A. Hald S. Johansen

On de Moivre's Recursion Formulae for the Duration of Play

Preprint
October
1982
8

Institute of Mathematical Statistics
University of Copenhagen

A. Hald and S. Johansen

ON DE MOIVRE'S RECURSION FORMULAE FOR THE DURATION OF PLAY

Preprint 1982 No. 8

INSTITUTE OF MATHEMATICAL STATISTICS
UNIVERSITY OF COPENHAGEN

SEPTEMBER 1982

On de Moivre's Recursion Formulae for the Duration of Play

by

A. Hald and S. Johansen

Institute of Mathematical Statistics
University of Copenhagen

Summary.

De Moivre gave two recursion formulae for calculating the probability of the Duration of Play. However, he did not prove these formulae. This was first done by Laplace. In the paper we first present de Moivre's results and then we reproduce Laplace's proofs in modern notation. Inspired by this, some new combinatorial results are proved and the recursion formulae are obtained as special cases.

Key words. History of probability. Duration of Play. De Moivre. Laplace. Recursion formulae. Difference equations. Combinatorial methods.

1. INTRODUCTION.

De Moivre's greatest achievement in probability theory was his solution of the problem of the Duration of Play. His formulation of the problem was essentially as follows: Consider two players A and B having a and b counters, respectively. In each game A has probability p and B has probability q = 1-p of winning and the winner gets a counter from the loser. The play continues until one of the players has lost all his counters. What is the probability that the play ends at the nth game or before?

De Moivre gave recursion formulae for calculating this probability and from the recursion he derived an explicit expression. However, he did not prove the recursion formulae. This was first done by Laplace (1774,1776), who later gave a more direct derivation of the explicit solution, which is the one used today.

The purpose of the present paper is to reproduce Laplace's proof of de Moivre's recursion formulae in modern notation and inspired by this to give a new combinatorial proof.

For the reader who wants to compare our presentation with that of de Moivre we shall refer to the 3rd edition of the Doctrine of Chances (1756) even if de Moivre solved the problem earlier. The history of the problem is, however, well-known. We refer to Todhunter (1865) and the papers by Schneider (1968) and Kohli (1975).

The historical part of the paper has been written by A. Hald and the new combinatorial proof is due to S. Johansen.

2. DE MOIVRE'S RESULTS.

We shall first reformulate the problem as a random walk with two absorbing barriers. Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables taking on the values $+1$ and -1 with probabilities p and q, respectively, and set $S_n = X_1 + X_2 + \ldots + X_n$ for $n = 1,2,\ldots$ and $S_0 = 0$. The probability of a random walk starting from $(0,0)$ and ending in (n,x), $-a < x < b$, without on the way hitting the barriers at $-a$ and b, $a > 0$, $b > 0$, is

$$u_n(x) = P\{-a < S_i < b,\ i = 0,1,\ldots n-1,\ S_n = x\}, \quad -a < x < b, n \geqq 0,$$

and $u_0(0) = 1$.

Returning to the original problem $u_n(x)$ gives the probability that player A has $a + x$ counters after n games without neither A nor B having been ruined during these games.

Other probabilities of interest are easily found from $u_n(x)$. The probabilities of ruin at the n'th game for A and B, respectively, are

$$r_n = qu_{n-1}(-a+1) \quad \text{and} \quad r_n^* = pu_{n-1}(b-1).$$

The probability of continuation after n games is

$$U_n = \sum_{x=-a+1}^{b-1} u_n(x).$$

The probability of a duration of exactly n games is $d_n = r_n + r_n^*$ and the probability of a duration of at most n games is

$$D_n = \sum_{i=1}^{n} d_i = 1 - U_n.$$

De Moivre's main results may be summarized under the following four headings.

(1) An algorithm for successively calculating $u_1(x)$, $u_2(x), \ldots,$ which in modern notation may be written as the partial difference equation

$$u_n(x) = pu_{n-1}(x-1) + qu_{n-1}(x+1), \ -a < x < b, \ n \geqq 1, \tag{2.1}$$

with the boundary conditions

$$u_0(0) = 1 \ \text{ and } \ u_n(x) = 0 \ \text{ for } x > (b-1)\wedge n \ \text{ and } \ x < (-a+1)\vee(-n), \ n \geqq 0.$$

(2) An explicit expression for r_n in the form

$$r_{a+2m} = q^a (pq)^m \sum_{i=-[(a+m)/(a+b)]}^{[m/(a+b)]} \frac{a + 2i(a+b)}{a + 2m} \binom{a+2m}{m - i(a+b)}$$

and $r_{a+2m+1} = 0$, $m \geqq 0$.

(3) A recursion formula for r_n or equivalently for $u_n(-a+1)$ which we shall write as

$$\sum_{i=0}^{[(a+b-1)/2]} (-1)^i \binom{a+b-1-i}{i} (pq)^i \, u_{n-2i}(-a+1) = 0 \quad , \quad n \geqq a+b-1. \tag{2.2}$$

For $a = b$ de Moivre also gave the recursion for $u_n(x)$

$$\sum_{i=0}^{n\wedge[a/2]} (-1)^i \frac{a}{a-i} \binom{a-i}{i} (pq)^i \, u_{n-2i}(x) = 0 \ , \ n \geqq 1. \tag{2.3}$$

The same recursion obviously holds for U_n, r_n and r_n^*.

(4) Considering the recursion formula for U_n for $a = b$ as a homogeneous linear difference equation, de Moivre developed a method for solving such equations and thus derived his famous trigonometric expression for U_n.

De Moivre also gave similar formulae for a random walk with only one absorbing barrier, i.e. the case where one of the players has an infinite number of counters.

The basic tool for de Moivre was the recursion formula which he used to find U_n either by means of his theory for the summation of recurrent series or as indicated above by solving the corresponding difference equation.

De Moivre does not indicate how he found the recursion formulae. Presumably he used (2.1) to compute a great number of examples for small values of n, a and b and by studying the relations between consecutive terms of $u_n(x)$ he found the relation (2.3) and later on (2.2) by incomplete induction.

For the generation of mathematicians after de Moivre it naturally became a challenge to prove the recursion formulae. The young Laplace formulated the partial difference equation (2.1) and showed how this equation may be transformed to the linear difference equation in one variable (2.2). This step in the historical development is usually overlooked because Laplace later invented the method of generating functions for the direct solution of partial difference equations, which is the method used today.

It is clear that $u_n(x)$ may be written in the form

$$u_n(x) = c_n(x)\, p^{(n+x)/2}\, q^{(n-x)/2},$$

where $c_n(x)$ denotes the number of paths from (0,0) to (n,x) which do not hit the barriers. From (2.1) and (2.2) we then get

$$c_n(x) = c_{n-1}(x-1) + c_{n-1}(x+1) \tag{2.4}$$

with the corresponding boundary conditions, and

$$c_n(x) = \sum_{i=1}^{[(a+b-1)/2]} (-1)^{i-1} \binom{a+b-1-i}{i} c_{n-2i}(x) \quad \text{for} \quad n \geqq a+b-1, \tag{2.5}$$

where we have written x instead of (-a+1) since we shall prove the extension of de Moivre's result that (2.2) holds for $-a < x < b$.

The use of these formulae has been illustrated for a = 4 and b = 5 in Figure 1. The first part of the diagram, i.e. for $n < a$, is just a part of Pascal's triangle. For $n \geqq a$ the boundary conditions modify the binomial coefficients.

For $n \geqq a + b - 1 = 8$ we get the recursion from (2.5) as

$$c_n(x) = 7c_{n-2}(x) - 15c_{n-4}(x) + 10c_{n-6}(x) - c_{n-8}(x) \; .$$

x										
4	.	.	.	.	1	.	5	.	<u>20</u>	.
		.	.	1	.	5	.	20	.	75
2	.	.	1	.	4	.	15	.	<u>55</u>	.
	.	1	.	3	.	10	.	35	.	124
0	1	.	2	.	6	.	20	.	<u>69</u>	.
	.	1	.	3	.	10	.	34	.	117
-2	.	.	1	.	4	.	14	.	<u>48</u>	.
	.	.	.	1	.	4	.	14	.	48
-4	.	.	.	.	.	.	.	.	.	.

n

-1	.	10	.	-15	.	7	.	Coefficients
0		+10		-60		+98		= <u>48</u>
-1		+20		-90		+140		= <u>69</u>
0		+10		-60		+105		= <u>55</u>
0		+0		-15		+35		= <u>20</u>

<u>Figure 1.</u> A table of $c_n(x)$, the number of paths leading to (n,x) without hitting the boundary $-a = -4$ and $b = 5$, as constructed from de Moivre's algorithm (2.4). Also shown are the coefficients of the recursion formula (2.5) for $a = 4$ and $b = 5$ and the computation of $c_8(x)$. A dot means zero.

3. LAPLACE'S DERIVATION OF THE RECURSION FORMULAE.

Laplace discussed the problem in two papers (1774, 1776). In the first paper Laplace only discussed the symmetric case $a = b$. In the second paper he first solved the problem for $a = b$ and then gave the solution by the same method for any a and b.

To avoid repetitions and shorten the exposition we shall first give the general solution and then point out how to obtain the solution for $a = b$. We shall use modern notation, which also helps to shorten the proofs, but we shall keep faithfully to the ideas of Laplace. Before treating the general case it is practical to discuss two lemmas which form parts of Laplace's general proof.

Lemma 1. An expansion of the binomial. (Laplace, 1776).

$$(p+q)^a = p^a + q^a + \sum_{i=1}^{[a/2]} \gamma_i(a-1)(p+q)^{a-2i}, \tag{3.1}$$

where

$$\gamma_i(a-1) = (-1)^{i-1} \frac{a}{a-i} \binom{a-i}{i} (pq)^i. \tag{3.2}$$

Proof. Laplace argues that it is clear that the form of the expansion in (3.1) is correct, so that only the problem of finding the coefficients remains. Multiplying (3.1) by $(p+q)$ we get

$$(p+q)^{a+1} = \sum_{i=1}^{[a/2]} \gamma_i(a-1)(p+q)^{a-2i+1} + p^{a+1} + q^{a+1} + (pq)\left(p^{a-1} + q^{a-1}\right).$$

Eliminating $p^{a-1} + q^{a-1}$ by means of (3.1) we find

$$(p+q)^{a+1} = (\gamma_1(a-1) + pq)(p+q)^{a-1} + \sum_{i=2}^{[a/2]} (\gamma_i(a-1) - p\,q\,\gamma_{i-1}(a-2))(p+q)^{a-2i+1}$$
$$+ p^{a+1} + q^{a+1}\,.$$

Equating coefficients in this expression and the one given by (3.1) we get the difference equations

$$\gamma_1(a) = \gamma_1(a-1) + pq \text{ and } \gamma_i(a) = \gamma_i(a-1) - pq\, \gamma_{i-1}(a-2),\ i \geqq 2, \tag{3.3}$$

which have the solution (3.2).

Lemma 2. Random walk with only one absorbing barrier. (De Moivre, 1756, p. 210. Laplace, 1776).

The probability of absorption at $x = -a$ or A's probability of ruin is

$$r_{a+2i} = q^a (pq)^i \frac{a}{a+2i} \binom{a+2i}{i},\ i \geqq 0, \tag{3.4}$$

and zero elsewhere.

Proof. Laplace follows closely de Moivre's rule for computing these probabilities. Since ruin is impossible for $n < a$ we have $r_n = 0$ for $n < a$. Considering

$$(p+q)^a = q^a + \binom{a}{1} q^{a-1} p + \binom{a}{2} q^{a-2} p^2 + \binom{a}{3} q^{a-3} p^3 + \ldots$$

we see that $r_a = q^a$. Multiplying the remaining continuation probabilities by $q^2 + 2\,pq + p^2$ we get

$$\begin{aligned} &\binom{a}{1} q^{a+1} p + \binom{a}{2} q^a p^2 + \binom{a}{3} q^{a-1} p^3 \\ &\quad + 2\binom{a}{1} q^a p^2 + 2\binom{a}{2} q^{a-1} p^3 \\ &\quad + \binom{a}{1} q^{a-1} p^3 + \ldots \\ &= aq^{a+1} p + \frac{a(a+3)}{2} q^a p^2 + \binom{a+2}{3} q^{a-1} p^3 + \ldots, \end{aligned}$$

which shows that $r_{a+2} = q^{a+1} p\, a$. Continuing in this manner Laplace finds

$r_{a+4} = q^{a+2}p^2\, a(a+3)/2!$, $r_{a+6} = q^{a+3}p^3\, a(a+4)(a+5)/3!$ and then he says "and so on". It is easy to see that $r_{a+2i+1} = 0$ for $i \geqq 0$.

It is peculiar that Laplace did not give a proof based on the difference equation. He did not state that the solution above corresponds to the case with $b \to \infty$ but used his result only for the determination of the starting values for the recursion in the general case as we shall see later in (3.15).

Laplace's proof of the recursion for any a and b.

We shall next turn to the proof of the recursion formula (2.2) for a random walk with two absorbing barriers.

De Moivre's algorithm (1756, p.203) for finding the continuation probabilities is as follows: Let $a < b$, say. For $n < a, u_n(x)$ equals the corresponding term of the binomial expansion $(p+q)^n$. For $n = a$ we have to reject the extreme term $r_a = q^a$, which corresponds to the ruin of A. Multiply the remaining sum by $(p+q)$ and continue until $(p+q)^n$ is reached, rejecting one or both extremes after each multiplication if the difference between the exponents of p and q equals either $-a$ or b. Laplace (1776) expressed this sequential procedure in the form of the difference equation (2.1).

To transform the partial difference equation to an equation in one variable Laplace began by eliminating $u_{n-1}(x+1)$. Starting from $x = b-1$ he obtained in this way

$$u_n(b-1) = pu_{n-1}(b-2) \quad \text{for } n \geqq 1, \tag{3.5}$$

$$u_n(b-2) = pqu_{n-2}(b-2) + pu_{n-1}(b-3) \quad \text{for } n \geqq 2, \tag{3.6}$$

and so on. It follows that $u_n(x)$ may be written in the form

$$u_n(x) = \sum_{i=1} \alpha_i(x)u_{n-2i}(x) + \sum_{i=1} \beta_i(x)u_{n-2i+1}(x-1) \quad \text{for } n \geqq b-x, \tag{3.7}$$

for suitably chosen coefficients $\alpha_i(x)$ and $\beta_i(x)$, the total number of terms on the right-hand side being b-x. If b-x is even there are $(b-x)/2$ α's and β's and if b-x is odd there are $(b-x-1)/2$ α's and $(b-x+1)/$ β's. The summations are from $i = 1$ to the appropriate upper limits.

Setting $x = -a+1$ and noting that $u_n(-a) = 0$ we get

$$u_n(-a+1) = \sum_{i=1} \alpha_i(-a+1)\, u_{n-2i}(-a+1) \quad , \; n \geqq a+b-1, \tag{3.8}$$

which is a recursion formula of the form (2.2). The problem is only to find $\alpha_i(x)$.

Laplace's idea is to eliminate $u_{n-1}(x+1)$ using (2.1) in the form

$$qu_{n-1}(x+1) = u_n(x) - pu_{n-1}(x-1).$$

From (3.7) we obtain

$$\begin{aligned} qu_{n-1}(x+1) &= \sum_{i=1} \alpha_i(x+1) qu_{n-2i-1}(x+1) + q\sum_{i=1} \beta_i(x+1) u_{n-2i}(x) \\ &= \sum_{i=1} \alpha_i(x+1)\{u_{n-2i}(x) - pu_{n-2i-1}(x-1)\} + q\sum_{i=1} \beta_i(x+1) u_{n-2i}(x). \end{aligned}$$

Inserting this into (2.1) we find

$$u_n(x) = \sum_{i=1} \{\alpha_i(x+1) + q\beta_i(x+1)\} u_{n-2i}(x) + pu_{n-1}(x-1) - p\sum_{i=1} \alpha_i(x+1) u_{n-2i-1}(x-1).$$

Comparing with (3.7) we have

$$\alpha_i(x+1) + q\beta_i(x+1) = \alpha_i(x), \; i \geqq 1,$$

and

$$\beta_1(x) = p, \quad \beta_i(x) = -p\, \alpha_{i-1}(x+1), \; i \geqq 2,$$

so that

$$\alpha_1(x+1) - \alpha_1(x) = -pq \tag{3.9}$$

and

$$\alpha_i(x+1) - \alpha_i(x) = pq\ \alpha_{i-1}(x+2),\ i \geqq 2. \tag{3.10}$$

The boundary conditions follow from (3.5) and (3.6), which shows that $\alpha_i(b-1) = 0$, $i \geqq 1$, and $\alpha_i(b-2) = 0$, $i \geqq 2$. Repeated applications of (3.10) give that $\alpha_i(b+1-2i) = \beta_{i+1}(b-2i) = 0$, $i \geqq 1$.

The solution of (3.9) is

$$\alpha_1(x) = (b-1-x)pq \quad \text{since } \alpha_1(b-1) = 0.$$

Laplace also finds α_2 and α_3 from (3.10) and then says "and so on". It is easy to check that the general solution of (3.10) is

$$\alpha_i(x) = (-1)^{i-1}(pq)^i(b-i-x)^{(i)}/i!\ ,\ i \geqq 1. \tag{3.11}$$

Setting

$$\alpha_i = \alpha_i(-a+1) = (-1)^{i-1}\binom{a+b-1-i}{i}(pq)^i,\ i \geqq 1, \tag{3.12}$$

(3.8) becomes

$$u_n(-a+1) = \sum_{i=1}^{[(a+b-1)/2]} \alpha_i u_{n-2i}(-a+1)\ ,\ n \geqq a+b-1. \tag{3.13}$$

This is de Moivre's recursion formula (2.2).

From the definition of r_n it follows that the same recursion holds for r_n for $n \geqq a+b$. Since the results for player B are obtained from those for A by interchanging p and q and a and b the formulae also hold for $u_n(b-1)$ and r_n^*.

Let $R_n = \sum_{i=1}^{n} r_i$. The recursion formula then holds for $\Delta R_n = r_{n+1} = qu_n(-a+1)$ and by integration we get

$$R_n = \sum_{i=1}^{[(a+b-1)/2]} \alpha_i R_{n-2i} + C, \tag{3.14}$$

C being a constant of integration. Setting $n = a+b-1$ we get

$$C = R_{a+b-1} - \sum_{i=1}^{[(a+b-1)/2]} \alpha_i R_{a+b-1-2i} , \tag{3.15}$$

which may be computed by means of (3.4) because the upper boundary affects R_n only for $n \geqq a+b$. This completes the proof.

An extension of Laplace's proof.

Since $qu_{n-1}(-a+2) = u_n(-a+1)$ the recursion (3.13) also holds for $u_{n-1}(-a+2)$. Furthermore, since

$$qu_{n-1}(-a+3) = u_n(-a+2) - pu_{n-1}(-a+1)$$

the recursion also holds for $u_{n-1}(-a+3)$ and so on. This means that the recursion holds for all x such that $-a < x < b$.

Laplace's proof of the recursion for a = b.

De Moivre (1756, p. 198) gave a formula for this case which is different from the one obtained from (3.13) for $a = b$, compare also (2.2) and (2.3). He did not explain the relation between these formulae.

In his first paper Laplace (1774) only treats the symmetric case and he stricly follows de Moivre's algorithm by taking two steos at a time which leads him to the difference equation

$$u_n(x) = 2pq\, u_{n-2}(x) + p^2 u_{n-2}(x-2) + q^2 u_{n-2}(x+2) . \tag{3.16}$$

In the second paper (1776) he begins with the symmetric case but formulates the difference equation (2.1) from which (3.16) easily follows. In both papers he uses the same method of solution, namely a slight modification of the method used above.

Because of the symmetry we need only consider positive values of x. Instead of beginning from $u_n(a-1)$ as in (3.5) Laplace begins from

$$u_n(0) = pu_{n-1}(-1) + qu_{n-1}(1) = 2qu_{n-1}(1) \ , \ n \geqq 1,$$

and

$$u_n(1) = 2pqu_{n-2}(1) + qu_{n-1}(2) \ , \quad n \geqq 2.$$

The general solution may therefore be written as

$$u_n(x) = \sum_{i=1} \overset{*}{\alpha}_i(x)u_{n-2i}(x) + \sum_{i=1} \overset{*}{\beta}_i(x)u_{n-2i+1}(x+1) \ , \ n \geqq x+1 \ ,$$

which leads to the recursion

$$u_n(a-1) = \sum_{i=1} \overset{*}{\alpha}_i(a-1) \ u_{n-2i}(a-1) \ , \ n \geqq a \ , \tag{3.17}$$

because $u_n(a) = 0$.

Proceeding as in the general case we get the difference equations

$$\overset{*}{\alpha}_i(x+1) = \overset{*}{\alpha}_i(x) + p\overset{*}{\beta}_i(x) \ , \ i \geqq 1 \ ,$$

and

$$\overset{*}{\beta}_1(0) = 2q, \ \overset{*}{\beta}_1(x) = q, \ \overset{*}{\beta}_i(x) = -q\overset{*}{\alpha}_{i-1}(x-1) \ , \ i \geqq 2 \ ,$$

so that

$$\overset{*}{\alpha}_1(x+1) - \overset{*}{\alpha}_1(x) = pq \ , \ \overset{*}{\alpha}_1(1) = 2pq \ ,$$

and

$$\overset{*}{\alpha}_i(x+1) - \overset{*}{\alpha}_i(x) = -pq\overset{*}{\alpha}_{i-1}(x-1) \ , \ \overset{*}{\alpha}_i(i) = 0 \ , \ i \geqq 2 \ , \tag{3.18}$$

corresponding to (3.9) and (3.10). The solution is

$$\overset{*}{\alpha}_i(x) = (-1)^{i-1}(pq)^i(x+1)(x-i)^{(i-1)}/i! \tag{3.19}$$

so that

$$\alpha_i^* = \alpha_i^*(a-1) = (-1)^{i-1}(pq)^i \frac{a}{a-i} \binom{a-i}{i} , \; i \geqq 1 \qquad (3.20)$$

It will be seen that the difference equation (3.18) is the same as (3.3) and hence that $\alpha_i^*(a-1) = \gamma_i(a-1)$.

Like de Moivre Laplace does not comment on the relation between the coefficients given by (3.12) for $a = b$ and (3.20).

From the definition of r_n it follows that r_n and $d_n = r_n + r_n^*$ also satisfy the recursion formula and that $D_n = \sum_{i=1}^{n} d_i$ therefore satisfies

$$D_n = \sum_{i=1}^{[a/2]} \alpha_i^* D_{n-2i} + C \; , \; n \geqq a \; .$$

To determine C we note that $D_n = 0$ for $n < a$ and $D_a = p^a + q^a$ which means that $C = D_a$. It follows that $U_n = 1 - D_n$ satisfies

$$U_n = \sum_{i=1}^{[a/2]} \alpha_i^* U_{n-2i} + 1 - \sum_{i=1}^{[a/2]} \alpha_i^* - p^a - q^a = \sum_{i=1}^{[a/2]} \alpha_i^* U_{n-2i} \; , \; n \geqq a \; ,$$

the constant term being zero according to (3.1).

The relation between α_i for $a = b$ and α_i^* will be discussed in Theorem 5.

4. A COMBINATORIAL PROOF OF DE MOIVRE'S RECURSION FORMULAE.

The difference equation (2.1) shows how to obtain $u_n(x)$ from previous values, i.e. values of $u_m(y)$ with $m < n$.

This gives a way of deriving a variety of recursions and explicit results by using the equation (2.1) to eliminate values $u_m(y)$ until one reaches the boundary values, where $u_m(y)$ is known, or until one reaches those combinations of (m,y) which one wants to keep in the recursion.

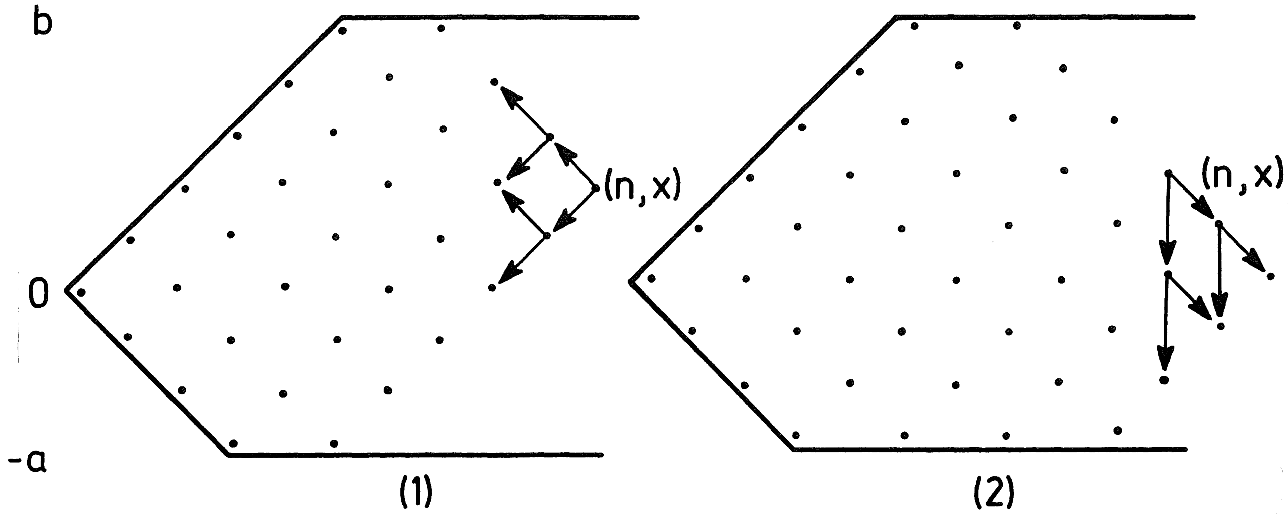

Figure 2.1 The recursion of de Moivre shows how $u_n(x)$ is expressed in terms of previous values, with lower n.

Figure 2.2 The recursion of Laplace shows how $u_n(x)$ is expressed in terms of later values with lower values of x.

A common property of these recursions is that the coefficients are positive. In fact $u_n(x)$ will be expressed as a convex combination of previous values. The idea of Laplace, however, was to solve (2.1) for $u_{n-1}(x+1)$ which leads to an equation of the form

$$qu_n(x) = u_{n+1}(x-1)-pu_n(x-2),$$

which gives a completely different type of recursion, since it expresses $u_n(x)$ by values $u_m(y)$, with $y < x$ and $m \geqq n$, and changing sign of the coefficients. By continuing the elimination from (n,x) down to $x = -a + 1$, we get a relation between $u_n(x)$ and $r_m = qu_{m-1}(-a+1)$ with $m \geqq n$, which for $x = b$ gives the relation (2.2), since $u_n(b) = 0$, see Figure 2. This argument also shows that (2.2) is satisfied by $u_n(x)$ itself. We can therefore prove a general result which contains (2.2) and a number of other results.

The basic tool in the proof of these relations is the method of inclusion and exclusion, a method used by de Moivre in other contexts. The new idea is to find a way of shortening and lengthening the paths of a random walk in such a way, that we get a relation between the ruin probabilities for different values of n. The main difficulty is to identify the coefficient $\binom{m-i}{i}$ as a solution of a combinatorial problem. We shall formulate this as a lemma. Consider first the numbers $1,\ldots,m$. We want to cover them by either doubletons, which cover two adjacent numbers, or singletons which cover just one number.

Lemma 3. The number of ways we can cover $(1,\ldots,m)$ by i doubletons and $m-2i$ singletons is $\binom{m-i}{i}$.

Proof. The total number of doubletons and singletons is $m-2i + i = m - i$. We just have to decide which of these is a singleton. This can be done in $\binom{m-i}{i}$ ways.

We shall meet the coefficient in the following context: let k_j, $j = 1,\ldots,i$, be the position of the first point of the j'th doubleton. Then we have that the coefficient $\binom{m-i}{i}$ is the number of terms in the summation over all $(k_1,k_2,\ldots,k_i)$ with the property that

$$1 \leqq k_1, k_1 + 1 < k_2, \ldots, k_{i-1} + 1 < k_i,\ k_i + 1 \leqq m .$$

Next we shall describe the <u>lengthening and shortening of paths</u>. Let $V_n(x)$ be the set of paths from (0,0) to (n,x), $-a < x < b$, which do not hit the barriers, i.e.

$$V_n(x) = \{(x_1,\ldots,x_n) \mid -a < S_i < b, i = 0,\ldots,n-1,\ S_n = x\}$$

so that $u_n(x) = P(V_n(x))$. The paths in $V_n(x)$ will be denoted by v.

<u>Lemma 4</u>. If $v \in V_n(x)$ contains a (+ −) at positions k and k + 1, say, $1 \leqq k < n$, then the path v' defined by

$$v'_j = \begin{cases} v_j, & 1 \leqq j < k \\ v_{j+2}, & k \leqq j < n-1 \end{cases}$$

is a path in $V_{n-2}(x)$.

<u>Proof</u>. We only have to note that the removal of a (+ −) does not change the values of the partial sums that define $V_n(x)$ and $V_{n-2}(x)$.

<u>Lemma 5</u>. If $v \in V_{n-2}(x)$ then the path v'' defined for $k \leqq n-1$ by

$$v''_j = \begin{cases} v_j, & 1 \leqq j < k \\ 1, & j = k \\ -1, & j = k+1 \\ v_{j-2}, & k+2 \leqq j \leqq n \end{cases}$$

is a path in $V_n(x)$ if $k > n-(b-x)$ or if $k < b$.

<u>Proof</u>. The transformation $v \to v''$ leaves the values of the partial sums unchanged except for $S''_k = S_{k-1} + 1 = x - \sum_{j=k}^{n-2} v_j + 1 \leqq x + 1 + (n-2-k+1) = n-k+x$. This value is less than b if $k > n-(b-x)$ which shows the first result. Another evaluation is $S''_k = S_{k-1} + 1 \leqq k-1+1 = k < b$, which shows the second result.

Thus we can lengthen a path in $V_{n-2}(x)$ to a path in $V_n(x)$ by inserting a

$(+-)$ after any of the last $b-x-2$ steps and obtain a path in $V_n(x)$ with a $(+-)$ among the last $b-x$ steps. A similar result holds for the first b steps.

Corollary 1. There is a one to one correspondence between the paths in $V_n(x)$ with a $(+-)$ among the last $b-x$ steps and the paths in $V_{n-2}(x)$. Similarly there is a one to one correspondence between the paths in $V_n(x)$ with a $(+-)$ among the first b steps and the paths in $V_{n-2}(x)$.

We can now formulate the main result.

Consider the set of paths of length n which have a $(+-)$ on the positions k and $k+1$, i.e.

$$A_k = \{X_k = 1, X_{k+1} = -1\} \quad , \quad 1 \leqq k < n ,$$

and the set of paths of length n with the property that they end with a string of $(-)$'s followed by a string of $(+)$'s, i.e.

$$B_k = \{X_{n-m+1} = \dots = X_k = -1,\ X_{k+1} = \dots = X_n = 1\}$$

for some $m \leqq b - x$ and $n-m \leqq k \leqq n$.

We then have

$$\bigcup_{n-m \leqq k \leqq n} B_k = [\bigcup_{n-m<k<n} A_k]^c$$

since any path will have either a $(+-)$ among the last m steps or they will consist of a string of $(-)$'s followed by a string of $(+)$'s.

Theorem 1. For $m \leqq b - x$ we have

$$\sum_{n-m \leqq k \leqq n} P(V_n(x) \cap B_k) = \sum_{i=0}^{\infty} (-1)^i \binom{(m\wedge n)-i}{i} (pq)^i P(V_{n-2i}(x)) .$$

Proof. We reduce the left-hand side using that the sets B_k are disjoint, and that their union equals the complement of the union of the A_k so that

$$\sum_{n-m\leqq k\leqq n} P(V_n(x)\cap B_k) = P(V_n(x)\cap[\bigcup_{n-m\leqq k\leqq n} B_k]) = P(V_n(x)\cap[\bigcup_{n-m<k<n} A_k]^c) ,$$

Using the method of inclusion and exclusion, see Feller (1957), we find the expression

$$P(V_n(x)) + \sum_{i=1}^{\infty} (-1)^i \sum_{n-m<k_1<\ldots<k_i<n} P(V_n(x)\cap A_{k_1}\cap\ldots\cap A_{k_i}) .$$

Now $P(V_n(x)\cap A_{k_1}\cap\ldots\cap A_{k_i})$ is zero unless

$$n-m < k_1,\ k_1+1 < k_2,\ldots,k_{i-1}+1 < k_i,\ k_i+1 \leqq n \tag{4.1}$$

and in this case we find, that the value is independent of $(k_1,\ldots,k_i)$. To see this, note that a path $v \in V_n(x)\cap A_{k_1}\cap\ldots\cap A_{k_i}$ has a $(+-)$ among the last $b-x$ steps, since $k_i > n-m \geqq n-(b-x)$.

Using Corollary 1 we find that

$$P(V_n(x)\cap A_{k_1}\cap\ldots\cap A_{k_i} \mid A_{k_i}) = P(V_{n-2}(x)\cap A_{k_1}\cap\ldots\cap A_{k_{i-1}})$$

and that

$$P(V_n(x)\cap A_{k_1}\cap\ldots\cap A_{k_i}) = pq\, P(V_{n-2}(x)\cap A_{k_1}\cap\ldots\cap A_{k_{i-1}})$$

$$=\ldots=(pq)^i P(V_{n-2i}(x)) .$$

Finally we just note that the number of non-zero terms satisfying (4.1) is $\binom{(m\wedge n)-i}{i}$ by Lemma 3. This completes the proof of Theorem 1.

<u>Theorem 2</u>. The ruin probabilities satisfy $r_n = 0$, $n < a$, $r_a = q^a$, and for $n > a$ we have

$$\sum_{i=0}^{\infty} (-1)^i \binom{(n\wedge(a+b-1))-i}{i} (pq)^i r_{n-2i+1} = 0. \tag{4.2}$$

Hence (2.2) is satisfied for $n \geqq a+b-1$.

<u>Proof</u>. That $r_n = 0$, $n < a$, and $r_a = q^a$ is obvious. For $n > a$ we choose in Theorem 1 $x = -a+1$, since then

$$V_n(-a+1) \cap B_k = \emptyset, \; n - m \leqq k < n \; ,$$

because a path that terminates at $-a+1$ without reaching $-a$ must end with a $(-)$.

The last term on the left-hand side is then

$$P(V_n(-a+1) \cap [\bigcap_{n-m<k\leqq n} \{X_k = -1\}]) = q^m u_{n-m}(-a+1+m)$$

which is interpreted as zero if $m > n$.

Thus we have the relation

$$q^{m+1} u_{n-m}(-a+1+m) = \sum_{i=0}^{\infty} (-1)^i \binom{(m \wedge n)-i}{i} (pq)^i r_{n+1-2i} \; .$$

Now take $m = a + b - 1$, then the left-hand side becomes zero since $u_{n-m}(b) = 0$. This proves (4.2).

If in the above expression we take $x = -a + 1 + m$ and choose $n' = n - m \geqq 0$, then we find

$$q^{x+a} u_{n'}(x) = \sum_{j=0}^{\infty} (-1)^j \binom{x+a-1-j}{j} (pq)^j r_{n'+x+a-2j} \; , \qquad (4.3)$$

which expresses $u_{n'}(x)$ by values of the ruin probabilities at later times. This relation will be used in

<u>Theorem 3</u>. The probabilities $u_n(x)$ satisfy

$$\sum_{i=0}^{\infty} (-1)^i \binom{a+b-1-i}{i} (pq)^i u_{n-2i}(x) = 0 \quad \text{for } n \geqq a + b - 1. \qquad (4.4)$$

<u>Proof</u>. Note that for $n \geqq a + b - 1$, then $n - 2i \geqq a + b - 1 - 2i \geqq 0$ which shows that we can use (4.3) with $n' = n - 2i$ so that the left-hand side of (4.4) equals

$$\sum_{i=0}^{\infty} (-1)^i \binom{a+b-1-i}{i} (pq)^i q^{-x-a} \sum_{j=0}^{\infty} (-1)^j \binom{x+a-1-j}{j} (pq)^j r_{n+x+a-2j-2i}$$

$$= \sum_{j=0}^{\infty} (-1)^j \binom{x+a-1-j}{j} (pq)^j \; q^{-x-a} \sum_{i=0}^{\infty} (-1)^i \binom{a+b-1-i}{i} (pq)^i r_{n+x+a-2i-2j} .$$

Now the inner summation can be reduced to zero by (4.2) provided we can prove that $n+x+a-2j-1 \geqq a+b-1$, but this is the case since the terms in the outer summation vanish unless $x+a-1 \geqq 2j$ and since we have chosen $n \geqq a+b-1$.

As a final point one can note that if the set $V_n(x)$ in Theorem 1, is replaced by the set of all paths, then one can choose $m = n$ and prove that

$$\frac{p^{n+1}-q^{n+1}}{p-q} = \sum_{i=0}^{\infty} (-1)^i \binom{n-i}{i} (pq)^i , \quad n \geqq 0, \qquad (4.5)$$

which can be considered the generating function for the coefficients $\binom{n-i}{i}$, $i \geqq 0$.

We can also derive the relation for binomial coefficients

$$\sum_{i=0}^{\infty} (-1)^i \binom{n-i}{i} \binom{n-2i}{x-i} = 1, \quad 0 \leqq x \leqq n,$$

by replacing the set $V_n(x)$ by the set $\{S_n = 2x-n\}$ and choosing $m = n$.

We next turn to the symmetric case where $a = b$. The result we want to prove is

<u>Theorem 4</u>. For $a = b$ the probabilities $u_n(x)$ satisfy

$$\sum_{i=0}^{\infty} (-1)^i \frac{a}{a-i} \binom{a-i}{i} (pq)^i u_{n-2i}(x) = 0, \quad n \geqq 1 . \qquad (4.6)$$

This relation proves (2.3).

<u>Proof</u>. The components of the proof are much the same as those of Theorem 1. This time, however, we decompose after the (+ −) configurations among the

first a steps. We define for $k = 0,1,\ldots,a$ the set of paths $C_{k,a}$ of length n, with the property that it starts with a string of (−)'s followed by a string of (+)'s , i.e.

$$C_{k,a} = \{X_1 = \ldots = X_k = -1,\ X_{k+1} = \ldots = X_a = +1\}$$

with a suitable interpretation for $k = 0$ and a.

Then

$$\bigcup_{0 \leq k \leq a} C_{k,a} = [\bigcup_{0<k<a} A_k]^c$$

which shows that, by the method of inclusion and exclusion, we get

$$\sum_{0 \leq k \leq a} P(V_n(x) \cap C_{k,a}) = \sum_{i=0}^{\infty} (-1)^i \sum_{0<k_1<\ldots<k_i<a} P(V(x) \cap A_{k_1} \cap \ldots \cap A_{k_i}) \quad (4.7)$$

By Corollary 1 and Lemma 3 this reduces to

$$\sum_{i=0}^{\infty} (-1)^i \binom{a-i}{i} (pq)^i u_{n-2i}(x). \qquad (4.8)$$

As for the left-hand side of (4.7) we note that since $V_n(x) \cap C_{0,a} = V_n(x) \cap C_{a,a} = \emptyset$ we get only contributions if $0 < k < a$. If $v \in V_n(x) \cap C_{k,a}$ we can cut away the (−+) at positions k and $k + 1$, and we then get a path v' in $V_{n-2}(x) \cap C_{k-1,a-2}$. Any path v' in this set can easily be expanded into a path v'' in $C_{k,a}$ by inserting the (−+) just after step $k-1$. The new path v'' is also in $V_n(x)$, since the values of the partial sums are unchanged, except for S_k'', but $S_k'' = S_{k-1}' - 1 \geq -(k-1) - 1 = -k > -a$. Hence

$$P(V_n(x) \cap C_{k,a}) = pq\ P(V_{n-2}(x) \cap C_{k-1,a-2}).$$

Thus the left-hand side of (4.7) becomes

$$pq \sum_{k=0}^{a-2} P(V_{n-2}(x) \cap C_{k,a-2})$$

which by (4.7) and (4.8) equals

$$pq \sum_{i=0}^{\infty} (-1)^i \binom{a-2-i}{i} (pq)^i u_{n-2-2i}(x) \ . \tag{4.9}$$

Equating (4.8) and (4.9) we find

$$0 = u_n(x) + \sum_{i=1}^{\infty} (-1)^i \left(\binom{a-i}{i} + \binom{a-i-1}{i-1}\right) (pq)^i u_{n-2i}(x)$$

$$= \sum_{i=0}^{\infty} (-1)^i \frac{a}{a-i} \binom{a-i}{i} (pq)^i u_{n-2i}(x)$$

which was to be proved.

Note also here, that if we had replaced the set $V_n(x)$ by the set of all paths we would have obtained the combinatorial identity (3.1)

$$p^a + q^a = \sum_{i=0}^{\infty} (-1)^i \frac{a}{a-i} \binom{a-i}{i} (pq)^i \tag{4.10}$$

which gives the generating function of the coefficients $\frac{a}{a-i} \binom{a-i}{i}$. This explains why the coefficients $\gamma_i(a-1)$ and $\alpha_i^*(a-1)$ of Section 3 are the same. Note also that the proofs of Lemma 1 and Theorem 4 are similar in some sense.

In this case too we could have chosen to replace $V_n(x)$ by the set $\{S_n = 2x-n\}$ in (4.7) and taken $a = n$. Then we would arrive at the identity

$$\sum_{i=0}^{\infty} (-1)^i \frac{n}{n-i} \binom{n-i}{i} \binom{n-2i}{x-i} = \begin{cases} 0 \ , & 0 < x < n, \\ 1 \ , & x = 0 \text{ and } n. \end{cases}$$

If we compare (2.2) or rather (4.4) for $a = b$ and (2.3) then we get the two relations

$$\sum_{i=0}^{\infty} (-1)^i \binom{2a-1-i}{i} (pq)^i u_{n-2i}(x) = 0 \tag{4.11}$$

and

$$\sum_{i=0}^{\infty} (-1)^i \frac{a}{a-i} \binom{a-i}{i} (pq)^i u_{n-2i}(x) = 0. \tag{4.12}$$

Now (4.12) has fewer terms (4.11) and it is to be expected that (4.11) can be derived from (4.12). This is indeed so since we have the following identity, which easily proves (4.11) from (4.12).

Theorem 5. The following identity holds

$$\sum_{i=0}^{\infty} \frac{a}{a-i} \binom{a-i}{i} \binom{a-m+i-1}{m-i} = \binom{2a-m-1}{m}, \quad 0 \leq m < a.$$

Proof. A simple proof of this convolution identity may be found by using the generating functions given earlier for the coefficients, see (4.5) and (4.10).

A combinatorial proof can be given as follows:

The right-hand side has the interpretation as the number of ways we can place m doubletons on the positions $(1,\ldots,2a-1)$. We shall decompose this number according to the number of doubletons i, say, which have a left end-point in $(1,\ldots,a)$. Hence $i \in (0,\ldots,m)$. There are two possiblities, either the i'th doubleton is at position $(a,a+1)$, in which case we have $(i-1)$ among $(1,\ldots,a-1)$ and $(m-i)$ in $(a+2,\ldots,2a-1)$. This gives $f(i) = \binom{a-1-i+1}{i-1} \binom{a-2-m+i}{m-i}$ possibilities, or if this is not the case then the i'th doubleton is on the set $(1,\ldots,a)$ and then we have the remaining $m-i$ doubletons on the set $(a+1,\ldots,2a-1)$. In this case the number of possibilities is $g(i) = \binom{a-i}{i} \binom{a-1-m+i}{m-i}$.

Hence we find

$$\binom{2a-m-1}{m} = \sum_{i=0}^{m} (f(i) + g(i)) = \sum_{i=0}^{m} (f(m-i+1) + g(i))$$

$$= \sum_{i=0}^{m} \left[\binom{a-m+i-1}{m-i} \binom{a-1-i}{i-1} + \binom{a-i}{i} \binom{a-1-m+i}{m-i}\right]$$

$$= \sum_{i=0}^{m} \binom{a-m+i-1}{m-i} \left[\binom{a-1-i}{i-1} + \binom{a-i}{i} \right]$$

$$= \sum_{i=0}^{m} \binom{a-m+i-1}{m-i} \frac{a}{a-i} \binom{a-i}{i} ,$$

which completes the proof.

5. REFERENCES.

De Moivre, A. (1756): The Doctrine of Chances. 3rd ed. London. Reprinted 1967 by Chelsea, New York.

Feller, W. (1957): An Introduction to Probability and its Applications. 2nd ed. Wiley, New York.

Kohli, K. (1975): Spieldauer: Von Jakob Bernoullis Lösung der fünften Aufgabe von Huygens bis zu den Arbeiten von de Moivre. Die Werke von Jakob Bernoulli, Band 3, 403-455. Birkhäuser, Basel.

Laplace, P.S. (1774): Mémoire sur les suites récurro-récurrentes et sur leurs usages dans la théorie des hasards. Oeuvres, 8, 5-24.

Laplace, P.S. (1776): Recherches sur l'intégration des équations differéntielles aux différences finies et sur leur usage dans la théorie des hasards. Oeuvres, 8, 69-197.

Schneider, I. (1968): Der Mathematiker Abraham de Moivre. Arch. History Exact Sci., 5, 177-317.

Todhunter, I. (1865): History of the Mathematical Theory of Probability. Macmillan, London.

PREPRINTS 1981

COPIES OF PREPRINTS ARE OBTAINABLE FROM THE AUTHOR OR FROM THE INSTITUTE OF MATHEMATICAL STATISTICS, UNIVERSITETSPARKEN 5, 2100 COPENHAGEN Ø, DENMARK.

No. 1 Johansen, Søren: Asymptotic Inference in Random Coefficient Regression Models.

No. 2 Asmussen, Søren: On the Role of a Certain Eigenvalue in Estimating the Growth Rate of a Branching Process.

No. 3 Lauritzen, Steffen.: Time Series Analysis in 1880. A Discussion of Contributions made by T.N. Thiele.

No. 4 Asmussen, Søren: Conditioned Limit Theorems Relating a Random Walk to its Associate, with Applications to Risk Reserve Processes and the GI/G/1 Queue.

No. 5 Johansen, Søren: The Statistical Analysis of a Markov Branching Process.

No. 6 Asmussen, Søren: Time-Dependent Approximations in some Queueing Systems with Imbedded Markov Chains Related to Random Walks.

No. 7 Skovgaard, Ib M.: A Second-Order Investigation of Asymptotic Ancillarity.

No. 8 Rootzén, Holger: The Rate of Extremes of Stationary Normal Sequences.

No. 9 Skovgaard, Ib M.: Large Deviation Approximations for Maximum Likelihood Estimators.

No. 10 Jensen, Ulla Funck: A Stochastic Projection Model with Implications for Multistate Demography and Manpower Analysis.

No. 11 Johansen, Søren: An Extension of Cox's Regression Model.

No. 12 Skovgaard, Ib M.: Edgeworth Expansions in Statistics.

PREPRINTS 1982

COPIES OF PREPRINTS ARE OBTAINABLE FROM THE AUTHOR OR FROM THE INSTITUTE OF MATHEMATICAL STATISTICS, UNIVERSITETSPARKEN 5, 2100 COPENHAGEN Ø, DENMARK.

No. 1 Holmgaard, Simon and Yu, Song Yu: Gaussian Markov Random Fields Applied to Image Segmentation.

No. 2 Andersson, Steen A., Brøns, Hans K. and Jensen, Søren Tolver: Distribution of Eigenvalues in Multivariate Statistical Analysis.

No. 3 Tjur, Tue: Variance Component Models in Orthogonal Designs.

No. 4 Jacobsen, Martin: Maximum-Likelihood Estimation in the Multiplicative Intensity Model.

No. 5 Leadbetter, M.R.: Extremes and Local Dependence in Stationary Sequences.

No. 6 Henningsen, Inge and Liestøl, Knut: A Model of Neurons with Pacemaker Behaviour Recieving Strong Synaptic Input.

No. 7 Asmussen, Søren and Edwards, David: Collapsibility and Response Variables in Contingency Tables.

No. 8 Hald, A. and Johansen, S.: On de Moivre's Recursion Formulae for Duration of Play.