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A Note on de Moivre's Limit Theorems: Easy Proofs

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A note on de Moivre's limit theorems: Easy proofs.

by

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We consider the standardized binomial distribution

$$f_n(x_{n,j}) = \sqrt{npq}\ \binom{n}{j} p^j q^{n-j} \quad ,$$

where

$$x_{n,j} = \frac{j-np}{\sqrt{npq}} \ , \ j = 0,1,\dots,n; \ n = 1,2,\dots \ ,$$

and p is fixed, $0<p<1$, $p+q=1$. Together with the points $(x_{n,j}, f_n(x_{n,j}))$, $j=-1,0,1,\dots,n,n+1$, we take the lines connecting these points and continue to call this broken line f_n. Now let n and j tend to infinity in such a way that $x_{n,j}$ tends to a number x, say. Then the sequence f_n has the Gaussian density $f(x) = \exp(-\frac{1}{2}x^2)/\sqrt{2\pi}$ as limit function. This is, of course, well-known; in fact, it is de Moivre's celebrated limit theorem. The proof is usually based on Stirling's formula or Fourier transformation. In what follows we give a proof based on the sequence f'_n of derivatives. We think that a rigorous proof along these lines ought to be available in the literature, but have

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only come across heuristic arguments: hence the present note.

We begin the proof by observing that it is possible to choose the sequence n and corresponding j such that for large n the given x is located between $x_{n,j}$ and $x_{n,j+1}$. Then the derivative $f_n'(x)$ is equal to the slope of the line segment connecting $(x_{n,j}, f_n(x_{n,j}))$ and $(x_{n,j+1}, f_n(x_{n,j+1}))$. The relative slope of this line segment is then

$$\frac{f_n(x_{n,j+1}) - f_n(x_{n,j})}{(x_{n,j+1} - x_{n,j}) f_n(x_{n,j})} = - \frac{npx_{n,j} + \sqrt{npq}}{np + \sqrt{npq}\; x_{n,j} + 1} ,$$

which tends to $-x$, when n and j tends to infinity in the manner described above, and the convergence is uniform on bounded intervals. [A heuristic proof might stop here by noting that $-x$ is also the relative derivative of the Gaussian density at x]. Hence it is enough to show that

$$\lim_{n\to\infty} \frac{f_n'(x)}{f_n(x)} = -x ,$$

uniformly on bounded intervals, implies f_n converges to the Gaussian density.

Since the convergence is uniform, the operations of taking limit and integration can be interchanged, i.e.

$$\begin{aligned} \lim_{n\to\infty} \int_0^u \frac{f_n'(x)}{f_n(x)}\, dx &= \int_0^u \lim_{n\to\infty} \frac{f_n'(x)}{f_n(x)}\, dx \\ &= \int_0^u (-x)\, dx \\ &= -\tfrac{1}{2} u^2 . \end{aligned}$$

On the other hand, f_n is continuous and f_n' is a step function, so that f_n and then $\log f_n$ can be recovered from their derivatives by integration. Hence

$$\lim_{n\to\infty} \int_0^u \frac{f_n'(x)}{f_n(x)}\, dx = \lim_{n\to\infty} \int_0^u d \log f_n(x)$$

$$= \lim_{n\to\infty} \log \frac{f_n(u)}{f_n(0)} \quad ,$$

and since log is continuous, we have

$$\log \lim_{n\to\infty} \frac{f_n(u)}{f_n(0)} = \lim_{n\to\infty} \log \frac{f_n(u)}{f_n(0)}$$

$$= -\tfrac{1}{2} u^2 \quad ,$$

i.e.

(1) $$\lim_{n\to\infty} \frac{f_n(u)}{f_n(0)} = e^{-\frac{1}{2} u^2}$$

Since the convergence in (1) is uniform on bounded intervals, it follows that

(2) $$\lim_{n\to\infty} \int_a^b \frac{f_n(u)}{f_n(0)}\, du = \int_a^b \lim_{n\to\infty} \frac{f_n(u)}{f_n(0)}\, du$$

$$= \int_a^b e^{-\frac{1}{2} u^2}\, du \quad .$$

In particular,

(3) $$\lim_{a\to\infty} \lim_{n\to\infty} \int_{-a}^a \frac{f_n(u)}{f_n(0)}\, du = \lim_{a\to\infty} \int_{-a}^a e^{-\frac{1}{2} u^2}\, du$$

$$= \sqrt{2\pi} \quad .$$

Since f_n is a probability density for a random variable with expectation and variance tending to the expectation and variance, 0 and 1 respectively, of the standardized binomial distribution, an application of Tchebycheff's inequality yields

$$1-\frac{1}{a^2} \leqq \liminf_{n\to\infty} \int_{-a}^{a} f_n(u)\,du \leqq \limsup_{n\to\infty} \int_{-a}^{a} f_n(u)\,du \leqq 1 .$$

Hence from (3)

$$(4) \qquad \lim_{n\to\infty} \frac{1}{f_n(0)} = \sqrt{2\pi} .$$

and the assertion now follows from (1).

In a short-hand notation,we have proved de Moivre's first result

$$\binom{n}{j} p^j q^{n-j} \sim \frac{1}{\sqrt{2\pi npq}}\, e^{-\frac{(j-np)^2}{2npq}} ,$$

where the symbol $\sim$ means that the ratio between the left-hand side and the right-hand side tends to 1.

It is easily seen that

$$\sum_{j:a\leqq x_{n,j}\leqq b} \frac{f_n(x_{n,j})}{\sqrt{npq}}$$

differs from

$$\int_a^b f_n(x)\,dx$$

by at most $\max_j f_n(x_{n,j})/\sqrt{npq}$. Hence, using (2) and (4), we have de Moivre's second result,

$$\sum_{j=a'}^{b'} \binom{n}{j} p^j q^{n-j} \sim \int_a^b \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2}\,dx$$

for

$$\frac{a'-np}{\sqrt{npq}} \to a , \quad \frac{b'-np}{\sqrt{npq}} \to b .$$

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