M. R. Leadbetter

Extremes and Local Dependence in Stationary Sequences

Preprint August 1982

Institute of Mathematical Statistics
University of Copenhagen
M.R. Leadbetter

EXTREMES AND LOCAL DEPENDENCE IN STATIONARY SEQUENCES

Preprint 1982 No. 5

INSTITUTE OF MATHEMATICAL STATISTIC
UNIVERSITY OF COPENHAGEN

August 1982
EXTREMES AND LOCAL DEPENDENCE IN STATIONARY SEQUENCES

by

M.R. Leadbetter*

Summary: Extensions of classical extreme value theory to apply to stationary sequences generally make use of two types of dependence restriction:

(a) a weak "mixing condition" restricting long range dependence

(b) a local condition restricting the "clustering" of high level exceedances.

The purpose of this paper is to investigate extremal properties when the local condition (b) is omitted. It is found that, under general conditions, the type of the limiting distribution for maxima is unaltered. The precise modifications and the degree of clustering of high level exceedances are found to be largely described by a parameter here called the "extremal index" of the sequence.

Key words: Extremes, maxima, stationary processes

* Research supported by the Danish Natural Science Research Council and the Air Force Office of Scientific Research Grant No. AFOSR F49620 82 C 0009.
1. INTRODUCTION

Classical Extreme Value Theory discusses the possible limiting laws for the maximum

\[ M_n = \max(\xi_1, \xi_2, ..., \xi_n) \]

of \( n \) independent identically distributed (i.i.d.) random variables (r.v.) as \( n \to \infty \). Specifically it is shown that if \( M_n \) has a non-degenerate distribution \( G \), i.e. if

\[ \lim_{n \to \infty} P\{a_n (M_n - b_n) \leq x\} \overset{d}{=} G(x) \]

for some constants \( a_n > 0, b_n \) then \( G \) must be one of the following classical types (in the sense that \( G(x) = H(ax+b) \) for some \( a > 0, b \) where \( H \) is one of the listed distributions):

- **Type I** \( H(x) = \exp(-e^{-x}) \) \(-\infty < x < \infty\)
- **Type II** \( H(x) = \exp(-x^{-\alpha}) \) \( x > 0 \) (\( \alpha > 0 \))
  \[ = 0 \quad x \leq 0 \]
- **Type III** \( H(x) = \exp(-(-x)^{\alpha}) \) \( x \leq 0 \) (\( \alpha > 0 \))
  \[ = 1 \quad x > 0 \]

It may be shown that this result remains true (cf [7], [9]) if the condition that the \( \xi_i \) be i.i.d. is replaced by the requirement that they form a stationary sequence satisfying a very weak dependence restriction. This restriction, here referred to as the **distributional mixing condition** \( D(u_n) \) is defined as follows.

Write \( F_{i_1...i_n}(x_1...x_n) = P(\xi_{i_1} \leq x_1, ..., \xi_{i_n} \leq x_n) \) for the joint distribution function of \( \xi_{i_1}, ..., \xi_{i_n} \), and, for brevity, \( F_{i_1...i_n}(u) = F_{1...n}(u,u...u) \) for each \( n, i_1, ..., i_n, u \). Let \( \{u_n\} \) be a sequence of constants. Then the sequence \( \{\xi_n\} \) is said to satisfy \( D(u_n) \) if for each \( n, \ell \) and each choice of integers \( i_1...i_p, j_1...j_p \), such that \( 1 \leq i_1 < i_2 < ... < i_p < j_1 < ... < j_p \leq n, j_1 - i_p \geq \ell \) we have
where $\alpha_n, \xi_n \to 0$ as $n \to \infty$ for some sequence $\{\xi_n\}$ with $\xi_n = o(n)$.

In spite of the slightly complicated definition this condition is clearly much weaker than the standard forms of mixing condition (such as strong mixing) in that it requires only approximate independence of events A "from the past" and B "from the future" having the special, simple forms

$$P_A = (\xi_1 \leq u_n), \quad B = (\xi_j \leq u_n).$$

The specific form of the theorem referred to above (proved in [7], [9]), is as follows.

Theorem 1.1. Let $\{\xi_n\}$ be a stationary sequence such that $M_n = \max\{\xi_1, \ldots, \xi_n\}$ has a non-degenerate limiting distribution $G$ as in (1.2) for some constants $a_n > 0, b_n$. Suppose that $D(u_n)$ holds for all sequences $u_n$ given by $u_n = x/a_n + b_n; -\infty < x < \infty$. Then $G$ is one of the three classical types given above.

Thus the condition $D(u_n)$ alone is sufficient to guarantee that the central classical result concerning the possible extremal types, holds also for stationary sequences.

It is also shown in [7] that if a further condition holds - there called $D'(u_n)$, viz.

$$D'(u_n): \limsup_{n \to \infty} \frac{[n/k]}{n} \sum_{j=2}^{[n/k]} P\{\xi_1 > u_n, \xi_j > u_n\} \to 0 \quad \text{as } k \to \infty,$$

(1.5) for each $u_n = x/a_n + b_n$, then the particular type which applies is the same as if the sequence $\{\xi_n\}$ were independent and identically distributed (i.i.d.) with marginal d.f. $F$, and the same normalizing constants may be used. In particular this means that the classical criteria for domains of attraction (cf. [9]) may be used to determine (on the basis of the behavior of the tail $1 - F(x)$ for
large x) which limiting law applies. These assertions result from making appropriate identifications (e.g. \( u_n = x/a_n + b_n \)) in the following theorem (cf. [7]) which generalizes a simple classical result.

**Theorem 1.2.** Let \( \{\xi_n\} \) be a stationary sequence and \( \{u_n\} \) a sequence of constants such that \( D(u_n), D'(u_n) \) hold. Let \( 0 \leq \tau < \infty \). Then

\[
(1.4) \quad P\{M_n \leq u_n\} \to e^{-\tau}
\]

if and only if

\[
(1.5) \quad n[1 - F(u_n)] \to \tau.
\]

Conditions similar to \( D'(u_n) \) have been used in virtually all studies of extremes of dependent sequences beginning with the early works of Watson [15] and Loynes [10] who showed in particular that (1.5) implies (1.4), using stronger dependence restrictions that \( D(u_n) \). However since Theorem 1.1 does not require \( D'(u_n) \) in limiting the extremal distributions to the classical types, it seems worthwhile to investigate the precise role of conditions of this kind.

It has in fact been shown by Chernick [3] (extending a result of Loynes [10]) that if for each \( \tau > 0 \), \( u_n = u_n(\tau) \) is defined to satisfy (1.5), then under \( D(u_n) \) conditions alone, any limit (function) for \( P\{M_n \leq u_n(\tau)\} \) must be of the form

\[
(1.6) \quad P\{M_n \leq u_n(\tau)\} \to e^{-\theta \tau}
\]

for some \( \theta \) with \( 0 \leq \theta \leq 1 \).

In the present paper we extend this result in various ways. It will then follow, as a consequence, that in virtually all cases of practical interest the condition \( D(u_n) \) alone is sufficient to guarantee that any asymptotic distribution for the maximum \( M_n \) is of precisely the same type as if the sequence \( \{\xi_n\} \) were i.i.d. with the same marginal df. F. In fact the only essential difference which appears in dropping the assumption \( D'(u_n) \) is that the normalizing constants in
(1.2) may have to be modified from those applying to the i.i.d. case. In obtaining these results we use some ideas from O'Brien ([12], [13]).

The parameter \( \theta \) in (1.6) is here (as in [9]) called the extremal index of the sequence \( \{\xi_n\} \). The main results concerning its existence are given in Section 2, with particular criteria and examples cited in Section 3. In Section 4 we look briefly at the role of \( D'(u_n) \) in obtaining a Poisson limit for the (time-normalized) point process of exceedances of the level \( u_n \) by the \( \xi_j \)'s. When \( D'(u_n) \) does not hold, the exceedances of \( u_n \) can occur in clusters, leading to multiple points in a limiting point process. As will be seen from Section 4 the degree of clustering is directly related to the extremal index \( \theta \).

2. Extremal results under \( D(u_n) \)

The basic technique of [7] for extending extremal theory to stationary cases is to show that

\[
P\{M_n \leq u_n\} - \frac{1}{r_n} \sum_{j=1}^{r_n} \xi_j \to 0
\]

for each \( k = 1, 2, \ldots \) when \( D(u_n) \) holds, where \( r_n = \lfloor n/k \rfloor \) (the integer part of \( n/k \)).

This clearly simply reflects a form of approximate independence of the submaxima in the \( k \) subsets of \( [n/k] = r_n \) integers \( (1, 2, \ldots r_n), (r_n + 1, \ldots, 2r_n), \ldots \) which together comprise essentially all of \( (1, 2, \ldots, n) \). Here we obtain a somewhat more general version of this result. The notation \( M(E) \) will be used (here and subsequently) to denote the maximum of \( \xi_j \) for \( j \) in the set \( E \) of integers.

**Lemma 2.1.** Let \( \{u_n\} \) be a sequence of constants and let \( D(u_n) \) be satisfied by the stationary sequence \( \{\xi_n\} \). Let \( \{k_n\} \) be a sequence of constants such that \( k_n = o(n) \) and, in the notation used in stating \( D(u_n) \), \( k_n \xi_n = o(n) \), \( k_n \xi_n = o_n, \xi_n \to 0 \). Then

\[
P\{M_n \leq u_n\} - \frac{1}{r_n} \sum_{j=1}^{r_n} \xi_j \to 0 \quad \text{as } n \to \infty
\]

where \( r_n = \lfloor n/k_n \rfloor \).
Proof: This will be sketched only since it is analogous to the proof of (2.1) given e.g. in [7], [9]. We shall also assume that \( n[1 - F(u_n)] \) is bounded, which is not necessary (cf. [7]) but simplifying (and holds via (1.5) in the applications to be made).

Let \( \{\ell_n\} \) be as in the definition of \( D(u_n) \). Divide the integers 1 \...\ n into intervals (i.e. sets of consecutive integers) \( I_1, I_1^*, I_2, I_2^* \... \ I_k, I_k^* \) where

\[
I_1 = \{1, 2, \ldots, \ell_n\}, \quad I_1^* = \{r_n - \ell_n + 1 \ldots, r_n\}, \quad I_2 = \{r_n + 1 \ldots, 2r_n - \ell_n\},
I_2^* = \{2r_n - \ell_n + 1 \ldots, 2r_n\}, \ldots, \ I_k = \{(k - 1)r_n + 1 \ldots, kr_n - \ell_n\}, \quad I_k^* = \{kr_n - \ell_n + 1 \ldots, n\}.
\]

Thus each interval \( I_j \) contains \( r_n - \ell_n \) integers, with each \( I_j^* \) except \( I_k^* \) having \( \ell_n \) integers, and \( I_k^* \) having \( n - kr_n + \ell_n \leq k + \ell_n \) (since \( r_n = [n/k_n] \)). It is readily seen that

\[
\begin{align*}
0 & \leq P_{\ell_n} \{M(I_j) \leq u_n\} - P_{\ell_n} \{M \leq u_n\} \\
& \leq \frac{k}{n} \{M(I_1^*) > u_n\} + P_{\ell_n} \{M(I_k^*) > u_n\} \\
& \leq \frac{k}{n} \ell_n + \frac{\ell_n + \ell_n}{n} \{M > u_n\} \\
& \leq \frac{k}{n} \frac{(\ell_n + 1)}{n} \rightarrow 0 \text{ as } n \rightarrow \infty
\end{align*}
\]

by virtue of the stated assumptions (\( K \) being a constant).

It follows from \( D(u_n) \) by a straightforward induction (cf. [7, Lemma 2.3]) that

\[
|P_{\ell_n} \{M(I_1) \leq u_n\} - P_{\ell_n} \{M \leq u_n\}| \leq k \alpha_{\ell_n, \ell_n}
\]

which tends to zero by assumption. Finally it is readily checked that

\[
\begin{align*}
|P_{\ell_n} \{M(I_1) \leq u_n\} - P_{\ell_n} \{M \leq u_n\}| \\
\leq k \frac{\ell_n}{n} \{M(I_1) > u_n\} - P_{\ell_n} \{M \leq u_n\} = k \frac{\ell_n}{n} \{M(I_1) > u_n \leq M(I_1^*)\} \\
\leq k \frac{\ell_n}{n} \{M \leq u_n\} \leq K \frac{\ell_n}{n} \rightarrow 0
\end{align*}
\]
The result now follows at once by combining (2.3), (2.4) and (2.5).

We suppose from now on that for each \( \tau > 0 \) a sequence \( \{u_n(\tau)\} \) is defined to satisfy (1.5), viz.

\[
(2.6) \quad n[1-F(u_n(\tau))] \to \tau
\]

This imposes a slight restriction on the marginal d.f. \( F \) of the \( \xi_n \), but one which will always be satisfied in the applications made. Of course if \( F \) is continuous, \( u_n(\tau) \) can be defined to give equality in (2.6). In any case it is necessary and sufficient for (2.6) to hold that

\[
(2.7) \quad \frac{1-F(x)}{1-F(x-)} \to 1 \quad \text{as } x \to \infty
\]

(cf. [9]), a condition which always holds for any \( F \) in any of the three classical domains of attraction. It is also evident that if there exists \( u_n(\tau) \) satisfying (2.6) for one fixed \( \tau > 0 \), then there exists such a \( u_n(\tau) \) for all \( \tau > 0 \) (e.g. if \( u_n(1) \) satisfies (2.6) with \( \tau = 1 \), define \( u_n(\tau) = u_n[n/\tau](1) \)).

The following result reformulates and extends Theorem 1.2.

**Theorem 2.2.** Let \( \{\xi_n\} \) be a stationary sequence and \( \{u_n(\tau)\} \) constants satisfying (2.6) and such that \( D(u_n(\tau_0)) \) holds for some \( \tau_0 > 0 \). Then there exist constants \( \theta, \theta' \), \( 0 \leq \theta \leq \theta' \leq 1 \) such that

\[
\limsup_{n \to \infty} P\{M_n \leq u_n(\tau)\} = e^{-\theta \tau}
\]

(2.8)

\[
\liminf_{n \to \infty} P\{M_n \leq u_n(\tau)\} = e^{-\theta' \tau}
\]

for \( 0 < \tau \leq \tau_0 \). Hence if \( P\{M_n \leq u_n(\tau)\} \) converges for some \( \tau, 0 < \tau \leq \tau_0 \), then \( \theta = \theta' \) and \( P\{M_n \leq u_n(\tau)\} \to e^{-\theta \tau} \) for all such \( \tau \).

**Proof:** Note first that it is readily shown (cf. [9]) that \( D(u_n(\tau)) \) holds for \( 0 < \tau \leq \tau_0 \) since it holds for \( \tau = \tau_0 \). Write \( \psi(\tau) = \limsup_{n \to \infty} P\{M_n \leq u_n(\tau)\} \) and let \( k_n = k \) be a fixed integer. Then it follows from Lemma 2.1 with \( k_n = k \) that
(2.9) \[ \limsup_{n \to \infty} P\{M_{n'} \leq u_n(\tau)\} = \psi^{1/k}(\tau) \]

where \( n' = [n/k] \). Now if \( u_n(\tau) \geq u_{n'}(\tau/k) \) it follows that

\[ 0 \leq P\{M_{n'} \leq u_n(\tau)\} - P\{M_{n'} \leq u_{n'}(\tau/k)\} \leq \sum_{j=1}^{n'} P\{ u_n(\tau/k) < \xi \leq u_{n'}(\tau) \} \]

\[ \leq n'[F(u_n(\tau)) - F(u_{n'}(\tau/k))] \]

This together with the corresponding inequality when \( u_n(\tau) < u_{n'}(\tau/k) \) show that

\[ |P\{M_{n'} \leq u_n(\tau)\} - P\{M_{n'} \leq u_{n'}(\tau/k)\}| \leq n'[\tau/k(1+o(1)) - \tau(1+o(1))] \]

by (2.6), and this tends to zero as \( n \to \infty \) since \( n' \to n/k \). But clearly

\[ \limsup_{n \to \infty} P\{M_{n'} \leq u_{n'}(\tau/k)\} = \psi(\tau/k), \]

and it thus follows that \( \limsup_{n \to \infty} P\{M_{n'} \leq u_n(\tau)\} = \psi(\tau/k) \).

Combining this with (2.9) we see that

(2.10) \[ \psi(\tau/k) = \psi^{1/k}(\tau) \quad 0 < \tau \leq \tau_0, \quad k = 1, 2 \ldots \]

Now \( P\{M_{n'} \leq u_{n'}(\tau/k)\} \geq 1 - n'P[\xi > u_n(\tau)] \to 1 - \tau/k \) as \( n \to \infty \) so that by taking kth powers and using Lemma 2.1, it follows that \( \liminf_{n \to \infty} P\{M_n \leq u_n(\tau)\} \geq (1 - \tau/k)^k \), and letting \( k \to \infty \) that

(2.11) \[ \liminf_{n \to \infty} P\{M_n \leq u_n(\tau)\} \geq e^{-\tau} \]

In particular this implies that \( \psi(\tau) \) is strictly positive. It is also non-increasing since if \( \tau' < \tau \) it is clear that \( u_n(\tau') > u_n(\tau) \) when \( n \) is sufficiently large. But the only strictly positive non-increasing solution to the functional equation (2.10) is \( \psi(\tau) = e^{-\theta \tau} \) for some \( \theta \geq 0 \). That is \( \limsup_{n \to \infty} P\{M_n \leq u_n(\tau)\} = e^{-\theta \tau} \) with \( \theta \geq 0 \).

Similarly it may be shown that \( \liminf_{n \to \infty} P\{M_n \leq u_n(\tau)\} = e^{-\theta' \tau} \) where clearly \( \theta' \geq \theta \).
By (2.11) \( \theta' \leq 1 \) and hence \( 0 \leq \theta \leq \theta' \leq 1 \) as asserted. Thus the relations (2.8) follow and the final statements of the theorem are immediate from these.

If \( P\{M_n \leq u_n(\tau)\} \to e^{-\theta \tau} \) for each \( \tau > 0 \) with \( u_n(\tau) \) satisfying (2.6), we shall say that the sequence \( \{\xi_n\} \) has extremal index \( \theta \) (cf. Section 1). Use of this terminology will simplify statements of later results, and in particular gives the following obvious restatement of part of the above theorem.

**Corollary 2.3.** Let \( \{\xi_n\} \) be stationary and satisfy \( D(u_n(\tau)) \) for each \( \tau > 0 \) where \( n[1-F(u_n(\tau))] \to \tau \). If for some \( \tau_0 > 0 \), \( P\{M_n \leq u_n(\tau_0)\} \) converges to a limit \( \alpha \) then \( \{\xi_n\} \) has extremal index \( \theta = -1/\log \alpha \) so that \( P(M_n \leq u_n(\tau)) \to e^{-\theta \tau} \) for all \( \tau > 0 \). □

In the next section we shall show that the addition of the condition \( D'(u_n) \) (cf. § 1) implies that \( \theta = 1 \), and give other criteria determining \( \theta \) when \( 0 \leq \theta < 1 \). However here we proceed with the more general theory, showing that if \( \{\xi_n\} \) has a non-zero extremal index \( \theta \), then any limiting distribution for the maximum must be of the same type as if the terms were i.i.d. with the same normalizing constants if \( \theta = 1 \), and simply modified constants for \( 0 < \theta < 1 \). The basic result generalizes a theorem proved by O'Brien [13] under strong mixing assumptions. Here in addition to the previous notation we write

\[
\hat{M}_n = \max(\hat{\xi}_1, \hat{\xi}_2, ..., \hat{\xi}_n)
\]

where \( \hat{\xi}_1, \hat{\xi}_2, ... \) are i.i.d. random variables with the same d.f. \( F \) as each of the stationary sequence \( \hat{\xi}_1, \hat{\xi}_2, ... \) (following Loynes [10] we call \( \hat{\xi}_1, \hat{\xi}_2, ... \) the "associated independent sequence"). We note the well known (and easily proved) result that for any \( \tau > 0 \) and sequence \( \{u_n\} \),

\[
P\{M_n \leq u_n\} = F^n(u_n) \to e^{-\tau}
\]

if and only if

\[
n[1-F(u_n)] \to \tau
\]
Theorem 2.4. Suppose that the stationary sequence \( \{ \xi_n \} \) has extremal index \( \theta \), \( 0 \leq \theta \leq 1 \). Let \( \{ v_n \} \) be any sequence of constants and \( \rho \) any constant with \( 0 \leq \rho \leq 1 \).

Then

(i) If \( \theta > 0 \)

\[
P\{ M_n \leq v_n \} \to \rho \quad \text{if and only if} \quad P\{ M_n \leq v_n \} \to \rho^\theta
\]

(ii) If \( \theta = 0 \)

(a) If \( \liminf_{n \to \infty} P\{ M_n \leq v_n \} > 0 \), then \( P\{ M_n \leq v_n \} \to 1 \)

(b) If \( \limsup_{n \to \infty} P\{ M_n \leq v_n \} < 1 \), then \( P\{ M_n \leq v_n \} \to 0 \)

Proof: (i) Suppose \( \theta > 0 \) and \( P\{ M_n \leq v_n \} \to \rho \) where \( 0 < \rho \leq 1 \). Choose \( \tau > 0 \) such that \( e^{-\tau} < \rho \). Then

\[
P\{ M_n \leq v_n (\tau) \} \to e^{-\tau}, \quad P\{ M_n \leq v_n \} \to \rho > e^{-\tau}
\]

so that \( v_n > u_n (\tau) \) for sufficiently large \( n \), and hence

\[
\liminf_{n \to \infty} P\{ M_n \leq v_n \} \geq \lim_{n \to \infty} P\{ M_n \leq v_n (\tau) \} = e^{-\theta \tau}
\]

Since this holds for any \( \tau \) such that \( e^{-\tau} < \rho \) it follows that \( \liminf_{n \to \infty} P\{ M_n \leq v_n \} \geq \rho^\theta \).

It also follows in particular that if \( \rho = 1 \) then \( P\{ M_n \leq v_n \} \to 1 = \rho^\theta \).

Similarly by taking \( e^{-\tau} > \rho \) it may be shown that \( \limsup_{n \to \infty} P\{ M_n \leq v_n \} \leq \rho^\theta \) when \( 0 \leq \rho < 1 \). Hence \( P\{ M_n \leq v_n \} \to 0 \) when \( \rho = 0 \), and for \( 0 < \rho < 1 \), \( P\{ M_n \leq v_n \} \to \rho^\theta \) by combining the inequalities for the upper and lower limits. The proof of the converse is similar so that (i) follows.

To prove (ii) we assume \( \theta = 0 \), so that \( P\{ M_n \leq u_n (\tau) \} \to 1 \) as \( n \to \infty \) for each \( \tau > 0 \).

If \( \liminf_{n \to \infty} P\{ M_n \leq v_n \} = \rho > 0 \), choose \( \tau \) with \( e^{-\tau} < \rho \) and hence \( P\{ M_n \leq u_n (\tau) \} \to e^{-\tau} < \rho \) so that \( v_n \geq u_n (\tau) \) for sufficiently large \( n \). Thus
\begin{align*}
\liminf_{n \to \infty} P\{M \leq v_n \} & \geq \lim_{n \to \infty} P\{M \leq u_n (\tau)\} = 1, \\
giving (a).
\end{align*}

To show (b) note that if \( \limsup_{n \to \infty} P\{M \leq v_n \} < 1 \) we must have \( v_n < u_n (\tau) \) for sufficiently large \( n \) and hence
\[
\limsup_{n \to \infty} P\{M \leq v_n \} \leq \lim_{n \to \infty} P\{M \leq u_n (\tau)\} = e^{-\tau}
for each \( \tau \). The conclusion (b) follows by letting \( \tau \to \infty \).

This result now enables us to give conditions in terms of the extremal index under which \( M_n \) has a limiting distribution if and only if \( M_n \) does. This of course implies that in such cases, the classical domain of attraction criteria may be used in the dependent situation.

**Theorem 2.5.** Let the stationary sequence \( \{\xi_n\} \) have extremal index \( \theta > 0 \). Then \( M_n \) has a non-degenerate limiting distribution if and only if \( \hat{M}_n \) does, and these are then of the same type based on the same normalizing constants. In the case \( \theta = 1 \) the limiting distributions for \( M_n \) and \( \hat{M}_n \) are identical.

**Proof:** If \( P\{a (\hat{M}_n - b_n) \leq x\} \to G(x) \), non-degenerate, then Theorem 2.4 (i) shows (with \( v_n = x/a_n + b_n \)) that \( P\{a (M_n - b_n) \leq x\} \to G^\theta(x) \). But \( G \) is an extreme value distribution and it is well known (and easily checked from the possible functional forms) that 
\( G^\theta \) is of the same type as \( G \) in the sense of Section 1 that \( G^\theta(x) = G(ax+b) \) for some \( a > 0, b \).

The converse follows similarly, noting that if \( P\{a (M_n - b_n) \leq x\} \to H(x) \), non-degenerate, then \( P\{a (\hat{M}_n - b_n) \leq x\} \to H^{1/\theta}(x) \). As a limiting distribution for maxima from an i.i.d. sequence, \( H^{1/\theta} \) must be of extreme value type and \( H = (H^{1/\theta})^\theta \) must be of the same type as \( H^{1/\theta} \).

The final remark for \( \theta = 1 \) is obvious.

For the case \( 0 < \theta < 1 \) the same normalizing constants give limits e.g. \( G(x) \),
$G^\theta(x) = G(ax+b)$ for $\hat{M}_n$ and $M_n'$. Of course a simple change of the set of normalizing constants for $M_n$ will lead to the same limit $G(x)$.

It is, of course, also of interest to explore the situation when the extremal index is zero. An argument of R. Davis [4] shows (using also Theorem 2.4 (ii)) that $M_n$ and $\hat{M}_n$ cannot both have non-degenerate limiting distributions based on the same normalizing constants. This is stated precisely as follows, without proof.

**Theorem 2.6.** Let the stationary sequence $\{\xi_n\}$ satisfy $D(u_n(\tau))$ where for each $\tau > 0$ $u_n(\tau)$ satisfies (2.6). If $\{\xi_n\}$ has extremal index $\theta = 0$, then $M_n$ and $\hat{M}_n$ cannot both have non-degenerate limiting distributions based on the same normalizing constants. That is, it is not possible to have $P\{a_n(\hat{M}_n - b_n) \leq x\} \rightarrow G(x)$, $P\{a_n(M_n - b_n) \leq x\} \rightarrow H(x)$ for non-degenerate $G, H$.

Further comments on the (perhaps somewhat pathological) case when $\theta = 0$ will be given in Section 3.

3. Some criteria for the extremal index, and examples.

The first result has perhaps more theoretical than practical interest but serves as a means of extending the condition $D'(u_n)$ to apply to more dependent cases with $\theta < 1$. By way of convenient notation we again write $n' = \lceil n/k \rceil$ for fixed $k, n = 1, 2, \ldots$. Also as previously $F_{i_1, \ldots, i_r}(u)$ will denote the joint d.f. of $\xi_{i_1, \ldots, i_r}$ evaluated at $(u, u, \ldots, u)$.

**Theorem 3.1.** Let the stationary sequence $\{\xi_n\}$ satisfy $D(u_n(\tau))$ for each $\tau > 0$ where $u_n(\tau)$ satisfies (2.6). Then $\{\xi_n\}$ has extremal index $\theta$ ($0 \leq \theta \leq 1$) if and only if

\[(3.1) \quad k \limsup_{n \rightarrow \infty} |1 - F_{i_1, \ldots, i_r}(u_n) - \theta \tau_0/k| \rightarrow 0 \text{ as } k \rightarrow \infty\]

for some $\tau_0 > 0$. Equivalently this holds if and only if

\[(3.2) \quad 1 - F_{i_1, \ldots, i_r}(u_n) \rightarrow \theta \tau_0/k + \lambda_k \text{ as } n \rightarrow \infty\]

where $k\lambda_k \rightarrow 0$ as $k \rightarrow \infty$. 
Proof: For simplicity of notation we take $\tau_0 = 1$ and write $u_n = u_n(1)$. If $\{\xi_n\}$ has extremal index $\theta$, and since (2.1) holds by Lemma 2.1,

$$\mathbb{P}_1 \ldots n_i(u_n) = \mathbb{P}(M_{n_1} \leq u_n) + e^{-\theta/k} = 1 - \theta/k + o(1/k)$$

from which (3.2) (and hence obviously (3.1)) follow.

Conversely if (3.1) holds then

$$\limsup_{n \to \infty} \mathbb{P}(M_{n} \leq u_n) = \limsup_{n \to \infty} \left[ \mathbb{P}_1 \ldots n_i(u_n) - 1 + \theta/k \right] + 1 - \theta/k$$

$$\leq 1 - \theta/k + \limsup_{n \to \infty} \left[ 1 - \mathbb{P}_1 \ldots n_i(u_n) + \theta/k \right]$$

$$= 1 - \theta/k + o(1/k)$$

Hence again by (2.1)

$$\limsup_{n \to \infty} \mathbb{P}(M_{n} \leq u_n) \leq (1 - \theta/k + o(1/k))^k$$

for all $k$ giving, on letting $k \to \infty$,

$$\limsup_{n \to \infty} \mathbb{P}(M_{n} \leq u_n) \leq e^{-\theta}$$

The opposite inequality for $\liminf \mathbb{P}(M_{n} \leq u_n)$ follows similarly so that

$$\mathbb{P}(M_{n} \leq u_n) \to e^{-\theta}.$$ Thus we have convergence of $\mathbb{P}(M_{n} \leq u_n(\tau))$ to $e^{-\theta\tau}$ at $\tau = \tau_0 = 1$ and the result follows from Corollary 2.3.

The condition $D'(u_n)$ given by (1.3) limits the probability of one exceedance of $u_n$ being followed "closely" by another. One obvious generalization is to permit (with high probability) no more than some specified number of exceedances to occur together. One specific such restriction is to limit the quantity

$$(3.2) \quad E_{n, k}^{(r)} = \sum_{1 \leq i_1 < i_2 \ldots < i_r \leq n'} \mathbb{P}(\xi_{i_1} > u_n, \xi_{i_2} > u_n, \ldots, \xi_{i_r} > u_n)$$

for some $r$. For example the assumption $D'(u_n)$ limits $E_{n, k}^{(2)}$ so that in fact

$$\limsup_{n \to \infty} E_{n, k}^{(2)} \to 0$$

as $k \to \infty$, from which it follows (cf. [7]) that (1.4) holds so that
\{\xi_n\} has extremal index 1. Generalizations of this are clearly possible to allow a non-zero limit for \(\limsup_{n,k} E_{n,k}^{(\tau)}\) as \(k \to \infty\) for some values of \(\tau \geq 2\) as the following simplest case beyond \(D'(u_n)\) shows.

**Corollary 3.2.** Let the stationary sequence \(\{\xi_n\}\) satisfy \(D(u_n(\tau))\) for each \(\tau > 0\) where \(u_n(\tau)\) satisfies (2.6). Suppose that for some \(\tau_0 > 0\), \(u_n = u_n(\tau_0)\) and some \(\theta, 0 \leq \theta \leq 1\),

\[
\limsup_{n,k} |E_{n,k}^{(2)} - \tau_0(1-\theta)| \to 0 \text{ as } k \to \infty
\]

and

\[
\limsup_{n,k} E_{n,k}^{(3)} \to 0 \text{ as } k \to \infty
\]

Then \(\{\xi_n\}\) has extremal index \(\theta\).

**Proof:** Since \(1 - F_1 \ldots n'(u_n) = \mathbb{P}\{\bigcup_{j=1}^{n'} (\xi_n > u_j)\}\) it follows by standard Bonferroni inequalities that

\[
n'[1-F(u_n)] - k^{-1}E_{n,k}^{(2)} \leq 1 - F_1 \ldots n'(u_n)
\]

\[
\leq n'[1-F(u_n)] - k^{-1}E_{n,k}^{(2)} + k^{-1}E_{n,k}^{(3)}
\]

and hence

\[
kn'[1-F(u_n)] - \tau_0(1-\theta) - |E_{n,k}^{(2)} - \tau_0(1-\theta)| \leq k[1-F_1 \ldots n'(u_n)]
\]

\[
\leq kn'[1-F(u_n)] - \tau_0(1-\theta) + |E_{n,k}^{(2)} - \tau_0(1-\theta)| + E_{n,k}^{(3)}
\]

Since \(u_n = u_n(\tau_0)\), letting \(n \to \infty\) with \(k\) fixed yields

\[
\theta \tau_0 - \limsup_{n,k} |E_{n,k}^{(2)} - \tau_0(1-\theta)| \leq \liminf_{n,k} [1-F_1 \ldots n'(u_n)]
\]

\[
\leq \limsup_{n,k} [1-F_1 \ldots n'(u_n)] \leq \theta \tau_0 + \limsup_{n,k} |E_{n,k}^{(2)} - \tau_0(1-\theta)| + \limsup_{n,k} E_{n,k}^{(3)}
\]

from which it follows simply that

\[
\limsup_{n,k} [1-F_1 \ldots n'(u_n)] - \theta \tau_0 \leq \limsup_{n,k} |E_{n,k}^{(2)} - \tau_0(1-\theta)| + \limsup_{n,k} E_{n,k}^{(3)}
\]
which tends to zero as $k \to \infty$, giving (3.1) and hence the desired conclusion by the theorem.

The condition (3.3) may be restated in an obvious way to give the following alternative version of the corollary.

**Corollary 3.3.** The result of Corollary 3.2 holds if (3.3) is replaced by

$$(3.5) \quad \sum_{j=2}^{n'} (1-j/n') P\{\xi_{j-n} > u_n | \xi_1 > u_n\} + 1 - \theta + \lambda_k \text{ as } n \to \infty,$$

where $\lambda_k \to 0$ as $k \to \infty$.

**Proof:** It is simply checked that

$$E_n^{(2)} - \tau_0(1-\theta) = n[1-F(u_n)]^{\sum_{j=2}^{n'} (1-j/n') P\{\xi_{j-n} > u_n | \xi_1 > u_n\} - \tau_0(1-\theta)} + \lambda_k \text{ as } n \to \infty,$$

from which (3.3) follows at once.

By way of a very simple illustration of the use of results of this type consider i.i.d. random variables $\eta_1, \eta_2 \ldots$ with d.f. $F$ and define the sequence $\{\xi_n : n \geq 1\}$ by

$$(\xi_1, \xi_2, \xi_3 \ldots) = (\eta_1, \eta_2, \eta_2, \eta_3, \eta_3 \ldots)$$

or $(\eta_1, \eta_1, \eta_2, \eta_2 \ldots)$

each with probability $\frac{1}{2}$. It is readily checked that $P\{\xi_{2-n} > u_n | \xi_1 > u_n\} + \frac{1}{2}$

and $P\{\xi_{j-n} > u_n | \xi_1 > u_n\} = 1 - F(u_n)$ for $j > 2$ so that

$$\sum_{j=2}^{n'} (1-j/n') P\{\xi_{j-n} > u_n | \xi_1 > u_n\} = \frac{1}{2}(1+o(1)) + (1-F(u_n))^{\sum_{j=3}^{n'} (1-j/n')}$$

$$+ \frac{1}{2} + \tau_0/(2k) \text{ as } n \to \infty,$$

so that (3.5) holds with $\theta = \frac{1}{2}$. (3.4) is also clearly satisfied as is $D(u_n(\tau))$ so that $\{\xi_n\}$ has extremal index $\theta = \frac{1}{2}$. 

While "repeated limit conditions" such as (3.1) can be useful in practice, it may sometimes be more convenient to use conditions depending on a single limit only, and we shall show briefly how this may be done, giving an alternative form for Theorem 3.1.

The condition $D(u_n)$ requires that the quantity $\alpha_n \ell_n \to 0$ as $n \to \infty$ for some $\ell_n = o(n)$. It is clearly possible to obtain $k_n \to \infty$ such that both

$$k_n \alpha_n \ell_n \to 0$$

and

$$k_n \ell_n = o(n)$$

hold (e.g. taking $k_n = \min(\alpha_n \ell_n, (n/\ell_n)^k)$). Using such a sequence $k_n$ we have the following variant of Theorem 3.1.

**Theorem 3.4.** Let the stationary sequence $\{\xi_n\}$ satisfy $D(u_n(\tau))$ for each $\tau > 0$ where $u_n(\tau)$ satisfies (2.6). For some $\tau_0 > 0$ let $k_n \to \infty$ be such that (3.6) and (3.7) hold with $u_n = u_n(\tau_0)$. If, writing $r_n = \lfloor n/k_n \rfloor$,

$$k_n [1-F_{\ell_n (u_n)}] = \theta \tau_0 \quad \text{as} \quad n \to \infty \quad (0 \leq \theta \leq 1)$$

then $\{\xi_n\}$ has extremal index $\theta$. Conversely if $\{\xi_n\}$ has extremal index $\theta$ then (3.8) holds for each $\tau_0 > 0$ and each $k_n \to \infty$ satisfying (3.6) and (3.7) with $u_n = u_n(\tau_0)$.

**Proof:** If (3.8) holds then

$$P\{M_{r_n u_n} \leq u_n\} = F_{r_n u_n} = 1 - \theta \tau_0 (1+o(1))$$

so that

$$P_{r_n u_n} = [1 - \frac{\theta \tau_0}{k_n} + o(\frac{1}{k_n})] k_n \to e^{-\theta \tau_0}$$

and hence $P\{M_n \leq u_n\} = e^{-\theta \tau_0}$ by Lemma 2.1 showing that $\{\xi_n\}$ has extremal index $\theta$ by Corollary 2.3.
Conversely if \( \{ \xi_n \} \) has extremal index \( \theta \), and \( \tau_0 > 0 \), \( k_n \to \infty \) satisfying (3.6) and (3.7) then Lemma 2.1 shows that \( P^n(M_{r_n} \leq u_n) \to e^{-\theta \tau_0} \) with \( u_n = u_n(\tau_0) \). follows simply that \( F_{r_n} \to e \) and

\[
\log[1-(1-F_{r_n}(u_n))] = -\frac{\theta \tau_0}{k_n} (1+o(1))
\]

so that

\[
-[1-F_{r_n}(u_n)](1+o(1)) = -\frac{\theta \tau_0}{k_n} (1+o(1))
\]

giving (3.8) as required.

A simply expressed sufficient condition for (3.8) may be given as in the following corollary. In this we write \( E_n(s) \) for \( E_{n,k_n}(s) \) where this is given by (3.2) i.e.

\[
E_n(s) = k_n \sum_{1 \leq i_1 < \cdots < i_s \leq r_n} P\{ \xi_{i_1} > u_n \ldots \xi_{i_s} > u_n \}
\]

(3.9)

(\text{where } r_n = \lfloor n/k_n \rfloor).

\textbf{Corollary 3.5.} Let the stationary sequence \( \{ \xi_n \} \) satisfy \( D(u_n(\tau)) \) for each \( \tau > 0 \), where \( u_n(\tau) \) satisfies (2.6) For some \( \tau_0 > 0 \) let \( k_n \to \infty \) be such that (3.6) and (3.7) hold with \( u_n = u_n(\tau_0) \). Suppose that for each \( s = 1, 2 \ldots \) the \( E_n(s) \) defined by (3.9) satisfy

\[
(3.10) \quad E_n(s) \to \alpha_s
\]

where \( \alpha_s \to 0 \) as \( s \to \infty \). Then \( \{ \xi_n \} \) has extremal index

\[
\theta = \tau_0^{-1} \sum_{r=1}^{\infty} (-1)^{r-1} \alpha_r
\]

Proof: Write \( \lambda_n = k_n \sum_{1 \leq i \leq r_n} \sum_{j=1}^{i} P\{ \xi_j > u_n \} \). Then using Bonferroni Inequalities we have for \( s \) odd, \( n > s \),

\[
E_n(1) - E_n(2) + E_n(3) - \cdots + E_n(s) \geq \lambda_n \geq E_n(1) - E_n(2) + E_n(3) - \cdots - E_n(s+1)
\]

Writing \( \lambda = \liminf_{n \to \infty} \lambda_n \), \( \overline{\lambda} = \limsup_{n \to \infty} \lambda_n \) and letting \( n \to \infty \), we obtain, for each odd \( s \),
Since the extreme terms differ by \( \alpha_{s+1} \) which tends to zero as \( s \to \infty \), it follows that \( \lambda = \lambda, \) say. If \( T_s = \sum_{r=1}^{s} (-1)^{r-1} \alpha_r \), it further follows from (3.11) that
\[
0 \leq T_s - \lambda \leq T_s - T_{s+1} = \alpha_{s+1} \quad \text{for } s \text{ odd}
\]
and similarly \( 0 \leq \lambda - T_s \leq T_{s-1} - T_s = \alpha_s \quad \text{for } s \text{ even}, \) so that in both cases
\[
|T_s - \lambda| \leq \alpha_s + \alpha_{s+1} \to 0 \text{ as } s \to \infty
\]
Hence \( \sum_{r=1}^{\infty} (-1)^{r-1} \alpha_r \) converges to the value \( \lambda \) and (3.8) holds with \( \theta T_0 = \sum_{r=1}^{\infty} (-1)^{r-1} \alpha_r \), giving the desired result.

Finally in this section we cite some examples of sequences exhibiting all the possible types of behavior relative to the extremal index. In each of these cases \( D(u_n(T)) \) is satisfied.

The most common case is where \( D'(u_n(T)) \) holds leading to the extremal index \( \theta=1 \).

For example this is so for a stationary normal sequence \( \{ \xi_n \} \) with covariance sequence \( \{\rho_n\} \) satisfying the condition of S.M. Berman, [2], viz. \( r_n \log n \to 0 \) - an obviously weak condition indeed.

We have given a simple example of a case when \( \theta = \frac{1}{2} \) in the discussion above. An example where a series of values of \( \theta \) is possible through parameter choice in an auto-regressive scheme, has been given by Chernick [3]. The stable processes considered by Rootzen [14], can have any value of \( \theta \) in the range \( 0 < \theta \leq 1 \). A simple example due to L. de Haan also exhibiting this behavior is the sequence
\[
\xi_n = \max_{k \geq 0} \rho^k \eta_{n-k}
\]
where \( 0 < \rho < 1 \) and \( \{ \eta_n \} \) is an i.i.d. sequence with common d.f. \( \exp(-1/x) \). This yields an extremal index \( \theta = 1 - \rho \).

An example of Denzel and O'Brien [5] exhibits a "chain dependent" sequence \( \{ \xi_n \} \) with extremal index \( \theta = 0 \). In this case \( M_n \) has a Type II limiting distribution, but
we do not know whether $M_n$ has any sort of limiting distribution.

A further example of L. de Haan, however, provides a case where $\theta = 0$ and $M_n^\wedge, M_n$ both have limiting distributions. Specifically the sequence $\{\xi_n\}$ is defined by

$$\xi_n = \max_{k \geq 0} (\eta_n - k)$$

where $\eta_n$ are i.i.d. with common d.f. $\exp(-x^{-\alpha})$ for $x > 0, (\alpha > 1)$. In this case $M_n$ has a Type III limit with parameter $\alpha$ and norming constants $a_n = n^{-1/\alpha}, b_n = 0$ whereas $M_n^\wedge$ has a Type II limit with parameter $\alpha - 1$ and norming constants $a_n = n^{-1/(\alpha-1)}, b_n = 0$.

Finally an example of O'Brien [12] exhibits a case in which $\{\xi_n\}$ has no extremal index at all. In this each $\xi_n$ is uniform over the interval $[0,1], \xi_1, \xi_3 \ldots$ being independent and $\xi_{2n}$ a certain function of $\xi_{2n-1}$ for each $n$.

4. Point process of clusters.

As noted in Section 1, when $n[1-F(u_n)] \rightarrow \tau$ and $D(u_n)$ and $D'(u_n)$ both hold, the (time normalized) instants at which the sequence exceeds $u_n$ take on a Poisson character as $n$ becomes large. More specifically let $N_n$ denote the point process on the unit interval $(0,1]$ consisting of those points $j/n$ such that $\xi_j > u_n$. Then under the conditions above it may be shown ([8]) that $N_n$ converges weakly to a Poisson Process with intensity $\tau$ on $(0,1]$.

When $D'(u_n)$ does not hold, the exceedances of $u_n$ may tend to occur in clusters, leading to the simultaneous occurrence of multiple events i.e. a "compounding" in the limiting point process. A complete description of the limiting point process has been given by Rootzén [14] in the case where the underlying sequence $\{\xi_n\}$ belongs to a class of stable processes (cf. the above discussion in Section 3).

Again under a (multidimensional type of) strengthening of the condition $D(u_n)$, and assuming $D'(u_n)$, it is possible to obtain a "complete Poisson theorem" (cf. [1], [9]). This involves convergence of the point process in the plane with points at
(j/n, (ξ_j - b_n)/a_n), with appropriate a_n, b_n, to a certain Poisson process in the plane. Results of this type allow rather complete descriptions of (joint) asymptotic distributional results for extreme order statistics.

It is also of interest to determine the effect of eliminating the condition $D'(u_n)$ in results of this type. For example Mori [11] has shown that under strong mixing the limiting point processes are confined to a certain class (and it seems likely that this is true also under the weaker $D(u_n)$-type of condition).

We shall not investigate limiting results of these types in detail here. However it does seem interesting and useful to give the simplest of convergence results - involving the Poisson limit for the point process "positions" of the "clusters" of exceedances of high levels. This is analogous to a result of Rootzén in [14] for stable processes.

One very simple means of defining clusters of exceedances is to take a sequence $r_n$ and consider that events occurring within a distance $r_n$ of each other belong to the same cluster. $r_n$ should of course be chosen so that it is at least as large as (virtually all) cluster "lengths" but small compared with cluster "separation." For many usual situations this still leaves considerable flexibility in the choice of $r_n$, while leading to unique results as we shall see.

More specifically we shall suppose that $D(u_n)$ holds for $u_n = u_n(τ)$ satisfying (2.6), a sequence $k_n → ∞$ is chosen to satisfy (3.6) and (3.7) and $r_n = [n/k_n]$. A point process $N_n$ is defined on the unit interval $(0,1]$ as follows. If for given $s = 1, 2 ... k_n$ there is an exceedance of $u_n$ by $ξ_j$ for at least one $j$ such that $(s-1)r_n < j ≤ sr_n$, then $N_n$ has a single event at the point $t = sr_n/n$. That is any group of exceedances between $(s-1)r_n$ and $sr_n$ is replaced by a single event - after time-scaling - at $sr_n/n$, "representing" the original group. We refer to $N_n$ as the "point process of cluster positions." With this construction the
following result holds.

**Theorem 4.1.** Let the stationary sequence \( \{\xi_n\} \) satisfy \( D(u_n(\tau)) \) for each \( \tau > 0 \) where \( u_n(\tau) \) satisfies (2.6). Let \( k_n \to \infty \) be chosen to satisfy (3.6) and (3.7) and let \( \{\xi_n\} \) have extremal index \( \theta \) \((0 < \theta \leq 1)\). Then the point process \( N_n \) of cluster positions for exceedances of \( u_n(\tau) \) converges in distribution to a Poisson Process \( N \) on \((0,1]\) with intensity parameter \( \theta \tau \).

**Proof:** As in previous proofs of similar results (cf. [8]) it is by a theorem of Kallenberg [6] only necessary to show that

\[
(4.1) \quad E\{N((a,b])\} = E\{N[(a,b)]\} \quad \text{for} \quad 0 < a < b \leq 1
\]

and

\[
(4.2) \quad P(N_n(E) = 0) + P(N(E) = 0)
\]

for each finite disjoint union \( E \) of sets \((a_i,b_i]\) \( \subset (0,1] \).

If \( v_n \) denotes the number of (integer) intervals \((s-1)r_n, sr_n] \) completely contained in \([(na_i],[nb_i])\) it is clear that \( v_n \sim nr_n^{-1}(b-a) \sim k_n(b-a) \) and further that

\[
E\{N((a,b])\} \sim \sum_{i=1}^{r} P\{v_n(\xi_i > u_n)\}
\]

\[
\sim k_n(b-a)[1-F_i \ldots r_n(u_i)]
\]

\[
\to (b-a)\theta \tau
\]

by (3.8). But this is just \( E\{N((a,b])\} \) so that (4.1) follows.

To show (4.2) we write \( E = \cup_{i=1}^{p} (a_i,b_i] \) and write \( B_j \) for the integers in \( [(na_j],[nb_j])\). Then it is readily seen that

\[
P(N_n(E) = 0) = P\{\bigcap_{j=1}^{p} (M(B_j) \leq u_n)\} + o(1)
\]

\[
= \prod_{j=1}^{p} \{P(M(B_j) \leq u_n)\} + [P\{\bigcap_{j=1}^{p} (M(B_j) \leq u_n)\} - \prod_{j=1}^{p} P(M(B_j) \leq u_n)] + o(1)
\]
By a straightforward induction, the difference in square brackets does not exceed \( p\alpha_{n,n}\lambda \) in absolute value where \( \lambda \) is the minimum separation of the intervals \((a_j,b_j)\) (\( \lambda \) can be taken non-zero since abutting intervals can be combined). But \( \alpha_{n,\ell} \) may be taken non-increasing in \( \ell \) (cf. [9]) and it follows from \( D(u_n) \) that \( \alpha_{n,n}\lambda \to 0 \) as \( n \to \infty \). Since \( \{\xi_n\} \) has extremal index \( \theta \) it follows in an obvious way that

\[
P\{M(B_j) \leq u_n\} \to e^{-\theta T (b_j - a_j)}
\]

and hence

\[
P\{N_n(E) = 0\} \to \prod_{j=1}^{p} P\{N(a_j,b_j) = 0\} = P\{N(E) = 0\}
\]

proving 4.2.

It is of interest to note an intuitively appealing interpretation of the extremal index as the inverse of mean cluster size. This may be seen even in terms of the simple approach above. For the mean cluster size can be interpreted as the (limiting) mean number of exceedances in an interval of length \( r_n \), given at least one exceedance in that interval i.e. if \( Z \) denotes the number of exceedances of \( u_n(\tau) \) in an interval of length \( r_n \),

\[
E(Z \mid Z \geq 1) = \sum_{s=1}^{\infty} s P(Z = s \mid Z \geq 1)
\]

\[
= E_Z / P\{Z \geq 1\}
\]

\[
= r_n (1 - F(u_n)) / F_1 \ldots r_n (u_n)
\]

\[
\to \theta^{-1}
\]

Finally it should be noted that the limiting distributions of extreme order statistics will be affected in a more complicated way by the clustering than the maximum. These distributions would emerge from the more complete limiting result for individual exceedances. However use of the simple Poisson result given above will result in the distributions for the heights of the "kth highest clusters"
rather than the kth extreme order statistics, in an obvious way. This of course is analogous to consideration of kth highest local maxima in continuous parameter situations.

ACKNOWLEDGEMENTS

It is a pleasure to express my gratitude to Holger Rootzén, Georg Lindgren and Laurens de Haan for many stimulating conversations concerning this and related topics.
REFERENCES


PREPRINTS 1981

COPIES OF PREPRINTS ARE OBTAINABLE FROM THE AUTHOR OR FROM THE INSTITUTE OF MATHEMATICAL STATISTICS, UNIVERSITETSPARKEN 5, 2100 COPENHAGEN Ø, DENMARK.

No. 1 Johansen, Søren: Asymptotic Inference in Random Coefficient Regression Models.

No. 2 Asmussen, Søren: On the Role of a Certain Eigenvalue in Estimating the Growth Rate of a Branching Process.


No. 4 Asmussen, Søren: Conditioned Limit Theorems Relating a Random Walk to its Associate, with Applications to Risk Reserve Processes and the GI/G/1 Queue.

No. 5 Johansen, Søren: The Statistical Analysis of a Markov Branching Process.

No. 6 Asmussen, Søren: Time-Dependent Approximations in some Queueing Systems with Imbedded Markov Chains Related to Random Walks.


No. 8 Rootzen, Holger: The Rate of Extremes of Stationary Normal Sequences.

No. 9 Skovgaard, Ib M.: Large Deviation Approximations for Maximum Likelihood Estimators.

No. 10 Jensen, Ulla Funck: A Stochastic Projection Model with Implications for Multistate Demography and Manpower Analysis.

No. 11 Johansen, Søren: An Extension of Cox's Regression Model.

<table>
<thead>
<tr>
<th>No.</th>
<th>Authors</th>
<th>Title</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Holmgaard, Simon and Yu, Song Yu</td>
<td>Gaussian Markov Random Fields Applied to Image Segmentation.</td>
</tr>
<tr>
<td>3</td>
<td>Tjur, Tue</td>
<td>Variance Component Models in Orthogonal Designs.</td>
</tr>
<tr>
<td>4</td>
<td>Jacobsen, Martin</td>
<td>Maximum-Likelihood Estimation in the Multiplicative Intensity Model.</td>
</tr>
<tr>
<td>5</td>
<td>Leadbetter, M.R.</td>
<td>Extremes and Local Dependence in Stationary Sequences.</td>
</tr>
</tbody>
</table>