tau(u) \leqq T \mid V(0) = v) > 0$$

in view of $p_0 < \infty$, $P_{\theta_1}(\tau(u) < \infty) = 1$ follows by standard criteria like [5] Exercise 5.10.

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REFERENCES

[1] S. Asmussen, Conditioned limit theorems relating a random walk to its associate, with applications to risk reserve processes and the GI/G/1 queue, Adv. in Appl. Probab. 14 (1982) 143-170.

[2] S. Asmussen, Approximations for the probability of ruin within finite time, Submitted for publication (1982).

[3] O. Barndorff-Nielsen, Information and Exponential Families in Statistical Theory (Wiley, New York, 1978).

[4] R.E. Beard, T. Pentikäinen and E. Pesonen, Risk Theory (Methuen, London, 1969).

[5] L. Breiman, Probability (Addison-Wesley, Reading, 1968).

[6] R.B. Cooper, Introduction to Queueing Theory, 2nd Ed. (North Holland, New York Oxford, 1981).

[7] H. Cramér, Collective Risk Theory (The Jubilee Volume of Forsäkringsbolaget Skandia, Stockholm, 1955).

[8] M.A. Crane and D.L. Iglehart, Simulating stable stochastic systems I-IV, J. Assoc. Comput. Mach. 21 (1974) 103-113; ibid 114-123; Opns. Res. 23 (1975) 33-45; Management Sci 21 (1975) 1215-1224.

[9] M.A. Crane and A.J. Lemoine, An Introduction to the Regenerative Method for Simulation analysis (Springer, New York, 1977).

[10] J. Grandell, A class of approximations of ruin probabilities, Scand. Actuarial J., Suppl. (1977) 37-52.

[11] J. Grandell, A remark on "A class of approximations of ruin probabilities", Scand. Actuarial J. (1978) 77-78.

[12] J.M. Hammersley and D.C. Handscombe, The Monte Carlo Method (Methuen, London, 1964).

[13] J.M. Harrison and S.I. Resnick, The stationary distribution of a storage process with general release rule, Math. Opns. Res. 1 (1976) 347-358.

[14] D.L. Iglehart and G.S. Shedler, Regenerative Simulation of Response Times in Networks of Queues (Springer, New York, 1980).

[15] N.L. Johnson and S. Kotz, Distributions in Statistics. Continuous Unvariate Distributions - 1 (Houghton Mifflin, Boston, 1969)

[16] J.P.C. Kleijnen, Statistical Techniques in Simulation, Vol. I (Marcel Dekker, New York, 1974).

[17] N.U. Prabhu, On the ruin problem of collective risk theory, Ann. Math. Statist. 32 (1961) 757-764.

[18] N.U. Prabhu, Stochastic Storage Processes. Queues, Insurance Risk and Dams (Springer, New York Heidelberg Berlin, 1980)

[19] R.Y. Rubinstein, Simulation and the Monte Carlo Method (Wiley, New York, 1981).

[20] H.L. Seal, Simulation of the ruin potential of nonlife insurance companies (with discussion), Trans. Soc. Act. XXI (1969) 563-590.

[21] H.L. Seal, Numerical calculation of the probability of ruin in the Poisson/Exponential case, Mitt. Verein Schweiz. Versich. Math. 72 (1972) 77-100.

[22] H.L. Seal, Risk theory and the single server queue, Mitt. Verein Schweiz. Versich. Math. 72 (1972) 171-178.

[23] H.L. Seal, The numerical calculation of U(w,t), the probability of non-ruin in an interval (0,t), Scand. Actuarial J. (1974) 121-139.

[24] H.L. Seal, Survival Probabilities (Wiley, New York, 1978).

[25] D. Siegmund, Corrected diffusion approximations in certain random walk problems, Adv. in Appl. Probab. 11 (1979) 701-719.

[26] A.V. Skorokhod, Studies in the Theory of Random Processes (Addison-Wesley, Reading, 1965).

[27] J.L. Teugels, Estimation of ruin probabilities, Insurance: Mathematics and Economics 1 (1982) 163-175.

[28] O. Thorin and N. Wikstad, Numerical evaluation of the ruin probabilities for a finite period, Astin Bull. 7 (1973) 137-153.

[29] O. Thorin and N. Wikstad, Calculation of ruin probabilities when the claim distribution is lognormal, Astin Bull. 9 (1976) 231-246.

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