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1

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Summary Consider a random time τ determined by the evolution of a Markov chain X in discrete time and with discrete state space. Assuming that the pre-τ and post-τ processes are conditionally independent given $X_{\tau-1}$ and $0<\tau<\infty$, it is shown that: (i) the pre-τ process reversed is Markov and in natural duality to X if and only if τ is almost surely equal to a modified cooptional time; (ii) the pre-τ process itself is Markov and an h-transform of X if and only if τ is almost surely equal to a cooptional time with, in general, the possible starts for the pre-τ process restricted. Also, a result is presented characterizing those τ for which the reversed pre-τ process is Markov in natural duality to X, without the assumption of conditional independence.

Throughout this paper we shall maintain the same setup and notation as in Jacobsen and Pitman [2], hereafter referred to as BDC. Thus a Markov process with infinite lifetime is viewed as a Markov probability on the sequence space $\Omega = J^N$, J denoting a countable state space and N the nonnegative integers, while a process with finite lifetime is a Markov probability on the space Ω_Δ of sequences in $J \cup \{\Delta\}$ with the property that once they reach the coffin state Δ, they remain there forever. We shall write $X = (X_n, n \in N)$ for the coordinate process on Ω and $\mathcal{F}_n$ for the σ-algebra determined by $(X_k, k \leqq n)$, $\mathcal{F}$ for the σ-algebra generated by all X_n.

For τ a random time, let $K_\tau = (X_0, \ldots, X_{\tau-1}, \Delta, \Delta, \ldots)$ and $\theta_\tau = (X_\tau, X_{\tau+1}, \ldots)$ denote the pre-τ and post-τ processes respectively.

Let P be a Markov probability on Ω. In BDC characterizations were given of the class of regular birth times and the class of regular death times for P. Here e.g. τ is a regular birth time for P if θ_τ is again Markov and θ_τ and K_τ are conditionally independent given $X_\tau, \tau < \infty$.

From the class of regular birth times one subclass is of particular interest; the regular birth times with the property that θ_τ is Markov with the same transition function p as the given Markov chain P. It is well known and follows easily from the results in Section 3 of BDC, that τ belongs to this subclass if and only if it is P-a.s. equal to an optional time. (See also the introduction of [1]).

Thus in the terminology of Jacobsen [1], the properties of being regular birth and preserving the original transition function, amount to an operational characterization of optional (or stopping) times.

The main purpose of this paper is to present two operational characterizations of cooptional times, Theorems 2 and 3 below.

The first characterization is formulated by demanding that K_τ reversed from τ have a transition function in natural duality to p . For the characterization to work it is necessary to assume that K_τ and θ_τ be independent given $X_{\tau-1}, 0 < \tau < \infty$, which is the conditional independence property required of regular death times. As an intermediate step towards Theorem 2, Theorem 1 describes what happens, when this conditional independence assumption is dropped.

Theorem 2 emerges as the natural analogue of the characterization of optional times mentioned above, see the motivation below following the definition of cooptional times. However, it is also possible to obtain a characterization, working with K_τ in the forward direction of time. This second characterization is presented in Theorem 3.

The paper is concluded with Theorem 4, an analogue for birth times of Theorem 1.

From now on, let P be a given Markov probability on Ω with (stochastic) transition function p . Define

$$\xi(x) = \sum_{n=0}^{\infty} P(X_n = x) \qquad (x \in J) \ ,$$

the <u>occupation measure</u> for P . By discarding all states never reached by the process, we may and shall assume that $\xi(x) > 0$ for all $x \in J$.

Next, introduce for $x,y \in \{\xi < \infty\}$

$$\hat{p}(x,y) = \xi(y)p(y,x)\xi^{-1}(x) \ .$$

We shall call $\hat{p}$ the transition function in <u>natural duality to p</u> (in standard terminology it is in duality to p with respect to the occupation measure). We remind the reader that if $\xi(x) < \infty$,

$$\sum_{y:\xi(y)<\infty} \hat{p}(x,y) \leqq 1 \ ,$$

i.e. $\hat{p}$ is substochastic on $\{\xi < \infty\}$.

Recall from BDC, Section 5, that a random time τ is cooptional (algebraic definition) if either of the following three equivalent properties hold:

(i) $\tau = \sup\{n \geqq 1 : \theta_{n-1} \in F\}$ for some $F \in \mathcal{F}$;

(ii) $\tau \circ \theta_n = (\tau - n)^+$ for all $n \in N$;

(iii) $(\tau = n+1) = (\tau \circ \theta_n = 1)$ and
$(\tau = \infty) = (\tau \circ \theta_n = \infty)$ for all $n \in N$.

<u>Motivation</u> Before continuing with the build up towards Theorem 2, consider for a moment the special situation where P is Markov on Ω_Δ with $P(\zeta < \infty) = 1$, $\zeta = \inf\{n : X_n = \Delta\}$ denoting the lifetime. Defining ξ on J as above, we have $\xi < \infty$ and of course X reversed from ζ is Markov with transition function $\hat{p}$. Suppose now that $\tau \leqq \zeta$ is cooptional. In terms of the reversed process $\hat{X} = (X_{\zeta-1}, \dots, X_0, \Delta, \Delta, \dots)$, τ becomes an optional time $\hat{\tau}$, so by the strong Markov property for $\hat{X}$, $(\hat{X}_{\hat{\tau}}, \hat{X}_{\hat{\tau}+1}, \dots)$ is Markov $(\hat{p})$. But expressing this using X is exactly to say that K_τ reversed under P is Markov $(\hat{p})$. Since cooptional times are the only ones to become optional under timereversal the operational characterization of optional times quoted earlier, gives that K_τ reversed is Markov $(\hat{p})$ iff τ is P-a.s. equal to a cooptional time. Theorem 2 will show that this result carries over to chains with infinite lifetime, where direct timereversal is no longer possible. □

In order to formulate Theorems 1 and 2, we shall need to change the definition of cooptional times slightly, since we shall not be able to say anything about the detailed structure of the sets $(\tau = 0)$, $(\tau = \infty)$. Therefore, call τ a <u>modified cooptional time</u> if there exists τ' cooptional such that

$$(\tau = n) = (\tau' = n) \qquad (n \in N\setminus\{0\}) \ .$$

Note that τ is modified cooptional iff the first half of (iii) holds:

(iii)* $\quad (\tau = n+1) = (\tau \circ \theta_n = 1) \quad$ for all $n \in N$.

For τ an arbitrary random time, let J_τ denote the states that may be visited by K_τ on $(\tau < \infty)$:

$$J_\tau = \{x \in J : \sum_{n=0}^{\infty} P(X_n = x,\ n < \tau < \infty) > 0\} \ ,$$

and finally define the reverse of the pre-τ process as the process $Z = (Z_n, n \in N)$ given by

$$Z_n = \begin{cases} X_{\tau-1-n} & \text{on } (n < \tau < \infty) \\ \Delta & \text{on } (\tau \leqq n) \cup (\tau = \infty) \end{cases} \ .$$

The following two conditions will be needed for Theorems 1 and 2.

(1) $\qquad \xi(x) < \infty \qquad (x \in J_\tau)$,

(2) $\qquad x \in J_\tau,\ p(y,x) > 0 \Rightarrow y \in J_\tau$.

Of course (1) ensures that $\hat{p}$ makes sense on $J_\tau \times J_\tau$. Condition (2) states in a precise manner, that using $\hat{p}$, which is not defined everywhere, only states in J_τ can be reached from J_τ . The condition is important because it implies the following.

<u>Lemma</u> Suppose that (1) and (2) hold. Denote by $\hat{p}^{(m)}$ the m-step transitions determined from $\hat{p}$ on J_τ, i.e.

$$\hat{p}^{(m)}(x,y) = \sum_{z \in J_\tau} \hat{p}^{(m-1)}(x,z)\hat{p}(z,y) \qquad (x,y \in J_\tau) \ ,$$

and by $p^{(m)}$ the m-step transitions determined from p on all of J . Then

$$\hat{p}^{(m)}(x,y) = \xi(y)p^{(m)}(y,x)\xi^{-1}(x) \qquad (x,y \in J_\tau) ,$$

$$\hat{p}(y,\Delta) = \mu(y)\xi^{-1}(y) \qquad (y \in J_\tau) ,$$

where $\hat{p}(y,\Delta) = 1 - \sum_{z \in J_\tau} \hat{p}(y,z)$, $\mu(y) = P(X_0 = y)$. □

The proof is very easy and is left to the reader.

We are now able to state Theorems 1 and 2. In both P is a given Markov probability on Ω. The assumption $P(0 < \tau < \infty) > 0$ in the two theorems is introduced of course to make Z a non-trivial process.

<u>Theorem 1</u> (a) Suppose τ is a random time with $P(0 < \tau < \infty) > 0$ such that (1) and (2) hold. If the reversed pre-τ process Z is Markov with transition function $\hat{p}$ on J_τ, then there exists a function $\phi: J \to [0,1]$ such that for all $n \geqq 1$

(3) $$P(\tau = n | \mathcal{F}_{n-1}) = \phi(X_{n-1}) \qquad P\text{-a.s.}$$

(b) Conversely, if $P(0 < \tau < \infty) > 0$ and (3) holds for some ϕ, then (1) and (2) are satisfied and Z is Markov with transition function p on J_τ. □

<u>Theorem 2</u> (a) Suppose that in addition to the assumptions of part (a) of Theorem 1, it is assumed that K_τ and θ_τ are independent given $X_{\tau-1}$, $0 < \tau < \infty$. Then, if Z is Markov $(\hat{p})$, there exists a modified cooptional time τ' such that $\tau = \tau'$ P-a.s.

(b) Conversely, if $\tau = \tau'$ P-a.s. with τ' modified cooptional, then K_τ and θ_τ are independent given $X_{\tau-1}$, $0 < \tau < \infty$, (1) and (2) hold and Z is Markov $(\hat{p})$. □

<u>Remark</u> For (3) to hold for some ϕ defined on J, it is important only what happens on J_τ, since obviously $\phi = 0$ works on $J \setminus J_\tau$. □

<u>Proof of Theorem 1</u> (a) That Z is a Markov $(\hat{p})$ chain on J_τ (with finite

lifetime) means that for $n \geqq 1$, $x_0,\ldots,x_n \in J_\tau$

$$P(Z_0 = x_0,\ldots,Z_n = x_n) = P(Z_0 = x_0,\ldots,Z_{n-1} = x_{n-1})\hat{p}(x_{n-1},x_n) \quad ,$$

which using the definition of Z translates into

$$\begin{aligned} (4) \qquad & P(X_{\tau-1} = x_0,\ldots,X_{\tau-n-1} = x_n,\ n < \tau < \infty) \\ & = P(X_{\tau-1} = x_0,\ldots,X_{\tau-n} = x_{n-1}, n-1 < \tau < \infty)\hat{p}(x_{n-1},x_n) \ . \end{aligned}$$

Also as a consequence of the Markov assumption, we have that for $0 \leqq n < k, x_0,\ldots,x_n \in J_\tau$

$$\begin{aligned} (5) \qquad & P(X_{k-1} = x_0,\ldots,X_{k-n-1} = x_n, \tau = k) \\ & = P(X_{\tau-1} = x_0,\ldots,X_{\tau-n-1} = x_n, n < \tau < \infty)\hat{f}^{(k-n)}(x_n,\Delta) \ , \end{aligned}$$

where $\hat{f}^{(m)}(x,\Delta)$ is the probability that a Markov chain with transitions $\hat{p}$ and state space $J_\tau \cup \{\Delta\}$, given that it starts at $x \in J_\tau$ dies (reaches Δ for the first time) at time $m \geqq 1$. Since we are assuming that (1) and (2) hold, the lemma applies and a simple computation gives

$$\begin{aligned} (6) \qquad \hat{f}^{(m)}(x,\Delta) & = \sum_{y \in J_\tau} \hat{p}^{(m-1)}(x,y)\hat{p}(y,\Delta) \\ & = \xi^{-1}(x)P(X_{m-1} = x) \qquad (m \geqq 1, x \in J_\tau) \ . \end{aligned}$$

Using (5) as it stands, and with n replaced by $n-1$, (4) may be written

$$\begin{aligned} (7) \qquad & P(X_{k-1} = x_0,\ldots,X_{k-n-1} = x_n, \tau = k)\hat{f}^{(k-n+1)}(x_{n-1},\Delta) \\ & = P(X_{k-1} = x_0,\ldots,X_{k-n} = x_{n-1}, \tau = k)\hat{f}^{(k-n)}(x_n,\Delta)\hat{p}(x_{n-1},x_n) \ . \end{aligned}$$

Suppose now that $P(X_{k-1} = x_0,\ldots,X_{k-n-1} = x_n) > 0$. In particular then $P(X_{k-n-1} = x_n) > 0$, $P(X_{k-n} = x_{n-1}) > 0$, $p(x_n,x_{n-1}) > 0$, and (6) enables us to write (7) as

$$P(\tau = k | X_{k-1} = x_0, \ldots, X_{k-n-1} = x_n) = P(\tau = k | X_{k-1} = x_0, \ldots, X_{k-n} = x_{n-1}) \ .$$

Fixing k and letting $n, x_0, \ldots, x_n \in J_\tau$ vary, it follows that

$$P(\tau = k | \mathcal{F}_{k-1}) = P(\tau = k | X_{k-1}) \tag{8}$$

P-a.s. on $(X_j \in J_\tau,\ 0 \leqq j < k)$. Because of (2), this set equals $(X_{k-1} \in J_\tau)$ P-a.s. and since both sides of (8) vanish on $(X_{k-1} \notin J_\tau)$, (8) holds P-a.s. everywhere on Ω. To obtain the stronger conclusion (3) consider (4) with $n = 1$ and use (5) to rewrite it as

$$\begin{aligned} &P(X_{k-1} = x_0, X_{k-2} = x_1, \tau = k) \hat{f}^{(\ell)}(x_0, \Delta) \\ &\qquad = P(X_{\ell-1} = x_0, \tau = \ell) \hat{f}^{(k-1)}(x_1, \Delta) \hat{p}(x_0, x_1) \end{aligned} \tag{9}$$

for $k \geqq 2, \ell \geqq 1$, $x_0, x_1 \in J_\tau$. Think of x_0 as fixed and suppose that $k \geqq 2, \ell \geqq 1$ are chosen so that $P(X_{k-1} = x_0) > 0$, $P(X_{\ell-1} = x_0) > 0$. Then choose x_1 so that $P(X_{k-1} = x_0, X_{k-2} = x_1) > 0$ and use (6), (8) to reduce (9) to

$$P(\tau = k | X_{k-1} = x_0) = P(\tau = \ell | X_{\ell-1} = x_0) \ ,$$

in other words, for each $x \in J_\tau$, the conditional probability $P(\tau = k | X_{k-1} = x)$ does not depend on k so we may write

$$P(\tau = k | X_{k-1}) = \phi(X_{k-1}) \ ,$$

with $\phi = 0$ on $J \backslash J_\tau$, and (3) follows.

(b) With ϕ given so that (3) holds, we get for $n \in N$, $x_0, \ldots, x_n \in J_\tau$

$$\begin{aligned} &P(Z_0 = x_0, \ldots, Z_n = x_n) \\ &\quad = \sum_{k=n+1}^{\infty} P(X_{k-1} = x_0, \ldots, X_{k-n-1} = x_n, \tau = k) \\ &\quad = \sum_{k=n+1}^{\infty} P(X_{k-1} = x_0, \ldots, X_{k-n-1} = x_n) \phi(x_0) \end{aligned}$$

$$(10) \qquad = \xi(x_n)P^{x_n}(X_1 = x_{n-1},\dots,X_n = x_0)\phi(x_0) .$$

Now, if $x \in J_\tau$ it is possible to find $n, x_0,\dots,x_{n-1} \in J_\tau$, $x_n = x$ so that this probability is > 0. Choosing k so that the k'th term in the sums is > 0, it emerges that $P^{x_n}(X_1 = x_{n-1},\dots,X_n = x_0) > 0$, $\phi(x_0) > 0$, wherefore (10) forces $\xi(x) = \xi(x_n) < \infty$. It is then immediate that Z is Markov $(\hat{p})$.

It remains to establish (2). Introducing the potential kernel (which may be ∞)

$$u(x,y) = \sum_{n=0}^{\infty} p^{(n)}(x,y) \qquad (x,y \in J) ,$$

it is easy to deduce from (3) that

$$P(n < \tau < \infty | \mathcal{F}_n) = U\phi(X_n) ,$$

where U is the potential operator:

$$Uf(x) = \sum_{y \in J} u(x,y)f(y) ,$$

in particular $U\phi < \infty$. But then $z \in J_\tau$ iff

$$0 < \sum_{n=0}^{\infty} P(X_n = z, n < \tau < \infty) = \xi(z)U\phi(z) ,$$

i.e. iff $U\phi(z) > 0$. Therefore, if $x \in J_\tau, y \in J$ with $p(y,x) > 0$ it follows that

$$\sum_{n=0}^{\infty} P(X_n = y, n < \tau < \infty) \geqq \sum_{n=0}^{\infty} P(X_n = y, X_{n+1} = x, n+1 < \tau < \infty)$$

$$= \xi(y)p(y,x)U\phi(x) > 0 ,$$

so that $y \in J_\tau$. □

Proof of Theorem 2 (a) The additional assumption implies that (see Lemmas 3.12 and 5.5 of BDC) there exists $F_{n-1} \in \mathcal{F}_{n-1}$ for $n \geqq 1$ and $G \in \mathcal{F}$ such that

$$(11) \qquad (\tau = n) = F_{n-1}(\theta_{n-1} \in G) \qquad P\text{-a.s.}$$

for $n \geqq 1$, whence

(12) $$P(\tau = n \mid \mathcal{F}_{n-1}) = 1_{F_{n-1}} h(X_{n-1})$$

with $h(x) = P^x G$. Introducing $H = \{h > 0\}$ it is readily checked that

$$P(F_{n-1}, \theta_{n-1} \in G) = P(F_{n-1}, X_{n-1} \in H, \theta_{n-1} \in G) ,$$

so we may and shall assume that $F_{n-1} \subset (X_{n-1} \in H)$ P-a.s. in which case (12) gives

$$D_n := (P(\tau = n \mid \mathcal{F}_{n-1}) > 0) = F_{n-1} .$$

From Theorem 1 (a) we know that (3) holds so that also

$$D_n = (\phi(X_{n-1}) > 0) .$$

Thus $F_{n-1} = (\phi(X_{n-1}) > 0)$ P-a.s. and inserting this in (11) gives

$$(\tau = n) = (\theta_{n-1} \in G') \qquad \text{P-a.s.}$$

where $G' = (\phi(X_0) > 0)G$. But for $n \in \mathbb{N} \setminus \{0\}$, the sets $(\tau = n)$ are mutually disjoint, hence ignoring a P-null set, so are the sets $(\theta_{n-1} \in G')$, wherefore it follows that if τ' is the cooptional time

$$\tau' = \sup\{n \geqq 1 : \theta_{n-1} \in G'\} ,$$

we have $(\tau = n) = (\tau' = n)$ P-a.s. for $n \in \mathbb{N} \setminus \{0\}$, which is exactly to say that τ is P-a.s. equal to a modified cooptional time.

(b) Find τ'' cooptional so that $(\tau' = n) = (\tau'' = n)$ for $n \in \mathbb{N} \setminus \{0\}$. From Section 5 of BDC it is known that τ'' is a regular death time for P, in particular $K_{\tau''}$ and $\theta_{\tau''}$ are independent given $X_{\tau''-1}$, $0 < \tau'' < \infty$. But of course then the same conditional independence holds with τ'' replaced by τ. To complete the proof, merely observe that

$$\begin{aligned} P(\tau = n | \mathcal{F}_{n-1}) &= P(\tau' = n | \mathcal{F}_{n-1}) \\ &= P(\tau' \circ \theta_{n-1} = 1 | \mathcal{F}_{n-1}) \\ &= P^{X_{n-1}}(\tau' = 1) \ , \end{aligned}$$

i.e. (3) holds. Now apply Theorem 1 (b). □

<u>Example</u> We shall present a Markov probability P and a random time τ such that (3) holds but τ is not P-a.s. equal to a modified cooptional time.

The state space is $J = \{x,y,a\}$ and P is determined by some initial distribution with $P(X_0 = a) < 1$ and the transition matrix

$$\begin{array}{c c} & \begin{array}{ccc} x & y & a \end{array} \\ \begin{array}{c} x \\ y \\ a \end{array} & \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

Define τ by

$$(\tau = 1) = (X_1 = x, \sigma = 2)$$

$$(\tau = n) = (\sigma = n + 2) \qquad (n \geqq 2)$$

and $\tau = 0$ otherwise, where $\sigma = \inf\{n \geqq 0 : X_n = a\}$ is the time to absorption in a.

Since the sets $(\tau = n)$ and $(\tau \circ \theta_{n-1} = 1)$ for $n \geqq 2$ are different with respect to P, there is no modified cooptional τ' such that $\tau = \tau'$ P-a.s. On the other hand (3) is satisfied because

$$P(\tau = n | \mathcal{F}_{n-1}) = \begin{cases} P^{X_{n-1}}(\sigma = 3) = \left(\frac{1}{2}\right)^2 \cdot \frac{1}{2} = \frac{1}{8} & \text{on } (X_{n-1} \in \{x,y\}) \\ 0 & \text{on } (X_{n-1} = a) \end{cases}$$

for $n \geqq 2$, and

$$P(\tau = 1 | \mathcal{F}_0) = \begin{cases} P^{X_0}(X_1 = x, \sigma = 2) = \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8} & \text{on } (X_0 \in \{x,y\}) \\ 0 & \text{on } (X_0 = a) \ . \end{cases}$$

□

We shall now discuss what are the characteristic properties of the forward killed process K_τ, when τ is cooptional. Before stating Theorem 3, we need the following definition.

Let p be a (in general substochastic) transition function on J, let $J' \subset J$ and let q be a (substochastic) transition function on J'.

<u>Definition</u> The pair (q,J') is an <u>h-transform</u> of p if there exists a p-excessive bounded function $h: J \to [0,\infty)$ with $h > 0$ on J' and $h = 0$ on $\{x \in J \backslash J' : \sum_{y \in J'} u(y,x) > 0\}$ such that

$$q(x,y) = h^{-1}(x) p(x,y) h(y) \qquad (x,y \in J') \ .$$

□

<u>Remarks</u> Normally one would allow h to be unbounded, but in this paper we shall only encounter bounded excessive functions. Recall that h is p-excessive if

$$\sum_{y \in J} p(x,y) h(y) \leqq h(x) \qquad (x \in J) \ .$$

The set where h is demanded to vanish is of course the collection of states outside J', that may be reached from J' using p-transitions. □

If τ is a random time, denote by J_τ^∞ the state space for K_τ under the given Markov probability P:

$$J_\tau^\infty = \{x \in J : \sum_{n=0}^{\infty} P(X_n = x, \ \tau > n) > 0\} \ .$$

<u>Theorem 3</u> Suppose τ is a regular death time for P and let q be the tran-

sition function for K_τ. Then (q,J_τ^∞) is an h-transform of p if and only if there exists $H\subset J$, τ' cooptional such that P-a.s.

$$(13) \qquad (\tau = n) = (X_0 \in H, \tau' = n) \qquad (1 \leqq n \leqq \infty) .$$

Remark If (13) holds, τ is a regular death time for P, see Theorem 5.2 and the preceding remark of BDC or the statement following (4.1) of [1]. □

Proof The easier part consists in showing that (13) implies that (q,J_τ^∞) is an h-transform. Define $g(x) = P^x(\tau' > 0)$. If $x \in J_\tau^\infty$, find n so that $P(X_n = x, \tau > n) > 0$ and deduce from

$$P(X_n = x, \tau > n) = P(X_0 \in H, X_n = x, \tau' \circ \theta_n > 0)$$

that $g(x) > 0$. Next, let A_τ denote the part of $J\backslash J_\tau^\infty$ which can be reached from J_τ^∞ by p-transitions:

$$A_\tau = \{x \in J\backslash J_\tau^\infty : \sum_{y\in J_\tau^\infty} u(y,x) > 0\} .$$

If $x \in A_\tau$, find $y \in J_\tau^\infty$, n such that $p^{(n)}(y,x) > 0$. Also find m so that $P(X_m = y, \tau > m) > 0$. Then look at

$$P(X_m = y, X_{m+n} = x, \tau > m + n) = P(X_0 \in H, X_m = y) p^{(n)}(y,x) g(x) .$$

Since $x \notin J_\tau^\infty$, the left side is 0. But the choice of m forces $P(X_0 \in H, X_m = y) > 0$, hence $g(x) = 0$.

Thus $g > 0$ on J_τ^∞, $g = 0$ on A_τ, and it only remains to show that g is excessive, which follows from

$$g(x) = P^x(\tau' > 0) \geqq P^x(\tau' > 1) = P^x(\tau' \circ \theta > 0) = P^x g(X_1) ,$$

and to observe that $q(x,y) = g^{-1}(x) p(x,y) g(y)$ for $x,y \in J_\tau^\infty$.

For the converse, let τ be a regular death time for P, such that with

q the transition function for K_τ, (q, J_τ^∞) is an h-transform of p. By Theorem 5.2 of BDC and the remark preceding it, there exists $H \subset J$, $V \subset J \times J$ and $F \in \mathcal{F}$ such that P-a.s. $\tau = \tau_{HVF}$, where

$$(\tau_{HVF} = n) = (X_0 \in H, \tau_{VF} = n) \qquad (1 \leqq n \leqq \infty) ,$$

with $\tau_{VF} = \sup\{1 \leqq n \leqq \tau_V : \theta_{n-1} \in F\}$, $\tau_V = \inf\{n \geqq 1 : (X_{n-1}, X_n) \in V\}$. Furthermore, writing $g(x) = P^x(\tau_{VF} > 0)$, $V^c = (J \times J) \setminus V$,

$$q(x,y) = 1_{V^c}(x,y) g^{-1}(x) p(x,y) g(y) \qquad (x,y \in J_\tau^\infty) \tag{14}$$

and $g > 0$ on J_τ^∞.

As above, we let A_τ denote the states in $J \setminus J_\tau^\infty$ that may be reached from J_τ^∞ using p-transitions.

By assumption there exists h bounded and p-excessive with $h > 0$ on J_τ^∞, $h = 0$ on A_τ such that

$$q(x,y) = h^{-1}(x) p(x,y) h(y) \qquad (x,y \in J_\tau^\infty) . \tag{15}$$

We shall first show that without loss of generality it may be assumed that

$$H \subset J_\tau^\infty , \tag{16}$$

$$\{p > 0\} \cap (J_\tau^\infty \times J_\tau^\infty) \subset V^c \cap (J_\tau^\infty \times J_\tau^\infty) , \tag{17}$$

$$F \subset (X_0 \in J_\tau^\infty) . \tag{18}$$

To establish (16), define $H' = H \cap J_\tau^\infty$. For $1 \leqq n \leqq \infty$, $(\tau_{H'VF} = n) \subset (\tau_{HVF} = n)$ because $H' \subset H$. But the opposite inclusion holds P-a.s. because of the definition of J_τ^∞ and because $\tau = \tau_{HVF}$ P-a.s.

As for (17), from (15) we see that $x, y \in J_\tau^\infty$, $p(x,y) > 0$ forces $q(x,y) > 0$, so that by (14) also $(x,y) \in V^c$.

Defining $F' = F(X_0 \in J_\tau^\infty)$, clearly $\tau_{HVF'} \leqq \tau_{HVF}$ and since

$$(\tau_{HVF'} < \tau_{HVF}) \subset \bigcup_{n=1}^{\infty} (\tau \geqq n, \theta_{n-1} \in F \setminus F') = \bigcup_{n=1}^{\infty} (\tau \geqq n, X_{n-1} \in J \setminus J_\tau^\infty)$$

P-a.s., where by the definition of J_τ^∞, the last event has P-measure 0, we get $\tau_{HVF'} = \tau_{HVF}$ P-a.s. and (18) is proved.

To complete the proof, assume that (16) - (18) hold, introduce

$$\tau' = \sup\{n \geqq 1 : \theta_{n-1} \in F\} ,$$

and define a random time τ_1 by

$$(\tau_1 = n) = (X_0 \in H, \tau' = n) .$$

We shall show that $\tau_1 = \tau_{HVF}$ P-a.s. Clearly $\tau_{HVF} \leqq \tau_1$ and P-a.s. using (16) and the definition of A_τ

$$(\tau_{HVF} < \tau_1) \subset C$$

where, writing $B_\tau = J_\tau^\infty \cup A_\tau$,

$$C = (X_0 \in H,\ \tau_V < \infty,\ \tau' \circ \theta_{\tau_V} > 0,\ p(X_{\tau_V - 1}, X_{\tau_V}) > 0,\ (X_{\tau_V - 1}, X_{\tau_V}) \in B_\tau \times B_\tau) .$$

To see that $PC = 0$, consider $(x,y) \in V$ with $(x,y) \in B_\tau \times B_\tau$, $p(x,y) > 0$. Then $(x,y) \in J_\tau^\infty \times J_\tau^\infty$ is impossible by (17). Also, since h is excessive and $= 0$ on A_τ, from A_τ only transitions to $\{h = 0\}$ can occur, so $(x,y) \in A_\tau \times J_\tau^\infty$ is impossible. Therefore necessarily $y \in A_\tau$, but then $P^y(\tau' > 0) = P^y \bigcup_{n=0}^{\infty} (\theta_n \in F) = 0$ because of (18) and the fact that J_τ^∞ cannot be reached from A_τ, and an application of the strong Markov property now gives $PC = 0$.

□

<u>Remark</u> In the theorem we considered h-transforms for excessive h vanishing on A_τ. It would be nicer of course, if only h vanishing on all of $J \setminus J_\tau^\infty$ were needed, but that is not possible: if τ satisfies (13), as we have seen

$g(x) = P^x(\tau' > 0)$ is excessive. But the function g' defined by $g'(x) = g(x)$ on $J^\infty_\tau \cup \{g = 0\}$ and $= 0$ on $C_\tau = J \setminus (J^\infty_\tau \cup \{g = 0\})$ need not be excessive since it may be possible to have p-transitions from C_τ into J^∞_τ. □

Remark For the theorem to be true, it is essential that in the definition of h-transforms only h which are p-excessive on all of J are considered: Let $L \subset J$ and introduce

$$\tau = \inf\{n \geqq 0 : X_n \in L\} .$$

In general τ certainly does not satisfy (13) P-a.s. But τ is a regular death time for P with, obviously, $J^\infty_\tau \subset J \setminus L$, so the transitions for K_τ become

$$q(x,y) = p(x,y) \qquad (x,y \in J^\infty_\tau) ,$$

which looks exactly as an h-transform with $h(x) = 1_{J^\infty_\tau}(x)$. The point is of course, that $1_{J^\infty_\tau}$ is excessive on all of J iff p-transitions from $J \setminus J^\infty_\tau$ to J^∞_τ are impossible in which case P-a.s.

$$\tau = \sup\{n \geqq 1 : X_{n-1} \in J \setminus L\}$$

in agreement with the theorem. □

Remark With Theorem 3 in mind, a natural question to ask is whether all h-transforms (for bounded h) can be obtained by killing at cooptional times. The answer is no. To see this consider the example following the proof of Theorem 2. The important point now is that the state space is finite and that there is an absorbing state which is reached with certainty, since, as is easily seen, this implies that there is a countable collection $\mathcal{G}$ of $\mathcal{F}$-measurable sets, such that for $x \in J$, $F \in \mathcal{F}$, $P^x F > 0$ is possible only if $F \in \mathcal{G}$. Consequently, if $F \in \mathcal{F} \setminus \mathcal{G}$, $P(\theta_{n-1} \in F) = 0$ for all $n \geqq 1$ and $\sup\{n \geqq 1 : \theta_{n-1} \in F\} = 0$ P-a.s., i.e.

up to P-equivalence there are only countably many cooptional times. But it is readily verified, that even after normalizing with $h(x) = 1$, $h(a) = 0$, there are uncountably many bounded, excessive h. □

Remark If τ is given by (13) with τ' cooptional of course the transition functions for K_τ and $K_{\tau'}$ are the same: only the initial laws for the two killed processes are different. But while $K_{\tau'}$ reversed has transitions $\hat{p}$, the transitions for K_τ reversed are

$$\hat{p}_H(x,y) = \xi_H(y)p(y,x)\xi_H^{-1}(x) \qquad (x,y \in J_\tau) ,$$

where $\xi_H(z) = \sum_{n=0}^{\infty} P(X_0 \in H, X_n = z)$. Using Theorem 2 and the fact that τ does not have the homogeneity property (iii)* of all modified cooptional times, it is not difficult to see that $\hat{p}_H = p_H$ on $J_\tau \times J_\tau$ iff $\xi_H = \xi$ on J_τ. □

Although somewhat out of line with the main context of this paper, we shall conclude with a discussion of birth times that preserve the original transition function p without having the conditional independence property demanded of regular birth times.

Theorem 4 (a) Suppose τ is a random time with $P(\tau < \infty) > 0$ such that given τ within $(\tau < \infty)$, the post-τ process θ_τ is Markov (p). Then

(19) $$P(\tau = n | \theta_n) = P(\tau = n | X_n) \qquad (n \in N) .$$

(b) Conversely, if $P(\tau < \infty) > 0$ and (19) holds, then given τ within $(\tau < \infty)$, θ_τ is Markov (p).

Proof That θ_τ given τ is Markov (p) amounts to saying that

$$P(X_\tau = x_0, \ldots, X_{\tau+m} = x_m, \tau = n) = P(X_\tau = x_0, \ldots, X_{\tau+m-1} = x_{m-1}, \tau = n)p(x_{m-1}, x_m)$$

for all $n \geqq 0, m \geqq 1, x_0, \ldots, x_m \in J$. Rewriting the left as

$$P(X_n = x_0, \ldots, X_{n+m} = x_m) P(\tau = n | X_n = x_0, \ldots, X_{n+m} = x_m) ,$$

and the right in a similar fashion, it is clear that

$$P(\tau = n | X_n, \ldots, X_{n+m}) = P(\tau = n | X_n, \ldots, X_{n+m-1}) ,$$

and (19) follows. The converse is just as easy. □

Remark There does not appear to be any reasonable characterization of the τ that makes θ_τ Markov (p) without conditioning on the value of τ. If (19) holds, then certainly θ_τ is Markov (p), but the converse is not true: consider the example from p. 439 of BDC,

$$\tau = \inf\{n \geqq 1 : (X_n, \ldots, X_{n+k}) = (X_0, \ldots, X_k)\} .$$

As pointed out there, if P is recurrent, θ_τ is Markov (p). But it is easy to see that (19) does not hold in general. □

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