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Institute of Mathematical Statistics
University of Copenhagen

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By Søren Asmussen, University of Copenhagen.

ABSTRACT

A number of approximations for the probability of ruin before time T are surveyed, some new ones are suggested and numerical comparisons with the exact values are given for the Poisson/Exponential case. The approximations include normal ones and diffusion types. A variant and refinement of the classical diffusion approximation is derived and found to have a quite remarkable fit in the situations of main interest in risk theory.

1. INTRODUCTION

Consider a risk reserve process

$$R(t) = u + t - \sum_{n=1}^{N(t)} Y_n$$

with initial risk reserve $R(0) = u$, unit premium intensity, claim sizes $Y_1, Y_2, \ldots$ which are i.i.d. with $EY = 1$ and with claims arriving according to a Poisson process $\{N(t)\}_{0 \leq t < \infty}$ with intensity $\rho < 1$. Thus the interclaim times $Z_1, Z_2, \ldots$ are i.i.d. with $P(Z > z) = e^{-\rho z}$ and the safety loading is $\eta = (1-\rho)/\rho$. We are interested in the probabilities

$$\psi(u,T) = P(\inf_{0 \leq t \leq T} R(t) < 0), \quad \psi(u) = \psi(u,\infty) = P(\inf_{0 \leq t < \infty} R(t) < 0)$$

of ruin before time T, resp. of ultimate ruin (for interpretations in queueing and storage theory, see Section 2).

Even for the Poisson/Exponential (P/E) case $P(Y > y) = e^{-y}$, the explicit expressions for $\psi(u,T)$ are incomprehensible and their numerical evaluation requires some care, cf. e.g. Seal (1972a),(1974) and the concluding remarks in Section 7. For more general Y, such explicit expressions can usually only be found in terms of double Laplace transforms and their derivation requires much ingenuity. See Cramér (1955) for the general case as well as a number of papers in the literature for particular examples, e.g. Arfwedson (1953) and Thorin & Wikstad (1976). In view of the complications encountered, it is not surprising that approximations have received considerable attention. Some early expositions are in Cramér (1955) and Segerdahl (1959), whereas one of the main later developments relevant for the present paper is the introduction of diffusion approximations, see e.g. Iglehart (1969) and Grandell (1977).

Most approximations (though not all) are suggested by a limit theorem, say as $u \to \infty$, $T \to \infty$ or $\rho \uparrow 1$ with the other parameters fixed. This gives some rough guidelines as to when to use a particular approximation but clearly some empirical work is required for judging the robustness, i.e. the range in which the accuracy is reasonable. For examples of such investigations, see Cramér (1955) p. 45, Grandell & Segerdahl (1971), Beekman & Bowers (1972), Beard (1975) and Grandell (1977). Our aim here is twofold, first to undertake a more systematic numerical comparison (emphasizing the finite time problem) and next to improve upon some of the approximations studied by using the empirical results as guidelines.

The paper is organized as follows. Sections 2-3 contain some preliminaries, in particular a fundamental imbedding of the process in a whole class of processes defined in terms of an exponential family (i.e. of conjugate distributions or Esscher transforms). In Section 4, normal approximations are discussed with Segerdahl (1955) as point of departure. Section 5 deals with diffusion approximations. It is shown here as one of the main findings of the paper that ideas in Siegmund (1979), considered here for the first time in the risk theoretic setting, lead to new variants and refinements which seem superior to all approximations in the literature and also have an extremely good fit in a very wide range of parameters. In Section 6, we briefly discuss a classical formula put in its final form by Teugels (1982) and finally, in Section 7 we comment upon some of the computational aspects.

The numerical material for the P/E case has been extracted from two sets A,B of tables, which can be obtained in their whole upon request from the author. Set A gives $\psi(u,T)$ and the corresponding approximations as function of T for the set of values given in Table 1. A sample table and some further explanation can be found in the Appendix. The Figures 1-8 in the body of the paper are based on this set of tables.

Table 1 Parameters for Set A of tables

ρ	0.2	0.5	0.8	0.9
$\eta(\%)$	400	100	25	11
$\psi(u)\,(\%)$	10,2,1,0.2	25,5,2.5,0.5	40,8,4,0.8	45,9,4.5,0.9

Set B gives the relative error, as defined by

$$\frac{\text{approximation}}{\text{exact}} - 1 ,$$

as function of $\rho = 0.50 - 0.98$ (i.e. $\eta = 100\% - 2\%$) for the four values of u making $\psi(u) = 50\%$, 10%, 5% and 1% and corresponding T-values defined so as to make approximately $\psi(u,T)/\psi(u) \cong 25\%$, 50%, 75% and 95%.

We remark that this covers what has been argued to be the relevant range for risk theory, viz. with small safety loading (say $\eta < 20\%$) and the initial risk reserve u so large as to make $\psi(u)$ small, say in the range $1\% - 5\%$ (an important problem left open is, however, T-values so small that $\psi(u,T) \ll \psi(u)$) .

Concerning the presentation of the paper, we finally remark that since the point of view is to a large extent empirical, we have sometimes slightly relaxed from complete mathematical rigour. The aim is rather to motivate why the particular approximations are reasonable, and to present the calculations at such level of detail that the values of the constants can be checked.

2. SOME PRELIMINARY REMARKS

It will be convenient to express $\psi(u,T),\psi(u)$ in terms of

$$X(t) = R(0) - R(t) = \sum_{n=1}^{N(t)} Y_n - t, \quad M = \sup_{0 \leqq t < \infty} X(t) ,$$

$$M(T) = \sup_{0 \leqq t \leqq T} X(t), \quad \tau = \tau(u) = \inf\{t \geqq 0 : X(t) > u\}$$

(the dependence of τ on u is most often suppressed for notational convenience). Indeed, it is clear that

$$(2.1) \qquad \psi(u,T) = P(M(T) > u) = P(\tau \leqq T) ,$$

$$(2.2) \qquad \psi(u) = P(M > u) = P(\tau < \infty) ,$$

so that our risk theoretic problem is equivalent to the first passage time problem for the process $\{X(t)\}_{t \geqq 0}$. To this end, it is basic to note that $\{X(t)\}_{t \geqq 0}$ (as a compound Poisson process with a drift term) has stationary independent increments, i.e. represents the continuous time counterpart of a random walk. This suggests that many random walk results can be converted to the present setting. In some cases the translation is a straightforward application of the method of discrete skeletons (Kingman, 1963), in other cases some additional arguments can be required say to identify the value of certain constants. Examples abound in the literature and in the next sections.

We also point out the relation to queueing and storage theory, which at one hand shows that the discussion has a wider scope than just to risk theory alone and at the other sometimes is convenient by allowing the application of queueing and storage results. Indeed, if $V(t)$ is the virtual waiting time (residual work) of an initially empty M/G/1 queue with service times $Y_1, Y_2, \ldots$

and arrival intensity ρ , then

(2.3) $\psi(u,T) = P(V(T) > u)$, $\psi(u) = P(V(\infty) > u)$

(with $V(\infty)$ referring to the steady state). See Seal (1972b) or Prabhu (1961, 1980). The process $\{V(t)\}_{t\geq 0}$ can also be interpreted as the content of the infinite dam with compound Poisson input and constant release rate. Note that (in view of $EY = 1$) ρ is simply the traffic intensity of the queue. The time for the risk process has been chosen so as to make the connection to the queue as simple as possible but the translation to other cases is of course immediate. E.g. the connection to operational time T_{op} (with claims arriving at unit rate, a scaling standard in much of the literature) is simply given by $T_{op} = \rho T$. In particular, the present definition $\eta = (1-\rho)/\rho$ of the safety loading η coincides with the usual one for operational time, viz. the premiums $1/\rho$ received within one unit operational time minus the expected total claims = 1.

3. EXPONENTIAL FAMILIES AND THE FUNDAMENTAL IDENTITY OF SEQUENTIAL ANALYSIS

As motivation for introducing the parameters to follow below, we first recall two classical approximations, viz.

(3.1) $$\psi(u) \cong Ce^{-\gamma_1 u} \quad \text{as } u \to \infty \qquad \text{(Cramér-Lundberg)}$$

(3.2) $$\psi(u) - \psi(u,T) \cong C_1 u T^{-3/2} e^{-\gamma_0 u - \gamma_2 T} \quad \text{as } u, T \to \infty,\ u = o(T^{1/2}) ,$$

see e.g. Cramér (1955). To determine first γ_0, γ_1, let

$$\phi(\beta) = E\exp\{\beta Y\},\quad \kappa(\beta) = \rho(\phi(\beta) - 1) - \beta = \log E\exp\{\beta X(t)\}/t \ .$$

The slope of κ at 0 is negative since $\rho EY = \rho < 1$ and κ is strictly convex. Hence $\gamma_0, \gamma_1 > 0$ defined as solutions of

(3.3) $$\kappa'(\gamma_0) = \rho\phi'(\gamma_0) - 1 = 0,\quad \kappa(\gamma_1) = \rho(\phi(\gamma_1) - 1) - \gamma_1 = 0 ,$$

are necessarily unique. The existence of γ_0, γ_1 as well as of certain higher order derivatives of ϕ at γ_1 is, however, an assumption on the tail of Y. It will be made for the rest of the paper since it is minimal for the main results to be discussed, but it should be noted that it excludes certain claim size distributions encountered in the literature. Examples are the lognormal distribution and the Pareto distribution.

Examination of the expressions in Cramér (1955) for the remaining constants in (3.1), (3.2) reveal that these are given in terms of the moments of certain conjugate distributions associated with γ_0, γ_1. We shall here use the entire corresponding exponential family, which will be described in terms of probability measures P_θ governing risk processes with arrival intensities and claim size distributions given by

$$\rho_\theta = \rho\phi(\theta + \gamma_0),\quad \phi_\theta(\beta) = E_\theta \exp\{\beta Y\} = \frac{\phi(\beta+\theta+\gamma_0)}{\phi(\theta+\gamma_0)} \ .$$

Thus our initial process corresponds to $P = P_{\theta_0}$ with $\theta_0 = -\gamma_0$ and this choice of origin ensures that the probability $P_\theta(\tau < \infty)$ of ultimate ruin is < 1 when $\theta < 0$ and $= 1$ when $\theta \geqq 0$ (this is because of $\rho_\theta E_\theta Y \lesseqgtr 1$ precisely when $\theta \lesseqgtr 0$). The exponential family interpretation follows from

$$\kappa_\theta(\beta) = \log E_\theta \exp\{\beta X(t)\}/t = \rho_\theta(\phi_\theta(\beta) - 1) - \beta$$

$$= \rho_0(\phi_0(\theta + \beta) - \phi_0(\theta)) - \beta = \kappa_0(\theta + \beta) - \kappa_0(\theta), \text{ i.e.}$$

(3.4) $$E_\theta e^{\beta X(t)} = \frac{E_0 e^{(\theta+\beta)X(t)}}{E_0 e^{\theta X(t)}} .$$

See also Küchler & Küchler (1981) and the references there.

The following relation will be crucial in the remainder of the paper:

<u>Lemma 3.1</u> <u>Let</u> $B(u) = X(\tau(u)) - u$ <u>be the overshot</u>. <u>Then for all</u> θ', θ''

$$\psi_{\theta'}(u,T) = P_{\theta'}(\tau \leqq T) =$$

(3.5a) $$e^{(\theta'-\theta'')u} E_{\theta''}[\exp\{(\theta' - \theta'')B(u) - \tau[\kappa_0(\theta') - \kappa_0(\theta'')]\}; \tau \leqq T] =$$

(3.5b) $$e^{(\theta'-\theta'')u} E_{\theta''}[\exp\{(\theta' - \theta'')B(u) - \tau[\rho(\phi(\theta'+\gamma_0) - \phi(\theta''+\gamma_0)) - (\theta'-\theta'')]\}; \tau \leqq T]$$

<u>Proof</u> According to the fundamental identity of sequential analysis it holds for any stopping time τ^* and any event $F \subseteqq \{\tau^* < \infty\}$ in the usual pre-τ^*-σ-algebra that

$$P_{\theta'} F = E_{\theta''}[\exp\{(\theta' - \theta'')X(\tau^*) - \tau^*[\kappa_0(\theta') - \kappa_0(\theta'')]\}; F] .$$

Indeed, the discrete time case is treated at a number of places in the literature (e.g. Siegmund, 1979; see also von Bahr, 1974, and Asmussen, 1982) and the continuous time case follows easily by discrete approximations. To obtain the lemma, let $\tau^* = \tau$, $F = \{\tau \leqq T\}$ and note that $X(\tau) = u + B(u)$. □

Corollary 3.2 Let $\theta_1 = \theta_0 + \gamma_1$ with $\gamma_1 > 0$ defined by (3.3). Then

$$(3.6) \qquad \psi(u,T) = \psi_{\theta_0}(u,T) = e^{-\gamma_1 u} E_{\theta_1}[\exp\{-\gamma_1 B(u)\}; \tau \leqq T] .$$

Proof The coefficient to τ in (3.5b) vanishes if $\theta' = \theta_0, \theta'' = \theta_1$. □

The formula (3.6) is the starting point for von Bahr (1974), Siegmund (1975) and Asmussen (1982). It should be noted that in these and a number of other references the exponential family used is the one generated by $Y - Z$, i.e. obtained by replacing $X(t)$ by $Y - Z$ in (3.4). What regards the definitions of P_0 and P_{θ_1}, which are the most important P_θ's, this can be checked to be immaterial.

To apply formulas (3.5), (3.6), it is crucial to have some information on $B(u)$. An important fact is the existence of a limiting distribution, conveniently expressed by introducing a random variable $B(\infty)$ such that for any $\theta \geqq 0$ $B(u) \to B(\infty)$ as $u \to \infty$ in P_θ-distribution. This follows because the process $X(t)$ can only increase at the time t of claims where the values are the consecutive values of the random walk $S_n = \Sigma_1^n \{Y_k - Z_k\}$. Hence the $B(u)$-process is simply the overshot process of $\{S_n\}$, for which the existence of a limiting distribution is well-known. In view of the exponential distribution of Z_k it is even possible to evaluate the P_θ-distribution of $B(\infty)$ explicitly, cf. Feller (1971) Ch. XI. We shall not give the somewhat complicated expressions, since we shall only need simple functionals which can easily be evaluated directly. E.g. Feller (1971) pp. 377-378, 411, uses such an approach to compute the Cramér-Lundberg constant

$$(3.7) \qquad C = E_{\theta_1} e^{-\gamma_1 B(\infty)} = \frac{1-\rho}{(\gamma_1+\rho) E_{\theta_1} Y - 1}$$

(the first equality is a consequence of (3.6) and the last of Feller's expressions and integration by parts).

We conclude this section by considering the P/E case $P(Y > y) = e^{-y}$, i.e. $\phi(\beta) = (1-\beta)^{-1}$, where it follows by elementary calculus that

(3.8) $\gamma_0 = -\theta_0 = 1 - \rho^{\frac{1}{2}}, \quad \gamma_1 = \theta_1 - \theta_0 = 1 - \rho$,

(3.9) $P_0(Y > y) = e^{-\rho^{\frac{1}{2}} y}, \quad P_0(Z > z) = e^{-\rho^{\frac{1}{2}} z}$,

(3.10) $P_{\theta_1}(Y > y) = e^{-\rho y}, \quad P_{\theta_1}(Z > z) = e^{-z}$.

The exponential family approach leads here to even simpler expressions for the ruin probabilities. Indeed, from properties of the exponential distribution it is clear that if $\theta \geqq 0$, then $B(u)$ is independent of τ with $P_\theta(B(u) > b) = e^{-b/E_\theta Y}$. In particular, Corollary 3.2 yields at once the following result:

<u>Corollary 3.3</u> <u>In the P/E case</u>

(3.11) $\psi(u,T) = \psi_{\theta_0}(u,T) = \rho e^{-(1-\rho)u} P_{\theta_1}(\tau \leqq T)$.

That is, the problem of computing $\psi(u,T)$ is here equivalent to the study of the P_{θ_1}-distribution of the first passage time $\tau(u)$. As will be seen, a similar point of view for the general case underlies much of the present paper, the foundation being Corollary 3.2 and certain asymptotic independence properties of $B(u)$ and $\tau(u)$.

4. NORMAL APPROXIMATIONS

It was shown by Segerdahl (1955) that the time τ to ruin is asymptotically normal given $\{\tau<\infty\}$ as $u\to\infty$. The parameters are $(\lambda u,\omega^2 u)$ where

$$(4.1)\qquad \lambda=\frac{1}{\rho_{\theta_1}E_{\theta_1}Y-1}=\frac{1}{(\rho+\gamma_1)E_{\theta_1}Y-1}\ ,\quad \omega^2=\lambda^3\rho_{\theta_1}E_{\theta_1}Y^2=\lambda^3(\rho+\gamma_1)E_{\theta_1}Y^2\ .$$

In conjunction with Cramér's estimate (2.1) for the probability of the conditioning event this suggests the approximation

$$(4.2)\qquad \psi(u,T)=P(\tau\leq T)\simeq Ce^{-\gamma_1 u}\phi\left(\frac{T-\lambda u}{\omega u^{\frac{1}{2}}}\right)\ .$$

For queueing analogues, see Asmussen (1981,1982) and for a generalization to renewal arrivals of claims, von Bahr (1974). It should be noted that in these cases the determination of C presents a complicated random walk problem, in contrast to the simple expression (3.7) available here. An alternative proof of (4.2) was given independently by Siegmund (1975) and Asmussen (1982). The idea is to appeal to (3.6) and show that $(\tau,B(u))$ are asymptotically independent w.r.t. P_{θ_1} with the asymptotic distribution of τ as above. Letting $C=E_{\theta_1}e^{-\gamma_1 B(\infty)}$ then yields (4.2).

The numerical comparisons for the P/E case of (4.2) with the exact values, as given in Set A of tables, shows as expected that the fit improves as u increases. However, the fit is quite poor for the small value $\rho=0.2$ (see e.g. Figure 1 below). For $\rho=0.5$, 0.8 and 0.9 the situation is better, but not here either are the deviations negligible for even quite large u. Three examples are given in Figures 1 - 3, depicting $\psi(u,T)$ and the approximations as functions of T.

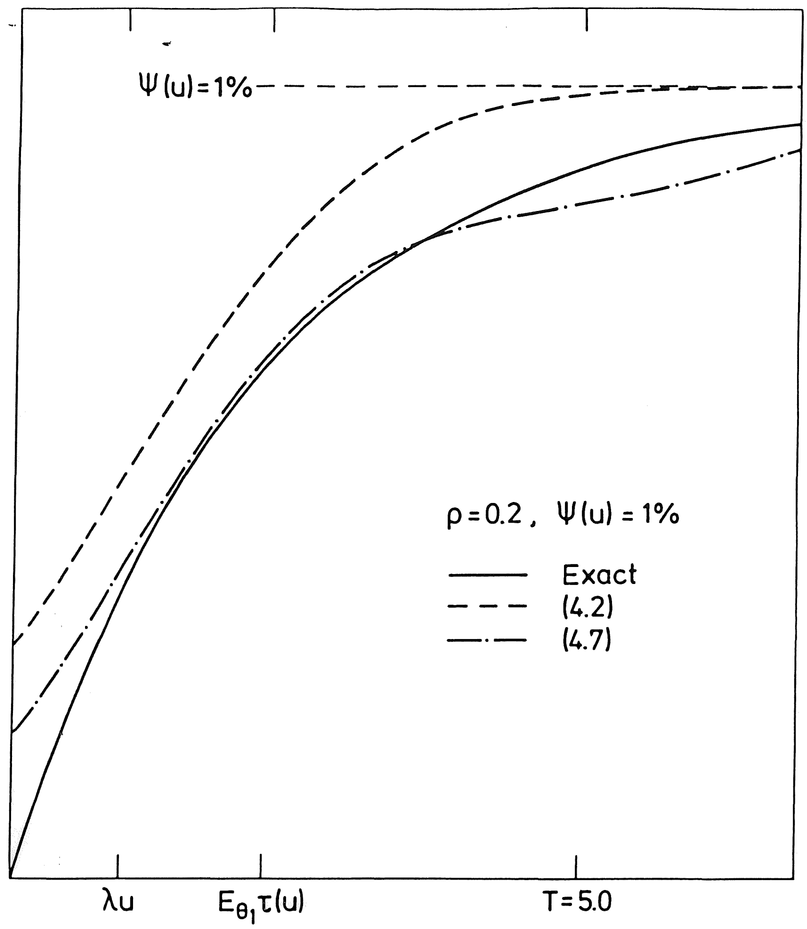


Figure 1

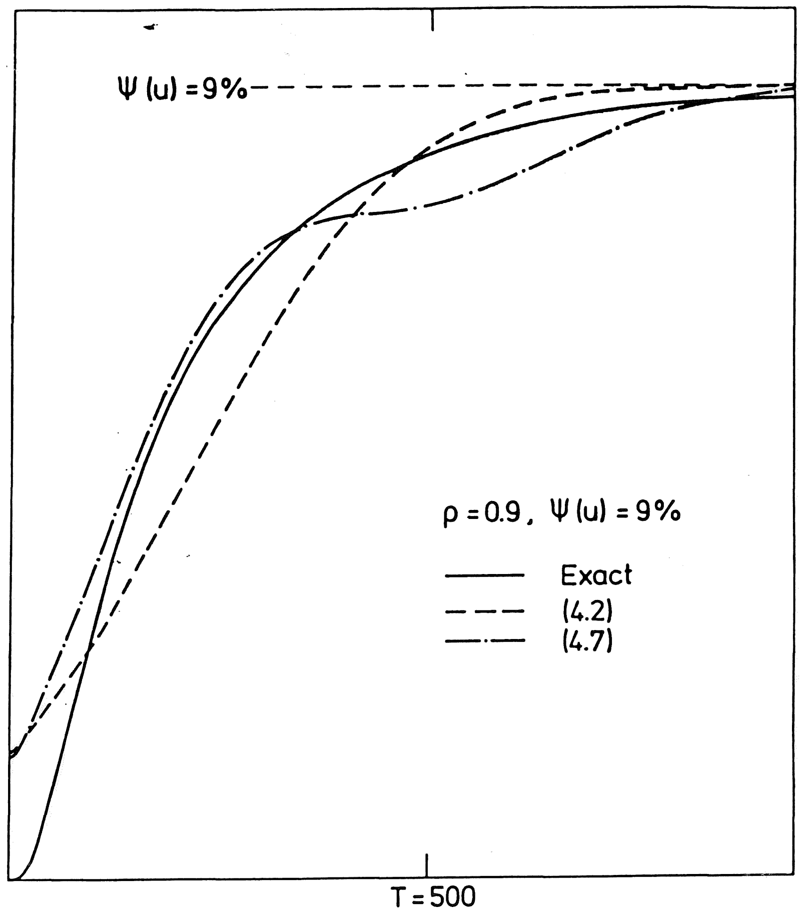
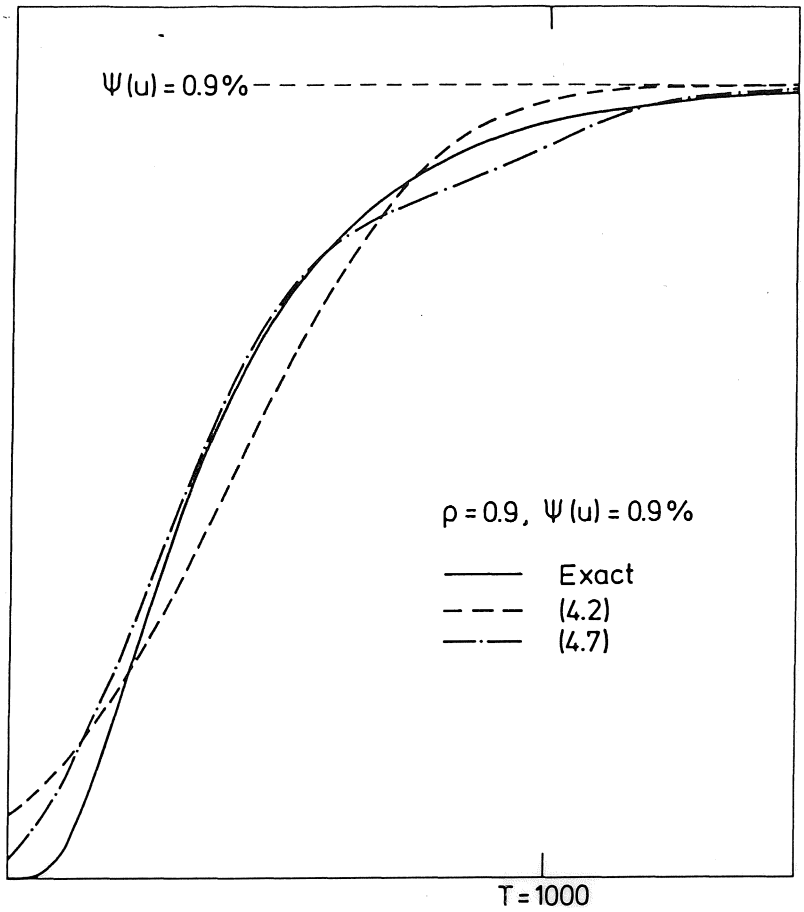


Figure 2

Figure 3



In order to arrive at a deeper understanding of the normal approximation, it is necessary to look closer into the dependence of the distribution of $\tau = \tau(u)$ on u :

Lemma 4.1 In the P/E case,

(4.3) $$E_{\theta_1} e^{-\beta\tau(u)} = e^{-\lambda u}(1 - \frac{\lambda}{\rho})$$

in a neighbourhood of $\beta = 0$, where $\lambda = \lambda(\beta) = (\rho - 1 - \beta + \sqrt{(1-\rho+\beta)^2 + 4\rho\beta})/2$.

Proof By the continuous time analogue of a standard random walk result (Neveu, 1972, IV-4) it follows that for λ real

$$Y(t) = \exp\{\lambda X(t) - t\kappa_{\theta_1}(\lambda)\}$$

is a P_{θ_1}-martingale and that

$$1 = E_{\theta_1} Y(\tau) = E_{\theta_1} \exp\{\lambda X(\tau) - \tau\kappa_{\theta_1}(\lambda)\} .$$

Writing $X(\tau) = u + B(u)$ and using the independence of τ and $B(u)$ yields

$$E_{\theta_1} \exp\{-\tau\kappa_{\theta_1}(\lambda)\} = e^{-\lambda u}(1 - \frac{\lambda}{\rho}) . \tag{4.4}$$

Hence substituting $\beta = \kappa_{\theta_1}(\lambda) = \lambda/(\rho - \lambda) - \lambda$, the result follows (the sign of $\sqrt{\cdots}$ is + since $\kappa_{\theta_1}(\lambda)$ is monotonely increasing at $\lambda = 0$). □

The formula (4.3) has a simple probabilistic interpretation. The factor $e^{-\lambda u}$ is the Laplace transform of a member H_u of a convolution semigroup $\{H_u\}_{u \geqq 0}$ whereas $1 - \lambda/\rho$ is the Laplace transform of a distribution K independent of u. H_u, K can be identified as follows. Consider a process $X^*(t)$ evolving as $X(t)$ but started at an exponential distributed initial value $X^*(0)$ with mean $1/\rho$, and the corresponding first passage times $\tau^*(u)$. Conditioning on $\tau(0)$ shows that $\tau(u) \overset{D}{=} \tau(0) + \tau^*(u)$ where the terms are independent. Similarly $\tau^*(u_1 + u_2)$ is the independent sum of random variables distributed as $\tau^*(u_1)$, resp. $\tau^*(u_2)$ and it follows that $H_u(t) = P_{\theta_1}(\tau^*(u) \leqq t)$, $K(t) = P_{\theta_1}(\tau(0) \leqq t)$.

Differentiating the transforms of H_u, K, it follows by elementary calculations that

<u>Corollary 4.2</u> <u>In the</u> P/E <u>case</u>

$$E_{\theta_1} \tau = \frac{\rho}{1-\rho} u + \frac{1}{1-\rho} , \quad \mathrm{Var}_{\theta_1} \tau = \frac{2\rho}{(1-\rho)^3} u + \frac{1+\rho}{(1-\rho)^3} \tag{4.5}$$

$$E_{\theta_1}(\tau - E_{\theta_1}\tau)^3 = \frac{6\rho(1+\rho)}{(1-\rho)^5} u + \frac{2\rho^2+8\rho+2}{(1-\rho)^5} . \tag{4.6}$$

That the fit of the normal approximation is not entirely satisfying may now be understood from the two following considerations:

1° The asymptotic normality of the distribution of τ, viz. $H_u * K$, stems from the central limit theorem for the convolution semigroup $\{H_u\}_{u \geqq 0}$. Besides the rate of convergence, it is also important that K should be small compared to H_u. A simple way to check this is to compare the means of H_u and $H_u * K$ as given by (4.5), cf. Table 2 based on the same parameters as Table 1.

Table 2 Comparison of the mean of the normal approximation with the true value

ρ	$\psi(u)$ (%)	λu	$E_{\theta_1} \tau(u)$
0.2	10.0	0.21	1.46
0.2	2.0	0.71	1.96
0.2	1.0	0.94	2.19
0.2	0.2	1.44	2.69
0.5	25.0	1.39	3.39
0.5	5.0	4.61	6.61
0.5	2.5	5.99	7.99
0.5	0.5	9.21	11.21
0.8	40.0	13.9	18.9
0.8	8.0	46.1	51.1
0.8	4.0	59.9	64.9
0.8	0.8	92.1	97.1
0.9	45.0	62.4	72.4
0.9	9.0	207	217
0.9	4.5	270	280
0.9	0.9	414	424

It is seen that, in particular for ρ small, K is by no means negligible compared to H_u. A comparison of (4.2) and (4.5) shows that the normal approximation on the average expects ruin to occur earlier than is actually the case. A striking illustration of these phenomena can be found in Figure 1, where the approximation is always larger than the true value and the deviation of the means considerable both absolutely and in units of the standard deviation.

2^o The fit of the normal approximation for H_u is well-known to depend heavily on the higher cumulants. The third one is given by the first term of (4.6) and a similar numerical study as in 1° shows that it is not negligible either.

Whereas we have no immediate suggestions for circumventing 1° (or even put the problem in a precise form for general claim size distributions), it seems obvious for 2° to invoke the higher cumulants of τ and use an Edgeworth expansion (cf. Gnedenko and Kolmogorov, 1954, pp. 192-193 for the non i.i.d. case) to produce correction terms for $\Phi(\cdot)$ in (4.2). We shall only consider the first-order correction, which suggest a formula of the type

$$\psi(u,T) \cong Ce^{-\gamma_1 u}\{\Phi(T(u)) + \frac{E_{\theta_1}(\tau - E_{\theta_1}(\tau))^3}{6(\mathrm{Var}_{\theta_1}\tau)^{3/2}}(1 - T(u)^2)\frac{1}{\sqrt{2\pi}}e^{-T(u)^2/2}\}, \tag{4.7}$$

$$T(u) = \frac{T - E_{\theta_1}\tau}{\mathrm{Var}_{\theta_1}\tau} .$$

In fact:

<u>Corollary 4.3</u> <u>In the</u> P/E <u>case</u> (4.7) <u>is valid as</u> $u\to\infty$ <u>up to terms which are</u> $o(e^{-\gamma_1 u}/u^{\frac{1}{2}})$.

This follows simply by writing

$$\psi(u,T) = Ce^{-\gamma_1 u}\int_0^\infty H_u(T-k)\,dK(k) ,$$

replace $H_u(T-k)$ by the first order Edgeworth expansion and use Taylor's formula. We omit the details.

It should be noted that for general claim size distributions the dependence between τ and $B(u)$ presents an additional problem, so that it is not even immediately clear here whether the constants in (4.7) are the proper ones.

The numerical results show that (4.7) improves somewhat upon (4.2), see again Figures 1 - 3. Generally, the agreement is quite good for intermediate values of T whereas discrepancies remain at the tails $T\to 0$, $T\to\infty$. The fit does not significantly change as ρ varies from 0.5 to 1, as illustrated in Table 3 of the relative error.

Table 3 Relative error (%) of (4.7) at the 50% fractile of (4.2)

ρ	η(%) \ $\alpha=\psi(u)$	50%	10%	5%	1%
0.50	100	–	6.4	4.1	2.1
0.60	67	65	5.1	3.6	2.0
0.70	43	24	4.6	3.3	1.9
0.80	25	17	4.3	3.2	1.9
0.85	18	15	4.2	3.1	1.9
0.90	11	14	4.2	3.1	1.8
0.92	8.7	14	4.2	3.1	1.8
0.94	6.4	14	4.1	3.0	1.8
0.96	4.2	14	4.1	3.0	1.8
0.98	2.0	14	4.0	3.0	1.8

This is to be understood as follows: For a given value of $\alpha = 50\%,\dots,1\%$ and $\rho = 0.50,\dots,0.98$ we choose first $u = u(\rho)$ to make $\psi(u) = \alpha$ and next $T = T(u(\rho),\rho)$ to make $\Phi((T-\lambda u)/\omega u^{\frac{1}{2}}) = 50\%$. Thus $\psi(u,T)/\psi(u) \cong 50\%$ and $\psi(u,T)/\psi(u) \to 50\%$ $\alpha \downarrow 0$. The reason for choosing the T-fractile approximate according to (4.2) and not exact is just computational convenience. However, it is clear from properties of (4.2) discussed so far that in some extreme cases T is quite far from the true median. E.g. the missing values for $\rho = 0.5, \alpha = 50\%$ are due to the fact that here $u(\rho) = 0$ so that also $T = \lambda u = 0$.

We shall leave the normal approximation at this place, not because we think that the topic has been exhausted but because ideas of the next section turn out to be what appears a rather more fruitful approach to the general problem of finding useful approximations for the ruin probabilities.

5. DIFFUSION APPROXIMATIONS

Our starting point is Grandell's (1977) paper (see also Bohman, 1974, and Grandell, 1974, for discrete time analogues). In the notation of the present paper, the idea is to first observe that if $\{W_\xi(t)\}_{0\leq t<\infty}$ is Brownian motion with unit variance and drift ξ, then as $u\to\infty$

$$(5.1)\qquad \{\frac{1}{u\sqrt{EY^2}}(X(tu^2/\rho)+\eta tu^2\}_{0\leq t<\infty}\to\{W_0(t)\}_{0\leq t<\infty}$$

in the sense of weak convergence in function space, cf. Billingsley (1968), Lindvall (1973) or Whitt (1980). Appealing to the operational time interpretation, the distribution of the l.h.s. of (5.1) is independent of ρ. Hence, if we fix the distribution of Y, let $\rho\uparrow 1$ (i.e. the safety loading $\eta\downarrow 0$) and $u\uparrow\infty$ in such a way that $\eta u\to\xi\in(0,\infty)$, then

$$(5.2)\qquad \{\frac{1}{u\sqrt{EY^2}}X(tu^2/\rho)\}_{0\leq t<\infty}\to\{W_{-\xi/\sqrt{EY^2}}(t)\}_{0\leq t<\infty}$$

Recalling that the time of the first passage of W_ξ to level c has distribution function

$$(5.3)\qquad G(t;\xi,c)=P(\max_{0\leq s\leq t}W_\xi(s)>c)=1-\Phi(ct^{-\frac{1}{2}}-\xi t^{\frac{1}{2}})+e^{2\xi c}\Phi(-ct^{-\frac{1}{2}}-\xi t^{\frac{1}{2}}),$$

cf. Skorohod (1965) p. 171, it follows from properties of weak convergence that

$$\psi(u,tu^2/\rho)=P(\max_{0\leq s\leq tu^2/\rho}X(s)>u)\to G(t;-\frac{\xi}{\sqrt{EY^2}},\frac{1}{\sqrt{EY^2}})$$

so that (using the relation $G(t;\xi,c)=G(t/c^2;\xi c,1)$) this suggest the approximation

$$(5.4)\qquad \psi(u,T)\cong G(\frac{TEY^2\rho}{u^2};-\frac{(1-\rho)u}{\rho EY^2},1)=G(\frac{T\phi''(0)\rho}{u^2};-\frac{(1-\rho)u}{\rho\phi''(0)},1),$$

cf. Grandell (1977) p. 5_2. As remarked by Grandell, the validity of (5.4) for

$T=\infty$ requires an extra argument. This is given in Grandell (1978) so that indeed

(5.5) $$\psi(u) \cong G(\infty, -\frac{(1-\rho)u}{\rho EY^2}, 1) = e^{-2(1-\rho)u/\rho EY^2} .$$

From the point of view of queues, this is simply the well-known heavy traffic approximation, see e.g. Whitt (1974).

Based upon a deeply interesting paper by Siegmund (1979), we shall now suggest first a variant of (5.4), (5.5), viz.

(5.6) $$\psi(u,T) \cong G(\frac{T\rho_0 E_0 Y^2}{u^2}; -\gamma_0 u, 1) = G(\frac{T\rho\phi''(\gamma_0)}{u^2}; -\gamma_0 u, 1) ,$$

(5.7) $$\psi(u) \cong e^{-2\gamma_0 u}$$

and next a refinement of these approximations,

(5.8) $$\psi(u,T) \cong G(\frac{T\rho_0 E_0 Y^2}{u^2} + \frac{E_0 Y^3}{3uE_0 Y^2}; -\frac{\gamma_1 u}{2}, 1 + \frac{E_0 Y^3}{3uE_0 Y^2})$$

$$= G(\frac{T\rho\phi''(\gamma_0)}{u^2} + \frac{\phi'''(\gamma_0)}{3u\phi''(\gamma_0)}; -\frac{\gamma_1 u}{2}, 1 + \frac{\phi'''(\gamma_0)}{3u\phi''(\gamma_0)}) ,$$

(5.9) $$\psi(u) \cong e^{-\gamma_1 E_0 Y^3/3E_0 Y^2} e^{-\gamma_1 u} = e^{-\gamma_1 \phi'''(\gamma_0)/3\phi''(\gamma_0)} e^{-\gamma_1 u} .$$

The conditions are of just the same flavour as for (5.4), (5.5), but need a slight twist from the mathematical point of view. In fact, we think of the P_0-distribution in the exponential family of Section 2 (i.e. of ϕ_0 and ρ_0) as fixed and consider the limit

(5.10) $\theta_0 \uparrow 0$, $u \uparrow \infty$ in such a way that $\theta_0 u \to \xi$

with $\xi \in (-\infty, 0)$. Note that

$$\eta_{\theta_0} u = \frac{1-\rho\phi'(\theta_0+\gamma_0)}{\rho\phi'(\theta_0+\gamma_0)}\, u \cong \frac{1-\rho\phi'(\gamma_0)-\rho\theta_0\phi''(\gamma_0)}{\rho\phi'(\gamma_0)}\, u = -\rho\phi''(\gamma_0)\xi \ .$$

I.e. the conditions for (5.6) - (5.9) as well as for (5.4), (5.5) both require that $u \uparrow \infty$, $\eta \downarrow 0$ in such a way that η and $1/u$ remain of the same order. The difference lies in the sets of other fixed and variable parameters.

Relations of the form (5.6) - (5.9) are highly expected from the random walk analogy and from Siegmund (1979). However, it is not obvious what the constants should be and to obtain the correct values, the calculations therefore have to be adopted to the present case.

We first note that as $\theta_0 \uparrow 0$,

$$\mathrm{Var}_{\theta_0} X(1) \cong \mathrm{Var}_0 X(1) = \kappa''(0) = \rho_0 E_0 Y^2 \ .$$

Using this relation, it is easy to deduce that

$$\left\{\frac{1}{u\sqrt{\rho_0 E_0 Y^2}}\left(X(tu^2) - tu^2\kappa'_{\theta_0}(0)\right)\right\}_{0 \leqq t < \infty} \to \{W_0(t)\}_{0 \leqq t < \infty}$$

in P_{θ_0}-distribution [we do not understand the corresponding step in Siegmund's proof; a formal proof may proceed by checking the conditions in Helland (1982)]. Hence

$$\left\{\frac{1}{u\sqrt{\rho_0 E_0 Y^2}}\left(X(tu^2) - tu^2\theta_0\kappa''_0(0)\right)\right\}_{0 \leqq t < \infty} \to \{W_0(t)\}_{0 \leqq t < \infty} \ ,$$

$$\left\{\frac{1}{u\sqrt{\rho_0 E_0 Y^2}} X(tu^2)\right\}_{0 \leqq t < \infty} \to \{W_{\xi\sqrt{\rho_0 E_0 Y^2}}(t)\}_{0 \leqq t < \infty} \ ,$$

(5.11) $$\psi(u, tu^2) \to G\left(t;\ \xi\sqrt{\rho_0 E_0 Y^2}, \frac{1}{\sqrt{\rho_0 E_0 Y^2}}\right) = G(t\rho_0 E_0 Y^2;\ \xi, 1)$$

which yields (5.6) by substituting $T = tu^2$. (5.7) now follows heuristically by letting $T \to \infty$. We omit the formal verification.

The derivation of (5.8) is much more technical. First note that in Laplace transform formulation (5.11) may be rewritten as

$$(5.12) \qquad E_{\theta_0} e^{-\lambda\tau\rho_0 E_0 Y^2/u^2} \to e^{-h(\lambda,\xi)} \quad ,$$

$$h(\lambda,\xi) = \sqrt{2\lambda+\xi^2} - \xi \; , \quad \tau = \tau(u) \quad .$$

The idea is now to improve upon (5.12) by invoking terms which are $O(u^{-1})$. By Lemma 3.1 we have for $\tilde{\theta} \geqq 0$ that

$$1 = P_{\tilde{\theta}}(\tau < \infty) = E_{\theta_0} \exp\{\tilde{\theta} - \theta_0)(u + B(u)) - \tau(\kappa_0(\tilde{\theta}) - \kappa_0(\theta_0))\} \; .$$

Replacing $\tilde{\theta}, \theta_0$ by $\tilde{\theta}/u$, ξ/u yields

$$e^{-(\tilde{\theta}-\xi)} = E_{\theta_0} \exp\{(\tilde{\theta} - \xi)B(u)/u - \tau(\kappa_0(\frac{\tilde{\theta}}{u}) - \kappa_0(\frac{\xi}{u}))\}$$

so if we let $\tilde{\theta} = (2\lambda + \xi^2)^{\frac{1}{2}} = h(\lambda,\xi) + \xi$ and use the relation

$$\kappa_0(\beta) = \rho_0\{\frac{\beta^2 E_0 Y^2}{2} + \frac{\beta^3 E_0 Y^3}{6} + \ldots\} \quad ,$$

we get up to $O(u^{-1})$ terms that

$$(5.13) \qquad e^{-h(\lambda,\xi)} \cong$$

$$E_{\theta_0}\{\exp h(\lambda,\xi)B(u)/u - \lambda\tau\rho_0 E_0 Y^2/u^2 - \tau\rho_0 \frac{E_0 Y^3}{6} (\tilde{\theta}^3 - \xi^3)/u^3\} \; .$$

To proceed from (5.13), we now need two Lemmas, the first of which shows that one of Siegmund's constants can be obtained in a more explicit form in the present case:

<u>Lemma 5.1</u> <u>The</u> $B(u), u \geqq 0$, <u>are uniformly integrable w.r.t.</u> P_0 . <u>Furthermore,</u>

$$\lim_{u\to\infty} E_0B(u) = E_0B(\infty) = \frac{E_0Y^3}{3E_0Y^2} .$$

Proof Recalling the random walk interpretation of the $B(u)$, cf. Section 3, it is well-known that

$$E_0B(\infty) = \frac{1}{E_0B(0)} \int_0^\infty bP_0(B(0) > b)db = \frac{E_0B(0)^2}{2E_0B(0)} = \frac{\iota''(0)}{2\iota'(0)} ,$$

where $\iota(\beta) = E_0e^{\beta B(0)}$. Since the descending ladder variable of $S_n = \Sigma_1^n\{Y_k - Z_k\}$ is exponential with mean $-1/\rho_0$, we have by Wiener-Hopf factorisation (Feller, 1971, p. 400) that

$$\phi_0(\beta)\frac{\rho_0}{\rho_0+\beta} = \iota(\beta) + \frac{\rho_0}{\rho_0+\beta} - \iota(\beta)\frac{\rho_0}{\rho_0+\beta} , \qquad \text{i.e.}$$

$$\iota(\beta) = \frac{\rho_0(\phi_0(\beta)-1)}{\beta} , \quad \iota'(\beta) = \frac{\rho_0(\beta\phi_0'(\beta)-\phi_0(\beta)+1)}{\beta^2}$$

so that by l'Hospital's rule

$$\iota'(0) = \rho_0 \frac{d^2/d\beta^2(\beta\phi_0'(\beta) - \phi_0(\beta) + 1)}{d^2/d\beta^2(\beta^2)}\Bigg|_{\beta=0} = \frac{\rho_0\phi_0''(0)}{2} = \frac{\rho_0E_0Y^2}{2} .$$

Differentiating once more, it follows in a similar manner by a threefold application of l'Hospital's rule that $\iota''(0) = \rho_0E_0Y^3/3$ so that indeed $E_0B(\infty)$ is as asserted. By renewal theory, $E_0B(u) \to E_0B(\infty)$ and uniform integrability follows. □

Lemma 5.2 Subject to (5.10), it holds for all continuous functions f,g with $f(\infty) = 0,\ g(b) = O(b)$ that

$$E_{\theta_0} f\left(\frac{\tau\rho_0E_0Y^2}{u^2}\right)g(B(u)) \to \int_0^\infty f(x)G(dx;\xi,1)E_0g(B(\infty)) .$$

<u>Proof</u> For $\theta_0 \downarrow 0$ (rather than $\theta_0 \uparrow 0$) and g bounded this is just the continuous time analogue of Lemma 3 of Siegmund (1979) (similar arguments can be found in Siegmund (1975) and Asmussen (1982)). The case $\theta_0 \uparrow 0$ can be treated in a similar manner by a conditioning argument or by invoking θ_1 and Corollary 3.1. We omit the details. That the conclusion also holds for $g(b) = O(b)$ now follows by the uniform integrability in Lemma 5.1. □

Returning to (5.13), the r.h.s. behaves like

$$E_{\theta_0} \exp\{-\lambda\tau\rho_0 E_0 Y^2/u^2\}[1 + h(\lambda,\xi)B(u)/u - \tau\rho_0 E_0 Y^3(\tilde{\theta}^3 - \xi^3)/6u^3] \cong$$

$$(5.14) \qquad E_{\theta_0} \exp\{-\lambda\tau\rho_0 E_0 Y^2/u^2\} + \frac{1}{u} e^{-h(\lambda,\xi)} h(\lambda,\xi) E_0 B(\infty)$$

$$- \frac{1}{u} \frac{E_0 Y^3}{6E_0 Y^2} (\tilde{\theta}^3 - \xi^3) E_{\theta_0} \frac{\tau\rho_0 E_0 Y^2}{u^2} \exp\{-\lambda\tau\rho_0 E_0 Y^2/u^2\} \quad ,$$

using Lemma 5.2. The last term is approximately

$$(5.15) \qquad \frac{1}{u} \frac{E_0 Y^3}{6E_0 Y^2} (\tilde{\theta}^3 - \xi^3) \frac{d}{d\lambda} e^{-h(\lambda,\xi)} =$$

$$- \frac{1}{u} \frac{E_0 Y^3}{6E_0 Y^2} [2\lambda + \xi^2 - \xi^3/(2\lambda + \xi)^{\frac{1}{2}}] e^{-h(\lambda,\xi)}$$

Combining (5.13) - (5.15) we get the desired refinement of (5.12),

$$E_{\theta_0} \exp\{-\lambda\tau\rho_0 E_0 Y^2/u^2\} = e^{-h(\lambda,\xi)} (1 - \frac{1}{u} h(\lambda,\xi) E_0 B(\infty))$$

$$+ \frac{1}{u} \frac{E_0 Y^3}{6E_0 Y^2} [2\lambda + \xi^2 - \xi^3/(2\lambda + \xi)^{\frac{1}{2}}] e^{-h(\lambda,\xi)} + o(\frac{1}{u})$$

$$(5.16) \qquad = \exp\{-h(\lambda,\xi)(1 + \frac{1}{u} \frac{E_0 Y^3}{3E_0 Y^2})\}$$

$$+ \frac{1}{u} \frac{E_0 Y^3}{6E_0 Y^2} [2\lambda + \xi^2 - \xi^3/(2\lambda + \xi)^{\frac{1}{2}}] e^{-h(\lambda,\xi)} + o(\frac{1}{u}) \quad .$$

To arrive at (5.8), let $\beta = E_0 Y^3 / 3E_0 Y^2$, $\gamma = E_0 Y^3 / E_0 Y^2$ in the derivation of (34) from (33) in Siegmund (1979) pp. 712-713. Again, (5.9) follows by letting $T \to \infty$ and a formal verification is omitted.

We now turn to the comparisons with the exact values for the P/E case, examples of which are given in Figures 4 - 6.

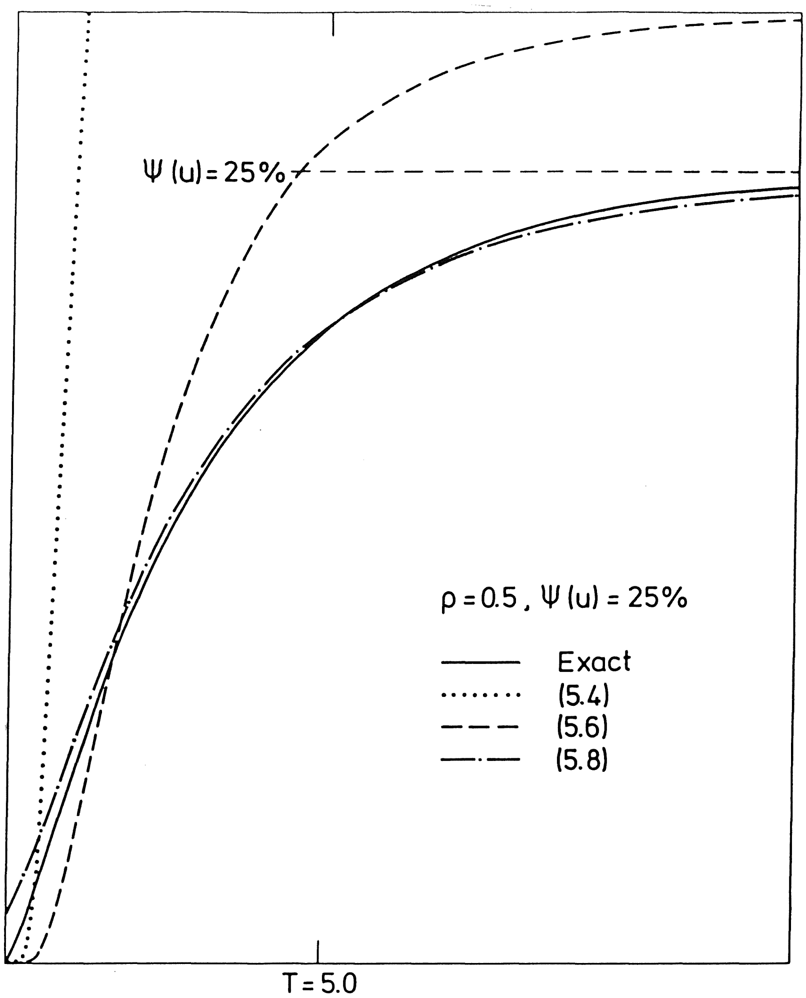


Figure 4

Figure 5

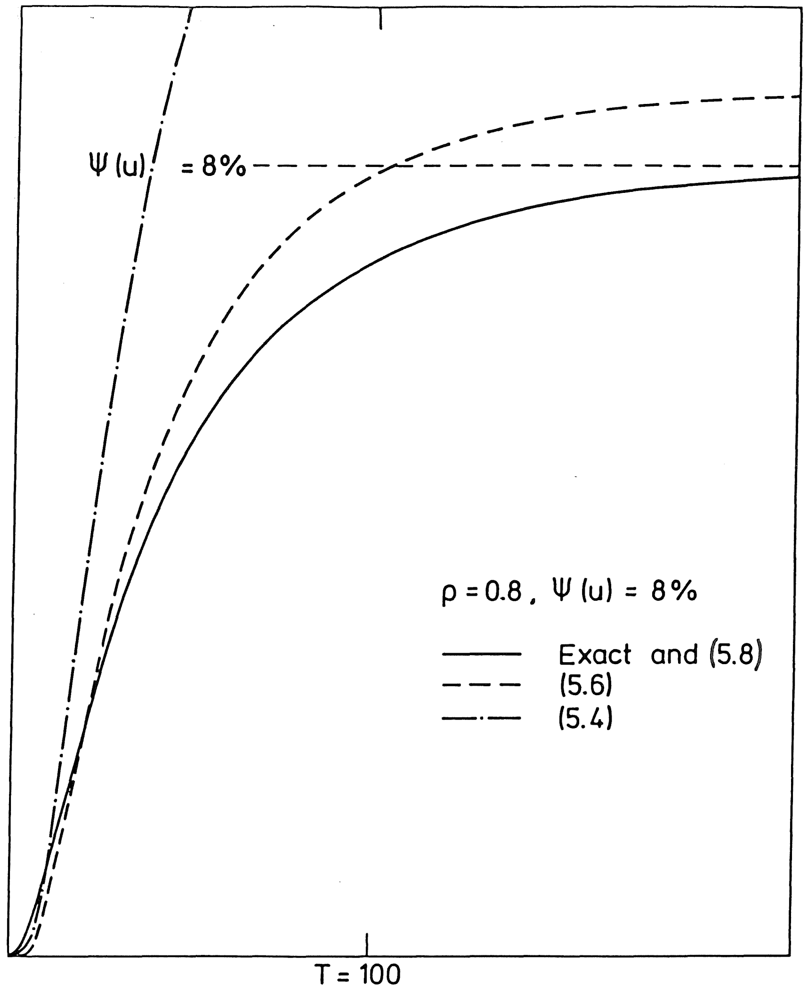
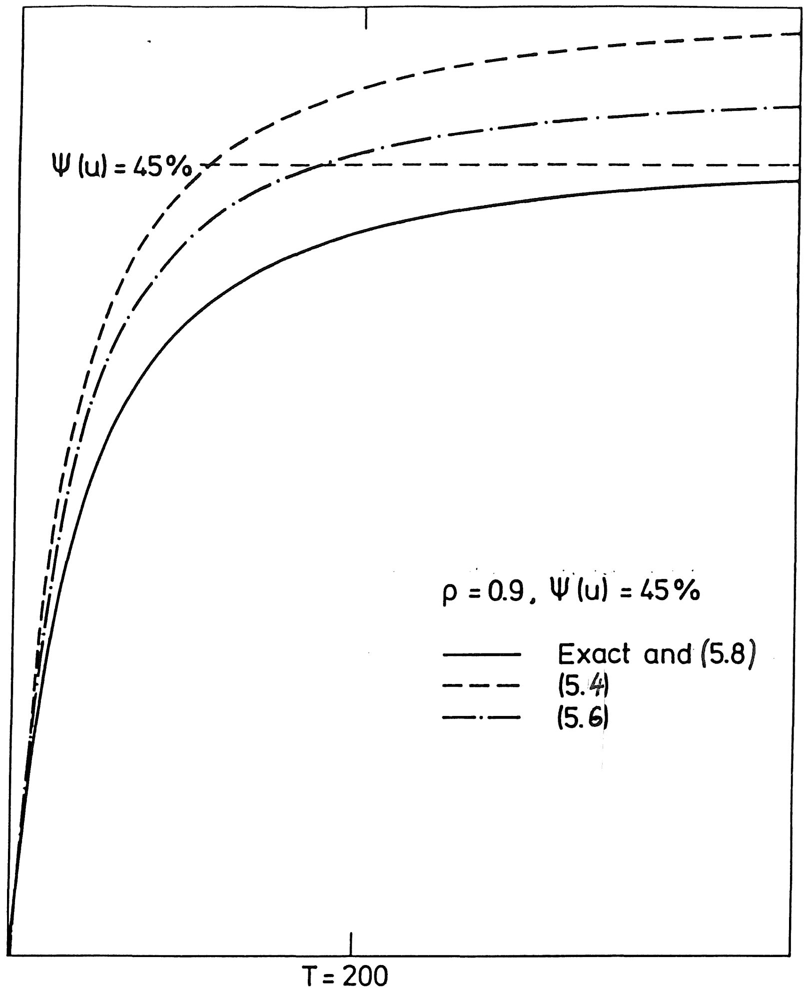


Figure 6



Considering first (5.4), (5.5), Grandell (1977) shows that the fit of (5.5) is best for u not too large (i.e., $\psi(u)$ not too small), but even then his numerical results are rather disappointing. The present investigation of (5.4) substantiates this. The fit is quite unacceptable if $\rho = 0.2$, $\rho = 0.5$ or u is large, somewhat better if $\rho = 0.8$ or 0.9 and u is moderate. The best agreement in the entire set of values considered is the one in Figure 6. The rate of convergence of the error of (5.4) to zero as $\rho \uparrow 1$ is given in Table 4 and is very slow.

Table 4 Relative error (%) of (5.4) at the 50% fractile of (4.2)

ρ	$\alpha=\psi(u)$ / η(%)	50%	10%	5%	1%
0.50	100	-	368	471	896
0.60	67	307	236	302	541
0.70	43	120	147	185	311
0.80	25	59	83	102	160
0.85	18	39	58	70	106
0.90	11	24	36	43	63
0.92	8.7	19	28	33	48
0.94	6.4	14	20	24	34
0.96	4.2	8.7	13	15	22
0.98	2.0	4.2	6.3	7.5	10

These findings are in agreement with similar studies of heavy-traffic approximations for queues. E.g. de Smit (1982) reports that the relative error of an approximation in a similar spirit for the mean wait in a certain multi-server queueing system is still 10% for the value $\rho = 0.99$ of the traffic intensity, which is quite unrealistically close to one.

The approximations (5.6), (5.7) behave much better. This may be somewhat surprising, since one might feel that they are not much more than a variant of (5.4), (5.5). We have no really satisfying explanation for this, but one might remark that the relevancy of the P_0-distribution for large deviation results is well-known, cf. e.g. Bahadur & Ranga Rao (1960). The fit is still quite unsatisfactory for $\rho = 0.5$, but rather good for $\rho = 0.8, 0.9$ and at least some values of u. Concerning tail behaviour, note that the ratio of (5.7) to the exact value is

$$(5.17) \qquad e^{-2(1-\rho^{\frac{1}{2}})u/\rho} e^{-(1-\rho)u} = e^{-(1-\rho^{\frac{1}{2}})^2 u/\rho} \quad .$$

That is, roughly the approximations give slightly too large values for u small and much too small values for u large. However, from this point of view all u-values considered in our examples are moderate.

The rate of convergence of the error to zero as $\rho \uparrow 1$ is illustrated by Table 5.

Table 5 Relative error (%) of (5.6) at the 50% fractile of (4.2)

ρ	$\alpha = \psi(u)$ / η(%)	50%	10%	5%	1%
0.50	100	-	80	46	- 1.9
0.60	67	270	47	28	- 3.7
0.70	43	95	28	17	- 3.8
0.80	25	43	15	9.0	- 3.1
0.85	18	28	11	6.2	- 2.4
0.90	11	16	6.5	3.8	- 1.7
0.92	8.7	12	5.0	2.9	- 1.4
0.94	6.4	8.9	3.7	2.1	- 1.1
0.96	4.2	5.7	2.4	1.4	- 0.7
0.98	2.0	2.7	1.2	0.7	- 0.4

The improvements (5.8), (5.9) have in contrast on quite outstanding fit. For $\rho = 0.8, 0.9$ they are hardly discernible from the true values and also for $\rho = 0.5$, which one feels is quite far from the underlying condition (5.10), the agreement is very good. These claims are further substantiated by an examination of Table 6.

Table 6 Relative error (%) of (5.8) at the 50% fractile of (4.2)

ρ	η(%) \ $\alpha = \psi(u)$	50%	10%	5%	1%
0.50	100	-	1.9	2.6	4.1
0.60	67	- 1.0	1.0	1.5	2.3
0.70	43	- 0.2	0.5	0.7	1.2
0.80	25	- 0.1	0.2	0.3	0.5
0.85	17	- 0.1	0.1	0.2	0.2
0.90	11	0.0	0.0	0.1	0.1
0.92	8.7	0.0	0.0	0.0	0.1
0.94	6.4	0.0	0.0	0.0	0.0
0.96	4.2	0.0	0.0	0.0	0.0
0.98	2.0	0.0	0.0	0.0	0.0

The fit does not significantly change with u, which can be understood by forming the ratio analogous to (5.17) which for the general case is

$$(5.18) \qquad e^{-2\gamma_1 E_0 Y^3/3E_0 Y^2} e^{-\gamma_1 u}/\psi(u) \to e^{-2\gamma_1 E_0 Y^3/3E_0 Y^2}/C, \quad u \to \infty$$

where C is the Cramér-Lundberg constant. That is, the functional dependence on u is asymptotically correct and the degree of accuracy to be expected for $\psi(u)$ small can be judged to some extent by comparing the r.h.s. of (5.18) to one.

These highly promising results for the Poisson/exponential case motivate, of course, strongly to look into general claim size distributions, at least for $T = \infty$. We shall here only treat one example, viz. the gamma case

(5.19) $\phi(\beta) = (1 - b\beta)^{-1/b}$

considered by Grandell & Segerdahl (1971). For $b = 10,100$ and $\eta = 10\%$ (i.e. $\rho = 1/1.1 = 0.91$) some comparisons can be found in Tables 7 - 8.

Table 7 $\psi(u)$ and approximations for $\eta = 10\%$ and (5.19) with $b = 10$

u	exact, C-L and (5.9)	(5.5)	(5.7)
100	0.17668	0.16232	0.19015
200	0.03530	0.02635	0.03616
300	0.00705	0.00428	0.00687
400	0.00141	0.00069	0.00131
500	0.00028	0.00011	0.00025

Table 8 $\psi(u)$ and approximations for $\eta = 10\%$ and (5.19) with $b = 100$

u	exact	C-L(3.1)	(5.5)	(5.7)	(5.9)
300	0.52114	0.52100	0.55208	0.58257	0.52101
600	0.30867	0.30866	0.30479	0.33939	0.30867
900	0.18287	0.18287	0.16827	0.19771	0.18287
1200	0.10834	0.10834	0.09290	0.11519	0.10834
1500	0.06418	0.06418	0.05129	0.06710	0.06418
1800	0.03803	0.03803	0.02832	0.03909	0.03803
2100	0.02253	0.02253	0.01563	0.02277	0.02253
2400	0.01335	0.01335	0.00863	0.01327	0.01335
2700	0.00791	0.00791	0.00476	0.00773	0.00791
3000	0.00468	0.00468	0.00263	0.00450	0.00468

The exact values and the Cramér-Lundberg approximation are taken from Grandell and Segerdahl (1971), (5.5) from Grandell (1977) while the rest of the tables have been produced on a desk calculator using Grandell and Segerdahl's computations of γ_1 and the explicit expression $\gamma_0 = (1 - \rho^{b/(b+1)})/b$ (the solution of the transcendental equations (3.3) may in general be somewhat laborious). These tables substantiate the above discussion. It is seen that (5.5) is only reasonable for moderate values of u, while the tail behaviour of (5.7) is considerably better. The Cramér-Lundberg approximation (which is exact in the P/E case considered so far) and (5.9) nearly coincide and both provide a beautiful fit.

6. THE ASYMPTOTIC BEHAVIOUR OF $\psi(u,T)$ as $T \to \infty$

We shall here briefly comment on a refinement of (3.2), viz. a formula of the type

$$\text{(6.1)} \qquad \psi(u,T) \cong \psi(u) - D(u)T^{-3/2}e^{-\gamma_0 u - \gamma_2 T} \qquad T \to \infty$$

where u and the other parameters are fixed. A relation of this type is highly expected also from random walk analogues (Veraverbeke and Teugels, 1972/73), from Teugels (1979) and from convergence rate results for queues as related to the concept of relaxation time (Cohen, 1982, pp. 179-181, 600-614 or Prabhu, 1965). A rigorous proof and an explicit evaluation of $D(u)$ was, however, first provided very recently by Teugels (1982).

The various illustrations of the shape of $\psi(u,T)$ presented so far indicate that (6.1) could hardly be an useful approximation for small or moderate T. E.g. (6.1) is a concave function of T and yields even negative values for small T. Also the problem of estimating $\psi(u) - \psi(u,T)$ accurately for large T is hardly practical since the relative error is then small anyway. In view of the role played by relations like (6.1) in the literature and their mathematical beauty it is, however, of some interest to present some numerical illustrations.

Considering again the P/E case, it is straightforward to check from Teugels (1982) that

$$\text{(6.2)} \qquad D(u) = \frac{\rho^{1/4}}{2\pi^{\frac{1}{2}}(1-\rho^{\frac{1}{2}})^2}(1+\rho^{\frac{1}{2}}u), \quad \gamma_2 = (1-\rho^{\frac{1}{2}})^2 .$$

The numerical comparisons show that indeed $\psi(u,T)$ has to be very close to $\psi(u)$ before (6.1), (6.2) provide any reasonable fit. Two examples are given in Figures 7 - 8 .

Figure 7

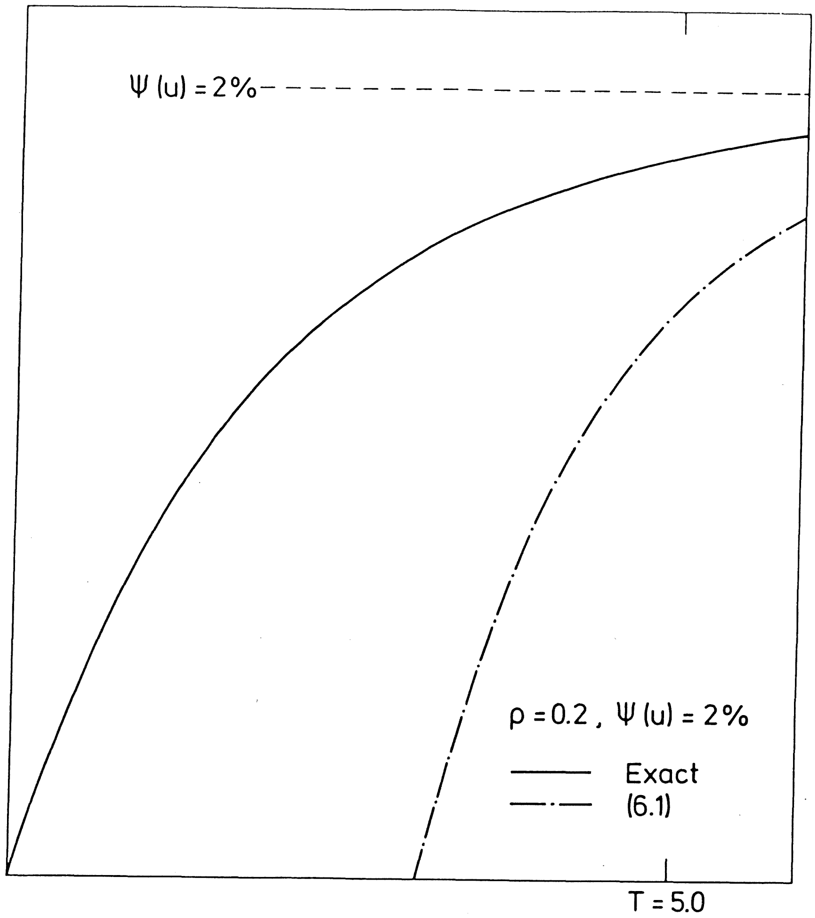
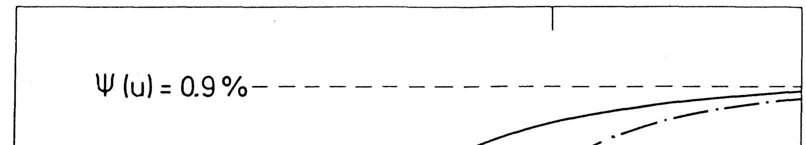


Figure 8



The corresponding figures for $\rho = 0.5, 0.8, 0.9$ and all values of $\psi(u)$ look all very much the same as Figure 8, though maybe the convergence becomes slightly faster as ρ increases and $\psi(u)$ decreases. Thus Figure 8 represents the best agreement in all set of values considered. The convergence is even slower if $\rho = 0.2$ as in Figure 7. This illustrates once more that while we have exhibited approximations (viz. (5.8),(5.9)) with a very good fit for $\rho \geqq 0.5$, the situation is less settled for small values of ρ . Presumably $\rho = 0.2$ is much too small to be of interest in risk theory but this is by no means so, however, in the queueing setting.

7. COMPUTATIONAL ASPECTS

Various methods are available for computing the true value of $\psi(u,T)$ for the P/E case, but they all invariably seem to require numerical integration. The approach of Seal (1972a) is to use an integral formula (e.g. Takacs, 1967, p. 152) involving Bessel functions which can then be computed by a further numerical integration or, as Seal does, by polynomial approximation. Another obvious idea would be to invert (4.3) numerically. We have here taken yet a third approach, viz. to express $\psi(u) - \psi(u,T) = \rho e^{-(1-\rho)u} - \psi(u,T)$ as an integral of elementary though complicated function (essentially iterated trigonometric functions),

$$(7.1) \qquad \psi(u) - \psi(u,T) = \frac{1}{\pi}\int_0^{\pi} f(\theta)\,d\theta \ ,$$

$$f(\theta) = \exp\{2\rho^{\frac{1}{2}}T\cos\theta - (1+\rho)T + u(\rho^{\frac{1}{2}}\cos\theta - 1)\}/(1-\rho-2\rho^{\frac{1}{2}}\cos\theta)$$

$$\cdot\,[\rho\sin(u\rho^{\frac{1}{2}}\sin\theta + 2\theta) - \rho^{\frac{1}{2}}\cos(u\rho^{\frac{1}{2}}\sin\theta + 2\theta)] \ .$$

To derive (7.1), it is convenient to rely on the queueing interpretation (2.3). If the queue length $Q(T)$ is N, then $V(T)$ is the residual service time of the customer being served plus the service times of the N-1 remaining ones, i.e. Erlangian with parameter N. Thus

$$(7.2) \qquad P(V(T) > u) = \sum_{N=1}^{\infty} P(Q(T) = N)\int_u^{\infty} \frac{x^{N-1}}{(N-1)!}e^{-x}dx$$

$$= \sum_{N=1}^{\infty} P(Q(T) = N)\sum_{k=0}^{N-1} e^{-u}\frac{u^k}{k!} = \sum_{k=0}^{\infty} e^{-u}\frac{u^k}{k!}P(Q(T) \geqq k+1) \ .$$

Now let

$$I_j(x) = \frac{1}{\pi}\int_0^{\pi} e^{x\cos\theta}\cos j\theta\,d\theta$$

denote the modified Bessel function of j^{th} order (Olver, 1965) and note that

$$\sum_{j=N}^{\infty} \rho^{j/2} \cos j\theta = \mathrm{Re} \sum_{j=N}^{\infty} (\rho^{\frac{1}{2}} e^{i\theta})^j = \frac{\rho^{N/2}(\cos N\theta - \rho^{\frac{1}{2}} \cos (N-1)\theta)}{1-\rho-2\rho^{\frac{1}{2}} \cos \theta} .$$

Hence (Prabhu, 1965)

$$(7.3) \qquad P(Q(T) \geqq N) = e^{-(1+\rho)T}\{ \sum_{j=N}^{\infty} \rho^{j/2} I_j(2\rho^{\frac{1}{2}}T) + \rho^N - \rho^N \sum_{j=N}^{\infty} \rho^{j/2} I_j(2\rho^{\frac{1}{2}}T)$$

$$= e^{-(1+\rho)T}\{\rho^N + \frac{1}{2\pi} \int_0^{\pi} e^{2\rho^{\frac{1}{2}}T \cos \theta} / (1 - \rho - 2\rho^{\frac{1}{2}} \cos \theta)$$

$$\cdot [\rho^{(N+1)/2} \cos (N+1)\theta - \rho^{(N+1)/2} \cos (N-1)\theta] d\theta \} .$$

A further application of Euler's formulas yields

$$(7.4) \qquad \sum_{k=0}^{\infty} \frac{u^k}{k!} \rho^{k/2} \cos (k+2)\theta = e^{u\rho^{\frac{1}{2}} \cos \theta} \cos (u\rho^{\frac{1}{2}} \sin \theta + 2\theta) .$$

Inserting (7.3) in (7.2), the $e^{-(1+\rho)T} \rho^{k+1}$ terms sum to $P(V(\infty)>u) = \psi(u)$ and a calculation similar to (7.4) for the last term in (7.3) produces (7.1).

The integrand $f(\theta)$ in (7.1) oscillates quite rapidly in particular for u large. A division was therefore made of $(0,\pi)$ into subintervals (a_k, b_k) in which $u\rho^{\frac{1}{2}} \sin \theta$ oscillates at most $\pi/2$ and $\int_{a_k}^{b_k} f$ was evaluated by the extended Simpson's rule, Davis and Polonsky (1965), with $S = 32, 64, 128, \ldots$ panels. To estimate the error, (a_k, b_k) was further subdivided into intervals $(a_{k\ell}, b_{k\ell})$ of each 32 panels of length h_k and an upper bound on the error judged by

$$(7.5) \qquad \sum_{k,\ell} \max_{a_{k\ell}<\theta<b_{k\ell}} \frac{16\, h_k^5}{90} |f''''(\theta)|$$

with $f''''(\theta)$ evaluated numerically at the grid points. The value of S was finalized when either $S = 512$ or the relative error as judged by the ratio δ

of (7.5) to the corresponding evaluated value of $\psi(u,T)$ came below a specified value ε. For Set A of tables, ε was set to $\frac{1}{2}\%$. The needed value of S was only in one of the 448 cases 512 and in most of the cases 64. For Set B of tables, ε was set to $\frac{1}{2}$ ‰.

The computations were carried out in Pascal (17 significant figures) at the Regional Computing Center, University of Copenhagen, on their Univac 1181 machine. As check, a testprogram reproducing Table 2 of Seal (1972a) for $1-\psi(u,T_{op})$ was written. Comparison showed only in a few cases discrepancies beyond the accuracy claimed by the two methods. This was mainly for large T_{op}, where we would think that the present method is superior since it more transparently reflects the convergence of $\psi(u,T)$ to $\psi(u)$. E.g. for $T_{op}=2000$ Seal reports a number of values with $\psi(u,T_{op})>\psi(u)$ which is impossible, while for $T_{op}=1000, u=110$ Seal has $1-\psi(u,T_{op})=0.99996$ whereas we got 0.99998 with an error estimate much below 10^{-5} (a further justification that this is the true value is the coincidence with (4.2) and u being very large). Here clearly the relative deviation on $1-\psi(u,T_{op})$ is unimportant, but that on $\psi(u,T_{op})$ not.

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Appendix. Sample from Set A of tables.

The upper value T^* of T is chosen as $\lambda u + 2.7\omega$, cf. (4.2), and the considered values $T = T_k$ are $k = 1,2,3,4,8,12,\ldots,96,100\%$ of T^*. This makes in all cases $\{\psi(u,T_k)\}$ cover $[0,\psi(u)]$ reasonably well.

Table 9 $\psi(u,T)/\psi(u)$, approximations/$\psi(u)$ (only values > 0), panels and estimate δ(‰) of the error on the true value. $\rho = 0.8$, $\psi(u) = 1\%$.

k	T_k	exact	(4.2)	(4.7)	(5.4)	(5.6)	(5.8)	(6.1)	S	2δ
1	3.4	0.000	0.096	0.046	0.000	0.000	0.000		256	3
2	6.8	0.001	0.105	0.059	0.000	0.000	0.002		128	3
3	10.3	0.003	0.114	0.073	0.000	0.000	0.005		128	1
4	13.8	0.009	0.124	0.088	0.002	0.000	0.012		64	4
8	27.5	0.071	0.171	0.158	0.091	0.028	0.077		64	0
12	41.3	0.181	0.227	0.244	0.359	0.121	0.187		64	0
16	55.1	0.305	0.293	0.341	0.713	0.250	0.310		64	0
20	68.8	0.423	0.366	0.441	1.069	0.379	0.427		64	0
24	82.6	0.527	0.444	0.538	1.392	0.494	0.530		64	0
28	96.4	0.614	0.525	0.625	1.672	0.590	0.617		64	0
32	110.1	0.687	0.605	0.697	1.909	0.668	0.689	0.231	64	0
36	123.9	0.746	0.680	0.752	2.107	0.730	0.748	0.447	64	0
40	137.7	0.794	0.749	0.792	2.273	0.779	0.795	0.595	64	0
44	151.4	0.833	0.809	0.820	2.411	0.818	0.834	0.699	64	0
48	165.2	0.864	0.859	0.840	2.526	0.849	0.865	0.773	64	0
52	179.0	0.890	0.900	0.856	2.621	0.873	0.890	0.828	64	0
56	192.7	0.910	0.931	0.871	2.701	0.892	0.911	0.868	64	0
60	206.5	0.927	0.954	0.887	2.768	0.908	0.927	0.898	64	0
64	220.3	0.940	0.971	0.904	2.825	0.920	0.941	0.920	64	0
68	234.0	0.951	0.982	0.922	2.872	0.929	0.951	0.938	64	0
72	247.8	0.960	0.989	0.939	2.912	0.937	0.960	0.951	64	0
76	261.6	0.967	0.994	0.955	2.945	0.943	0.967	0.961	64	0
80	275.3	0.973	0.997	0.968	2.974	0.947	0.973	0.969	64	0
84	289.1	0.978	0.998	0.978	2.998	0.951	0.978	0.975	64	0
88	302.9	0.982	0.999	0.986	3.018	0.954	0.982	0.980	64	0
92	316.6	0.985	1.000	0.991	3.035	0.957	0.985	0.984	64	0
96	330.4	0.988	1.000	0.995	3.050	0.959	0.987	0.987	64	0
100	344.2	0.990	1.000	0.997	3.063	0.960	0.990	0.990	64	0
	∞	1.000	1.000	1.000	3.140	0.967	1.000	1.000		

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