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TIME - DEPENDENT APPROXIMATIONS IN SOME QUEUEING SYSTEMS WITH IMBEDDED MARKOV CHAINS RELATED TO RANDOM WALKS*

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Approximations of the form $P(Q(T) \geq N) \cong C\delta^{-N}\Phi((T-\mu N)/\sigma N^{\frac{1}{2}})$ for the number $Q(T)$ of customers in the system at time T are derived for (i) the M/G/1 queue with bulk arrivals and batch service in four variants, viz. the transportation problem, the accessible batch model and two versions of delayed service (comprising E_p/G/1); (ii) the standard version of the GI/M/m queue.

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1. Introduction

Let $Q(T)$ be the number of customers in the system at time T in the $M/G/1$ queue with bulk arrivals and batch service, cf. Cohen [8] Ch. III. 2. The model involves amongst others the following quantities: The intensity α for arrivals of bulks of customers; the service times $U_1, U_2 \ldots$ with common distribution G; the probabilities $f_n (n=1,2,\ldots)$ of a bulk having size n; the batch capacities $X_1^{(2)}, X_2^{(2)}, \ldots$ and the $g_n = P(X_k^{(2)} = n)$ $(\sum_1^\infty g_n = 1)$. For simplicity, we exclude initial conditons with service in progress unless otherwise stated and define $\tau(0)=0$, $\tau(n)$ as the n^{th} departure instant and the imbedded Markov chain $Y_0, Y_1, \ldots$ by letting $Y_n = Q(\tau(n))$ be the number of customers in the system just after $\tau(n)$. The number of customers arriving during the n^{th} service period is denoted by $X_n^{(1)}$ (arrivals during idle periods are defined in terms of other r.v., c.f. $\mathcal{C},\mathcal{D}$ below). Thus

$$Y_n = Y_{n-1} - X_n^{(2)} + X_n^{(1)} \quad \text{on} \quad \{Y_{n-1} > X_n^{(2)}\} \tag{1.1}$$

and to complete the description of the model it only remains to specify the behaviour of the system if the batch capacity exceeds the number of customers when the server becomes idle. Four variants $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}$ (cf. [8] p. 369-370) will be considered. In $\mathcal{A},\mathcal{B}$ a new service period starts immediately at the end of the preceding one, whereas in $\mathcal{C},\mathcal{D}$ (of any of which the simple $M/G/1$ queue is a special case) service may be delayed:

$\mathcal{A}$ (the transportation problem [8] (ii)b). If $0 \le Y_{n-1} < X_n^{(2)}$, then the n^{th} batch contains Y_{n-1} customers (and may thus be empty), and new customers must await completion of service of this batch. Hence $Y_n = X_n^{(1)}$ and combining with (1.1), it follows that the transitions of $\{Y_n\}$ are completely described by

$$Y_n = (Y_{n-1} - X_n^{(2)})^+ + X_n^{(1)} \tag{1.2$\mathcal{A}$}$$

$\mathcal{B}$ (accessible batches [8] III 2.6). New customers can proceed immediately to service as long as the batch capacity is not fully utilized. Combining with (1.1), we thus have

$$Y_n = (Y_{n-1} - X_n^{(2)} + X_n^{(1)})^+ \tag{1.2$\mathcal{B}$}$$

Although somewhat too simple, this model incorporates features relevant for the study of queues at traffic lights (e.g. Newell [21]). A more realistic model (in some sense intermediate between $\mathcal{A}$ and $\mathcal{B}$) for that situation is offered some discussion at the end of Section 5.

$\mathcal{C}$ ([8] (ii) a) If $0 < Y_{n-1} < X_n^{(2)}$, then the system behaves as in $\mathcal{A}$. If $0 = Y_{n-1} < X_n^{(2)}$, then the server remains idle until the next group of customers of size (say) C_n arrives. Thus

$$Y_n = \begin{cases} X_n^{(1)} & \text{on } \{0 < Y_{n-1} < X_n^{(2)}\} \\ (C_n - X_n^{(2)})^+ + X_n^{(1)} & \text{on } \{0 = Y_{n-1} < X_n^{(2)}\} \end{cases} \tag{1.2$\mathcal{C}$}$$

$\mathcal{D}$ (delayed service [8] (i)) The server waits for arriving customers until the batch capacity is reached. This model will only be studied for bounded batch sizes, i.e. $P(X_n^{(2)} \in \{1,\ldots,p\}) = 1$ for some p. If $C_{n,i}$ are independent r.v., with $C_{n,i}$ having the distribution of the overshoot of level i in a renewal process governed by $\{f_n\}$, then for $i = 1,\ldots,p$

$$Y_n = C_{n,i} + X_n^{(1)} \quad \text{on} \quad \{X_n^{(2)} - Y_{n-1} = i\} \tag{1.2$\mathcal{D}$}$$

In the case $f_1 = 1$ of single arrivals and $g_p = 1$, $Q(t)$ has a standard interpretation as the number of phases present in $E_p/G/1$.

Finally we also consider the standard version of the GI/M/m queue (e.g. Takács [24] Ch. 2 or Kleinrock [17] Ch. 6). The introduction of the model is deferred to Section 6.

Defining $\lambda=EU_n$, the conditions

<u>Condition 1.1</u> $$\alpha\sum_{n=1}^{\infty} nf_n < \lambda^{-1}\sum_{n=1}^{\infty} ng_n \tag{1.3}$$

<u>Condition 1.2</u> <u>There is no</u> $h>1$ <u>with both</u> $\{f_n\}$ <u>and</u> $\{g_n\}$ <u>concentrated on</u> $\{h, 2h, 3h,\dots\}$

are well-known to ensure the ergodicity of $\{Y_n\}$,i.e. $Y_n \Rightarrow Y_\infty$ (with $\Rightarrow$ denoting convergence in distribution) for some Y_∞ with distribution independent of initial conditions. The analogue conclusion $Q(T) \Rightarrow Q(\infty)$ holds also if in addition

<u>Condition 1.3</u> <u>In models</u> A,B,G <u>is non-lattice</u>

holds (in some of the traffic applications, G would rather be deterministic and we treat this case in Section 5). Note that the l.h.s. of (1.3) represents the expected number of arrivals within one time unit and the r.h.s. the expected number of customers served when the system is working with full capacity. Another important interpretation of (1.3) is $EX_n=E(X_n^{(1)}-X_n^{(2)})<0$. In fact, (1.1) states that except at small values the increments of $\{Y_n\}$ are the same as those X_n of the random walk $\{S_n\} = \{X_1+\dots+X_n\}$ which will play a predominant role.

The object of the paper is to establish the type

$$\lim_{N\to\infty}\ \sup_{0\le T\le\infty} \left|\delta^N P(Q(T)\ge N) - C\Phi\left(\frac{T-\mu N}{\sigma N^{\frac{1}{2}}}\right)\right| = 0 \tag{1.4}$$

of tail behaviour. Here as usual Φ is the standard normal distribution function, $\delta>1,C,\ \mu,\sigma^2$ are constants to be determined later (with δ,μ,σ^2 the same in A,B,C,D , but C taking specific values C_A , C_B, C_B, C_D) and finally (1.4) has the obvious interpretation

$$P(Q(\infty)\geq N) \cong C\delta^{-N} \tag{1.5}$$

for the steady state $T=\infty$. Note also that it follows by Taylor's formula and the boundedness of $x\Phi'(x)$ that (1.4) implies

$$\lim_{N\to\infty} \sup_{0\leq T\leq\infty} \left| \delta^N P(Q(T)=N) - C(1-\delta^{-1})\Phi\left(\frac{T-\mu N}{\sigma N^{\frac{1}{2}}}\right) \right| = 0 \tag{1.6}$$

Relations similar to (1.4) for the actual and virtual waiting time in the GI/G/1 queue were obtained by the author [4], in part motivated from certain approximations in collective risk theory. It would seem reasonable to think that such relations hold in a great variety of queueing situations. However, the proofs are non-trivial already for the model in [4] (having a simple relation to random walks) as well as for the models of the present paper which, despite the fact that random walks come in a rather more complicated way than in [4], do exhibit the simplifying feature of imbedded Markov chains. The equilibrium case (1.5) is somewhat easier than (1.4). Here (1.5) was derived by Gaver [14] in models $\mathcal{B},\mathcal{C}$ with individual service $(g_1=1)$, using poles and residues under additional analyticity conditions (the author [3] gave a simple proof under minimal conditions in the simple M/G/1 case). Also in GI/M/m, the distribution of $Q(x)$ is well-known to be exact geometric modified in a finite number of terms. In practice, time-dependence is most often neglected and the steady state used as approximation. For a given $T<\infty$, a comparison of (1.4) and (1.5) might then provide some numerical tests of the accuracy of this procedure. Otherwise the interest in (1.4) arises largely from the simple functional dependence on N,T compared to the difficulties in studying exact solutions. Exact time-dependent expressions, viz. for $\int_0^\infty e^{-\beta t} Es^{Q(t)}\,dt$, are known in some cases (e.g. Takács [23] for $E_p/G/1$ and De Smit [10] for GI/M/m), and explicit expressions for Es^{Y_∞} or $Es^{Q(\infty)}$ in some further ones. They do not, however, reflect properties of the distributions in any transparent manner and, as Neuts [19] argues, "have largely been ignored by the practitioner".

Sections 2-5 of the present paper deal with the models $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}$, and Section 6 with the GI/M/m case, the study of which is essentially just a simplification of arguments from Sections 2-5. Section 2 gives some preliminaries. In particular a certain associated transient queueing system is introduced, based on the so-called associated or conjugate random walk (e.g. Feller [13] p. 406-407, [4], Keilson [15]). The expressions for μ,σ^2 and to some extent C will be in terms of the parameters of this system. In Section 3 the precise conditions for (1.4) are stated and the mainstream of proof given, with a number of technical steps left out to Section 4. The approach is probabilistic rather than based on traditional transform methods. Some of the basic tools are the application of Anscombe's Theorem ([2]) to a first passage problem, and extensions and applications (in a number of disguises) of ideas from renewal theory. A difficulty of the approach is that explicit expressions for C do not readily come out even in simple cases. Instead we attack (1.5) directly in Section 5 and use the validity for $T=\infty$ of the proof of (1.4) to identify C. Section 5 also has some further material on the imbedded Markov chain.

2. The associated transient queue

We recall the definition of f_n, g_n, G from Section 1 and let $\hat{f}(s)=\sum_0^\infty s^n f_n$ denote the p.g.f. and $\hat{G}(\beta) = \int_0^\infty e^{\beta x}dG(x)$ the m.g.f. It is well-known and readily checked that $\hat{h}(s)=Es^{X_n^{(1)}} = \hat{G}(\alpha[\hat{f}(s)-1])$.

The random walk associated with $\{S_n\}$ is defined by first solving the equation

$$1=E\delta^{X_n} = E\delta^{X_n^{(1)}} E\delta^{-X_n^{(2)}} = \hat{h}(\delta)\hat{g}(\delta^{-1})=\hat{G}(\gamma)\hat{g}(\delta^{-1}) \quad (\delta>1) \tag{2.1}$$

where we have put $\gamma=\alpha[\hat{f}(\delta)-1]$, and next define the associated probabilities by ${}^aP(X_n=i) = \delta^i P(X_n=i)$, cf. Feller [13] p. 406-407. Then $EX_n < 0$ implies ${}^aEX_n > 0$. We shall need

Condition 2.1 The equation (2.1) admits a solution $\delta > 1$ with the additional property $\hat{G}''(\gamma) < \infty$, $\hat{f}''(\delta) < \infty$.

This is essentially a restriction on the tails of G, $\{f_n\}$ and automatic, e.g., if the distributions have bounded support.

Theorem 2.1 The ${}^aP$-distribution of X_n is the same as in a queueing system with intensity ${}^a\alpha=\alpha\hat{f}(\delta)$ for arrivals of bulks, service time distribution ${}^aG(dx)= e^{\gamma x}G(dx)/\hat{G}(\gamma)$ and probabilities ${}^af_n=\delta^n/\hat{f}(\delta)$, ${}^ag_n = \delta^{-n}g_n/\hat{g}(\delta^{-1})$ for bulk size n, resp. batch capacity n.

Proof The p.g.f. of $X_n^{(1)}$, resp. X_n, in the system described in the theorem is

$$ {}^a\hat{G}({}^a\alpha[{}^a\hat{f}(s)-1])= \frac{\hat{G}(\gamma+{}^a\alpha[{}^a\hat{f}(s)-1])}{\hat{G}(\gamma)} = $$

$$ \frac{\hat{G}(\alpha[\hat{f}(\delta)-1]+\alpha[\hat{f}(\delta s)-\hat{f}(\delta)])}{\hat{G}(\gamma)} = \frac{\hat{G}(\alpha[\hat{f}(\delta s)-1])}{\hat{G}(\gamma)} , \quad \text{resp.} $$

$$ \frac{\hat{G}(\alpha[\hat{f}(\delta s)-1])}{\hat{G}(\gamma)}\ {}^a\hat{g}(s^{-1}) = \frac{\hat{G}(\alpha[\hat{f}(\delta s)-1])}{\hat{G}(\gamma)} \cdot \frac{\hat{g}(\delta^{-1}s^{-1})}{\hat{g}(\delta^{-1})} = $$

$$ \hat{G}(\alpha[\hat{f}(\delta s)-1])\ \hat{g}(\delta^{-1}s^{-1}) = E(\delta s)^{X_n} = \sum_{k=o}^{\infty} s^k\ {}^aP(X_n=k) . \quad \square $$

In view of ${}^aEX_n > 0$ this associated queueing system is easily seen to be transient, i.e. ${}^aP(Q(t)\to\infty)=1$ [when passing from P to ${}^aP$, we adapt the convention of letting the distribution of $Q(0)$ be unchanged] .

We shall also need an auxiliary process $\{Q^*(t)\}$ defined by allowing for negative values and ignoring the modifications needed when the batch capacity exceeds the queue length. That is, $Q^*(0)=Q(0)$ and $Q^*(t)-Q^*(0)= Q_+^*(t)-Q_-^*(t)$ where $\{Q_+^*(t)\}$ is compound Poisson (in fact, just the arrival process of $\{Q(t)\}$) and $Q_-^*(t)$ compound renewal with epochs $\tau^*(n) = U_1+\ldots+U_n$ $n\geq 1$ (except if the first epoch is specified to have a distribution $\neq G$). Hence $S_n = Q^*(\tau^*(n))-Q^*(0)$ and the paths of

$\{Q(t)\}$ and $\{Q^*(t)\}$ coincide on $[0,\tau(\underline{n}-1)) = [0,\tau^*(\underline{n}-1))$ (in models A, B even on $[0,\tau^*(\underline{n}))$ where

$$\underline{n} = \inf\{n \geq 1 : Y_{n-1} < X_n^{(2)}\} \tag{2.2}$$

We let F_n^* be the σ-algebra spanned by $Q^*(0)$, $U_1,\dots,U_n$, $X_1^{(2)}, \dots, X_n^{(2)}$ and the arrival process in $[0,\tau^*(n))$. Then $X_1^{(1)},\dots,X_n^{(1)}$, $S_1,\dots,S_n$ etc. are F_n^*-measurable and $\underline{n}$ a stopping time w.r.t. $\{F_n^*\}$. Also define the first passage time

$$\theta^*(N) = \inf\{t \geq 0 : Q^*(t) \geq N\} \qquad \text{and let}$$

$$\nu^*(N) = \inf\{n : \tau^*(n) \geq \theta^*(N)\},\ B_0^*(N) = Q^*(\tau^*(\nu^*(N))) - N.$$

Proposition 2.1 (cf. [4] Lemma 2.2) For any events $F_N \in F^*_{\nu^*(N)}$

$$PF_N = \delta^{-N}\ {}^{a}E[\delta^{Q(0)-B_0^*(N)}; F_N] \tag{2.3}$$

Proof. Let the T_n (say) arrivals in $[\tau^*(n-1),\tau^*(n))$ be at times $\tau^*(n-1)+D_n^j$ $(D_n^1 < \dots < D_n^{T_n})$ and let the corresponding bulk sizes be E_n^j $(j = 1,\dots,T_n)$. It then suffices to verify (2.3) for $F_n = \{Q^*(0) = q;\ G_N^1; \dots; G_N^k;\ \nu^*(N) = k\}$ where the G_N^n are of the form

$$G_N^n = \{U_n \leq u(n),\ X_n^{(2)} = x(n),\ T_n = t(n),$$
$$D_n^j \leq d(n,j),\ E_n^j = e(n,j)\ j=1,\dots,t(n)\}.$$

In fact, if q, k and the $x(n), t(n), e(n,j)$ are fixed, then this class of events spands a σ-algebra, viz. the trace of $F^*_{\nu^*(N)}$ on

$$\{Q^*(0) = q, \nu^*(N) = k,\ X_n^{(2)} = x(n),\ T_n = t(n),\ E_n^j = e(n,j)$$
$$n=1,\dots,k, j=1,\dots,t(n)\} \tag{2.4}$$

and the class of events of the form (2.4) forms a $F^*_{\nu^*(N)}$- measurable countable partition of the basic probability space. With $H_{u,t}(d(1),\ldots,d(t))$ the t-variate d.f. of the order statistics corresponding to t drawings from a uniform distribution on $[0,u)$, we have (suppressing n for brevity and letting $z = e(1) + \ldots + e(t)$)

$$PG_N = g_x \int_0^u e^{-\alpha y}\frac{(\alpha y)^t}{t!} H_{u,t}(d(1),\ldots,d(t)) f_{e(1)}\cdots f_{e(t)} G(dy) =$$

$$\delta^{x-z}\, {}^a g_x \int_0^u e^{-{}^a\alpha y}\frac{({}^a\alpha y)^t}{t!} H_{u,t}(d(1),\ldots,d(t))\, {}^a f_{e(1)}\cdots {}^a f_{e(t)}\, {}^a G(dy) =$$

$${}^a E[\delta^{X^{(2)}-X^{(1)}}; G_N] ,$$

performing some elementary manipulations to express the g_x, f_e, G etc. by the ${}^a g_x$, ${}^a f_e$, ${}^a G$. Now the occurrence of $\{\nu^*(N) = k\}$ depends solely on the values of $Q(0)$ and the $X_n^{(2)}$, T_n, E_n^j $n = 1,\ldots,k$. Thus either $F_N = \emptyset$, in which case both sides of (2.3) are zero, or

$$\{Q^*(0) = q;\ G_N^1;\ldots;G_N^k\} \subseteq \{\nu^*(N) = k\} \quad \text{so that}$$

$$PF_N = P(Q^*(0) = q;\ G_N^1;\ldots;G_N^k) = P(Q^*(0) = q)\ PG_N^1 \ldots PG_N^k =$$

$${}^a P(Q^*(0) = q)\prod_{n=1}^k {}^a E[\delta^{X_n^{(2)}-X_n^{(1)}}; G_N^n] = {}^a E[\delta^{-S_k}; Q^*(0)=q;\ G_N^1;\ldots;G_N^k] =$$

$${}^a E[\delta^{-S_{\nu^*(N)}}; F_N] = \delta^{-N}\, {}^a E[\delta^{Q(0)-B_0^*(N)}; F_N] . \qquad \square$$

To conclude this section, we shall state some formulae for the first and second moment of $\{Q^*(t)\}$ which will provide expressions for μ,σ^2, in (1.4). Define

$$\mu_+^{-1} = \lim_{t\to\infty} {}^a E(Q_+^*(t)\,|\,Q_+^*(0)=0)/t,\ \omega_+^2 = \lim_{t\to\infty} {}^a \mathrm{Var}(Q_+^*(t)\,|\,Q_+^*(0) = 0)/t , \tag{2.5}$$

μ_-^{-1}, ω_-^2, μ^{-1}, ω^2 in a similar manner so that $\mu^{-1} = \mu_+^{-1} - \mu_-^{-1}$, $\omega^2 = \omega_+^2 + \omega_-^2$. The existence of these quantities is well-known and easily checked, and it follows by elementary calculations that

$$\mu^{-1} = {}^a\alpha\ {}^aEX_n^{(1)} - {}^a\lambda^{-1}\ {}^aEX_n^{(2)} \tag{2.6}$$

$$\omega^2 = {}^a\alpha\ {}^a\mathrm{Var}\ X_n^{(1)} + ({}^a\alpha\,{}^aEX_n^{(1)})^2 + {}^a\lambda^{-1}\ {}^a\mathrm{Var}\ X_n^{(2)}$$
$$+\ {}^a\lambda^{-3}\ {}^a\mathrm{Var}\ U_n({}^aEX_n^{(2)})^2 \tag{2.7}$$

where ${}^a\lambda = {}^aEU_n$. Note that μ,ω^2 have explicit expressions in terms of α, $\hat{f}$, $\hat{g}$, $\hat{G}$ by means of formulae like

$${}^aEX_n^{(1)} = {}^a\lambda\,{}^a\alpha\,{}^a\hat{f}'(1) = {}^a\lambda\alpha\delta\hat{f}'(\delta),\ {}^a\lambda = {}^a\hat{G}'(0) = \frac{\hat{G}'(\gamma)}{G(\gamma)}$$

$${}^a\mathrm{Var}X_n^{(2)} = {}^a\hat{g}''(1) + {}^a\hat{g}'(1) - {}^a\hat{g}'(1)^2 = \frac{\delta^{-2}\hat{g}''(\delta^{-1})}{\hat{g}(\delta^{-1})}$$
$$+\ \frac{\delta^{-1}\hat{g}'(\delta^{-1})}{\hat{g}(\delta^{-1})} - \left(\frac{\delta^{-1}\hat{g}'(\delta^{-1})}{\hat{g}(\delta^{-1})}\right)^2 .$$

3. Statement of result and some main lemmata. Mainstream of proof.

We recall that different initial conditions obtain by letting the P-distribution of $Q(0)$ vary, but that always the server commences to work at time 0 [though actual service may first start later in models $\mathcal{C}$, $\mathcal{D}$ if $Q(0) < X_1^{(2)}$].

<u>Theorem 3.1</u> <u>Suppose that Conditions 1.1, 1.2 and 2.1 are in force.</u> <u>Then there is a</u> $C \in (0,\infty)$, <u>the value of which can be determined by algorithms to be discussed in Section 5, such that whenever</u> $E\delta^{Q(0)} < \infty$, <u>then</u> (1.4) <u>holds with</u> $\sigma^2 = \mu^3\omega^2$ <u>and</u> μ,ω^2 <u>given by</u> (2.6), (2.7).

We proceed to give the main steps in the proof. The more technical details are omitted in cases where the ideas are essentially the same as in [4], and carried out in Section 4 otherwise.

The argument is based on regenerative properties of the process as in [4], but the situation is somewhat more complicated. For the simple M/G/1 case, the obvious choice of the imbedded renewal process of regeneration points is the successive ends of the busy cycles, viz. the $\tau(n)$ with $Y_n = 0$, so that the first regeneration point is

$$\underline{c} = \tau(\underline{m}) \quad \text{with} \quad \underline{m} = \inf\{n \geq 1 : Y_n = 0\} \tag{3.1}$$

However, in general the concept of busy cycle is more ambiguous and the definition (3.1) of the first regeneration point does not always seem to lead to the simplest analysis. Thus we take this approach only for the model $\mathcal{B}$, whereas in $\mathcal{A}$, $\mathcal{C}$, $\mathcal{D}$ we are concerned with the $\tau(n)$ with $Y_{n-1} < X_n^{(2)}$ so that the first regeneration point is $\tau(\underline{n})$, cf. (2.2). For the ease of notation, we let $\underline{n}=\underline{m}$ in this and the following section.

For the models $\mathcal{A}$, $\mathcal{B}$, these instants form indeed a renewal process of regeneration points, whereas for $\mathcal{C}$, $\mathcal{D}$ we need to invoke the more general concept of semi-regenerativity, cf. Çinlar [7] Ch.10. In fact, if $Y_{n-1} < X_n^{(2)}$ then the development of the post-$\tau(n)$ process depends on the distribution of Y_n which may be of one of several types, say $i=1,\ldots,p$. In $\mathcal{C}$ we have $p=2$ and the two cases are described by (1.2$\mathcal{C}$) according to whether $Y_{n-1} > 0$ $(i=1)$ or $Y_{n-1} = 0$ $(i=2)$. In $\mathcal{D}$ we can take p as the maximal batch size and $i = X_n^{(2)} - Y_{n-1}$.

We let P_i, E_i $(i=1,\dots,p)$ refer to an initial distribution of type i, and use for convenience the same notation with $p = 1$ in $\mathcal{A}$, $\mathcal{B}$. Let for an arbitrary initial distribution P $H_j(du)$ be the probability that $\tau(\underline{n})$ occurs at time u and is of type j, and let $H_{i,j}(du)$ be the same quantity with P replaced by P_i (similar notations are used in the following without further notice), $\underline{H} = (H_{i,j})$, $\underline{U} = (U_{i,j}) = \Sigma_0^\infty \underline{H}^{*n}$. Then letting $\underline{\pi} = (\pi_j)$ be the stationary distribution for the Markov chain of transitions $i \to j$ and

$$Z_N(T) = P(Q(T) \geq N), \quad z_N(T) = P(Q(T) \geq N, \quad t < \tau(\underline{n})),$$

$$\xi_i = \sum_{j=1}^{p} \int_0^\infty u H_{i,j}(du), \; H*Z(T) = \int_0^T Z(T-u)H(du) \quad , \quad \text{we get}$$

$$Z_N(T) = z_N(T) + \sum_{j=1}^{p} H_j * Z_{j,N}(T) \tag{3.2}$$

$$Z_{i,N}(T) = \sum_{j=1}^{p} U_{i,j} * z_{j,N}(T) \tag{3.3}$$

$$Z_N(\infty) = \frac{1}{\underline{\pi}\,\underline{\xi}} \sum_{j=1}^{p} \pi_j \int_0^\infty z_{j,N}(t)\,dt, \tag{3.4}$$

cf.[7] p.346-347. The following two lemmata, to be proven in Section 4,

Lemma 3.1 If $E\delta^{Q(0)} < \infty$, then $\int_0^\infty u H_i(du) < \infty$ and $\lim_{N\to\infty} \delta^N \sup_{0 \leq T < \infty} z_N(T) = 0$,

Lemma 3.2 In any of the models $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ $E_i Q(0)\delta^{Q(0)} < \infty$ $i=1,\dots,p$,

will allow to restrict attention to the case $P = P^i$. In fact, once (1.4) is shown for this case, it follows that (uniformly in T)

$$\delta^N Z_N(T) \cong \sum_{j=1}^{p} \int_0^T C\Phi(\frac{T-u-\mu N}{\sigma N^{\frac{1}{2}}}) H_j(du) \cong$$

$$\sum_{j=1}^{p} \int_0^T C\Phi(\frac{T-\mu N}{\sigma N^{\frac{1}{2}}}) H_j(du) + \sum_{j=1}^{p} \int_0^T O(\frac{u}{N^{\frac{1}{2}}}) H_j(du) \cong$$

$$C\Phi(\frac{T-\mu N}{\sigma N^{\frac{1}{2}}}) \sum_{j=1}^{p} \int_0^\infty H_j(du) + 0 = C\Phi(\frac{T-\mu N}{\sigma N^{\frac{1}{2}}}) \quad ,$$

using Taylor's formula and the boundedness of Φ' for the second $\cong$, and dominated convergence for the third.

To study (3.3), write $z_N = u_N + v_N$ with

$$u_N(T) = P(Q(T) \geq N, t < \tau(\underline{n}-1)), \; v_N(T) = P(Q(T) \geq N; \tau(\underline{n}-1) \leq t < \tau(\underline{n})) \; .$$

In Section 4 we show

Lemma 3.3 If $E\delta^{Q(0)} < \infty$, then $\lim_{N\to\infty} \delta^N \sup_{0\leq T\leq\infty} U_{i,j} * v_N(T) = 0$

so that we can replace $z_{j,N}$ by $u_{j,N}$ in (3.3). Now u_N depends on the law of $\{Q^*(t)\}$ only and can be evaluated by conditioning upon $\{u = \Theta^*(N) < \tau(\underline{n}-1)\}$, the overshot $b_1 = B_1^*(N) = Q^*(\Theta^*(N)) - N = Q(\Theta^*(N)) - N$ and the residual service time $b_2 = B_2^*(N) = \tau^*(\nu^*(N)) - \Theta^*(N)$ at $\Theta^*(N)$. More precisely, defining

$$\tilde{K}_N(u,b_1,b_2) = P(\Theta^*(N) < u \wedge \tau(\underline{n}-1), B_1^*(N) \leq b_1, B_2^*(N) \leq b_2\},$$

$$K_N(u,b_1,b_2) = \tilde{K}_N(\mu N + \sigma N^{\frac{1}{2}} u, b_1, b_2) \;,$$

$$k_{N,b_1,b_2}(t) = P(Q^*(t) \geq N,\ t < \tau(\underline{n}-1) \mid Q^*(0) = N+b_1,\ \tau^*(1) = b_2\},$$

$$k_{b_1,b_2}(t) = \lim_{N\to\infty} k_{N,b_1,b_2}(t) = P(Q^*(t) \geq 0 \mid Q^*(0) = b_1, \tau^*(1) = b_2\} ,$$

we may write

$$u_N(t) = \int_0^t \int_0^\infty \int_0^\infty k_{N,b_1,b_2}(t-u)\tilde{K}_N(du,db_1,db_2) \tag{3.5}$$

$$U_{i,j} * u_{j,N}(T) = \int_0^T \int_0^\infty \int_0^\infty U_{i,j} * k_{N,b_1,b_2}(T-u)\tilde{K}_{j,N}(du,db_1,db_2) \tag{3.6}$$

$$= \int_{-\infty}^{\frac{T-\mu N}{\sigma N^{\frac{1}{2}}}} \int_0^\infty \int_0^\infty U_{i,j} * k_{N,b_1,b_2}(T-\mu N-\sigma N^{\frac{1}{2}}u)K_{j,N}(du,db_1,db_2)$$

interchanging the integrations w.r.t. $U_{i,j}$ and $\tilde{K}_{j,N}$ to derive the first identity in (3.6) from (3.5). We shall need the following three lemmata.

Let $\Rightarrow$ denote convergence in distribution and (more generally) weak convergence of bounded measures, i.e. convergence of all integrals of functions $f \in C_b(\mathcal{T})$ (the bounded continuous functions on the underlying metric space $\mathcal{T}$).

Lemma 3.4 There exist r.v. $V^*(\infty)$, $B_1^*(\infty)$, $B_2^*(\infty)$, with $EB_i^*(\infty)<\infty$ $i = 1,2$ and $V^*(\infty)$ standard normal and independent of the $B_i^*(\infty)$, such that $K(u,b_1,b_2) = P(V^*(\infty) \leq u, B_1^*(\infty) \leq b_1, B_2^*(\infty) \leq b_2)$ has the following property: For all P satisfying $E\delta^{Q(0)}<\infty$ there exists a constant $D(P)<\infty$ (with $D(P_i) > 0)$ such that

$$\delta^N K_N(du,db_1,db_2) \Rightarrow D(P)K(du,db_1,db_2) .$$

If furthermore $EQ(0)\delta^{Q(0)} < \infty$, then also

$$\delta^N(\varkappa_0+\varkappa_1 b_1+\varkappa_2 b_2)K_N(du,db_1,db_2) \Rightarrow D(P)(\varkappa_0+\varkappa_1 b_1+\varkappa_2 b_2)K(du,db_1,db_2) .$$

Lemma 3.5 Define $\bar{k}^j_{b_1,b_2} = \pi_j \int_0^\infty k_{b_1,b_2}(t)dt/\pi\underline{\xi}$.

Then it holds uniformly in b_1,b_2 on compact sets that

$$\lim_{N\to\infty} \sup_{t\geq \sigma N^{\frac{1}{4}}} |U_{i,j}*k_{N,b_1,b_2}(t) - \bar{k}^j_{b_1,b_2}| = 0 .$$

Lemma 3.6 There exist constants $\varkappa_0,\varkappa_1,\varkappa_2$ such that

$U_{i,j}*k_{N,b_1,b_2}(t) \leq \varkappa_0+\varkappa_1 b_1+\varkappa_2 b_2$ for all i,j,N,b_1,b_2,T .

The first step in the proof of Lemma 3.5 is to note that $U_{i,j}$ is a (delayed) renewal function with $U_{i,j}(t+a) - U_{i,j}(t) \to a\pi_j/\pi\underline{\xi}$ so that by the key renewal theorem $U_{i,j}*k(T)\to a\pi_j \int_0^\infty k(t)dt/\underline{\pi}\,\underline{\xi}$ whenever k is directly Riemann integrable. The rest of the argument as well as the proof of Lemma 3.6 follows [4] closely and is omitted. In contrast, the proof of Lemma 3.4 presents a key step and is given in full in Section 4.

We can now easily prove (1.4) with $P = P_i$ and thereby Theorem 3.1. The argument follows [4] closely, but in view of its central place we give it for the sake of self-containedness. Define $C = C_1 + \ldots + C_p$ with

$$C_j = D(P_j) \int_{-\infty}^\infty \int_0^\infty \int_0^\infty \bar{k}^j_{b_1,b_2} K(du,db_1,db_2) .$$

From $0<D(P_j)<\infty$, $EB_i^*(\infty) < \infty$ and Lemma 3.6 one easily checks $0 < C_j < \infty$, and it follows from (3.3) and the above discussion that it suffices to show $\delta^N U_{i,j}*u_{j,N}(T) \to C_j\Phi((T-Np)/\sigma N^{\frac{1}{2}})$ uniformly in T. Let

$$q_1(T,N) = \delta^N U_{i,j} * U_{j,N}(T) \qquad \text{(cf. (3.6))} ,$$

$$q_2(T,N) = \int_{-\infty}^{\frac{T-\mu N}{\sigma N^{\frac{1}{2}}} - N^{-\frac{1}{4}}} \int_0^\infty \int_0^\infty U_{i,j} * k_{N,b_1,b_2}(T-\mu N-\sigma N^{\frac{1}{2}}u)\, \delta^N K_{j,N}(du,db_1,db_2) ,$$

$$q_3(T,N) = \int_{-\infty}^{\frac{T-\mu N}{\sigma N^{\frac{1}{2}}} - N^{-\frac{1}{4}}} \int_0^\infty \int_0^\infty \bar{k}^j_{b_1,b_2}\, \delta^N K_{j,N}(du,db_1,db_2) ,$$

$$q_4(T,N) = \int_{-\infty}^{\frac{T-\mu N}{\sigma N^{\frac{1}{2}}}} \int_0^\infty \int_0^\infty \bar{k}^j_{b_1,b_2}\, \delta^N K_{j,N}(du,db_1,db_2) ,$$

$$q_5(T,N) = \int_{-\infty}^{\frac{T-\mu N}{\sigma N^{\frac{1}{2}}}} \int_0^\infty \int_0^\infty \bar{k}^j_{b_1,b_2} D(P_j) K(du,db_1,db_2) = C_j \Phi\left(\frac{T-N\mu}{\sigma N^{\frac{1}{2}}}\right) ,$$

$$\varepsilon_i = \overline{\lim_{N\to\infty}} \sup_{0\leq T\leq\infty} |q_i(T,N) - q_{i+1}(T,N)|$$

(with some obvious interpretations and simplifications for $T = \infty$, cf. e.g. (3.4), Lemma 3.5). From Lemma 3.6, the continuity of $V^*(\infty)$ and the last part of Lemma 3.4 we may conclude that

$$\varepsilon_1 \leq \overline{\lim_{N\to\infty}} \sup_{-\infty<x<\infty} \int_{x-N^{-\frac{1}{4}}}^{x} \int_0^\infty \int_0^\infty (\varkappa_0+\varkappa_1 b_1+\varkappa_2 b_2)\, \delta^N K_{j,N}(du,db_1,db_2) = 0$$

and $\varepsilon_3 = 0$ follows by the same argument combined with $\bar{k}^j_{b_1,b_2} \leq \varkappa_0+\varkappa_1 b_1+\varkappa_2 b_2$. For $\varepsilon_4 = 0$ we have also to combine with $\bar{k}^j_{b_1,b_2}$ being continuous in $b_1,b_2 \in \mathbb{N} \times [0,\infty)$ as is easily seen. Finally

$$\varepsilon_2 \leq \overline{\lim_{N\to\infty}} \int_{-\infty}^\infty \int_0^\infty \int_0^\infty \sup_{t\geq\sigma N^{\frac{1}{4}}} |U_{i,j} * k_{N,b_1,b_2}(t) - \bar{k}^j_{b_1,b_2}|\, \delta^N K_{j,N}(du,db_1,db_2) = 0$$

by tightness and Lemma 3.5 .

4. Details of proof.

The proof of Lemma 3.1 is deferred to the end of the section, so we start by the

Proof of Lemma 3.2. Since $X_{\underline{n}}^{(1)}$ is independent of $\underline{n}$ in all models, its p.g.f. is that of $X_n^{(1)}$ so that $EX_{\underline{n}}^{(1)}\delta^{X_{\underline{n}}^{(1)}} = \delta\hat{h}'(\delta) < \infty$. This proves the lemma in models $\mathcal{A}$, $\mathcal{B}$ as well as for the case $i=1$ in $\mathcal{C}$. For $i=2$ in $\mathcal{C}$,

$$E_2Q(0)\delta^{Q(0)} \leqslant E[(C_{\underline{n}}+X_{\underline{n}}^{(1)})\delta^{C_{\underline{n}}+X_{\underline{n}}^{(1)}}|Y_{\underline{n}-1}=0]=EC_n\delta^{C_n}E\delta^{X_n^{(1)}}+EX_n^{(1)}\delta^{X_n^{(1)}}EC_n < \infty$$

and for $\mathcal{D}$ it suffices similarly to show $EC_{n,i}\delta^{C_{n,i}} < \infty$. But since $f_0 = 0$, level i is reached in at most i steps so that $C_{n,i}$ is stochastically dominated by the sum T_i of i independent bulk sizes satisfying $ET_i\delta^{T_i} < \infty$. □

Proof of Lemma 3.3. We consider only the more complicated cases of models $\mathcal{C}$, $\mathcal{D}$. Let $\underline{\sigma}$ be the instant in $[\tau(\underline{n}-1),\tau(\underline{n}))$ where service starts and $I_1 = [\tau(\underline{n}-1),\underline{\sigma})$, $I_2 = [\underline{\sigma},\tau(\underline{n}))$, $M^k = \sup_{t\in I_k} Q(t)$, $v_N^k(t) = P(Q(t)\geqslant N, t\in I_k)$ so that $v_N = v_N^1 + v_N^2$. From $U_{i,j}$ being a delayed renewal function, it follows that there are k_1,k_2 such that $U_{i,j}(t+a)-U_{i,j}(t)\leqslant k_1+k_2a$ and hence

$$U_{i,j}*v_N^k(T) = E\int_0^T I(Q(T-t)\geqslant N,\ T-t\in I_k)\ U_{i,j}(dt) \leqslant EI(M^k\geqslant N)\{k_1+k_2|I_k|\} \tag{4.1}$$

(with $|\cdot|$ denoting Lebesgue measure) so that it suffices to show (4.1) being $o(\delta^{-N})$ for $k = 1,2$.

To deal with case C, note first that

$$P(\underline{Y}_{n-1} \geqslant N) \leqslant P(S_{n-1} \geqslant N, X_n^{(2)} > N \quad \text{for some} \quad n) \leqslant \tag{4.2}$$

$$\sum_{n=1}^{\infty} P(S_{n-1} \geqslant N) P(X_n^{(2)} > N) = o(1) E \sum_{n=1}^{\infty} I(S_{n-1} \geqslant N) = o(\delta^{-N})$$

because of the last expectation being $O(E\delta^{Q(0)-N})$ as follows from random walk theory. Since $M^1 = \underline{Y}_{n-1}$ is independent of the exponential r.v. $|I_1|$, the assertion for the case $k=1$ follows in C and it is also automatic in D since here $M^1 \leqslant p$ so that even (4.1) vanishes for $N > p$. For $k = 2$, we have in both models $M^2 = Q(\underline{\sigma}) + \underline{X}_n^{(1)}$ with $Q(\underline{\sigma})$ independent of $|I_2|$ so that it suffices to show $P(Q(\underline{\sigma}) \geqslant N) = o(\delta^{-N}), E\delta^{\underline{X}_n^{(1)}} |I_2| < \infty$. Conditioning upon the length t of the service time $|I_2|$, the last assertion follows at once from

$$E\delta^{\underline{X}_n^{(1)}} |I_2| = \int_0^{\infty} e^{\alpha t[\hat{f}(\delta)-1]} t dG(t) = \hat{G}'(\gamma) < \infty .$$

Finally $P(Q(\underline{\sigma}) \geqslant N) = o(\delta^{-N})$ follows in C from $Q(\underline{\sigma}) = \underline{Y}_{n-1} + \underline{C}_n$, (4.2) and $E\delta^{\underline{C}_n} = \hat{f}(\delta) < \infty$, and in D from

$$E(\delta^{Q(\underline{\sigma})} | \underline{X}_n^{(2)} - \underline{Y}_{n-1} = i) = E(\delta^{\underline{C}_{n,i}} | \underline{X}_n^{(2)} - \underline{Y}_{n-1} = i) \leqslant \hat{f}(\delta)^i .$$

□

The following two lemmata in conjunction with Proposition 2.1 constitute the main steps in the proof of Lemma 3.4 :

<u>Lemma 4.1</u> <u>There exist r.v.</u> $B_i^*(\infty)$ (i=0,1,2) <u>such that</u> $\underline{B}^*(N) = (B_0^*(N), B_1^*(N), B_2^*(N)) \Rightarrow \underline{B}^*(\infty)$ <u>w.r.t.</u> ${}^aP$. <u>Furthermore, for</u> i = 1,2 ${}^aEB_i^*(\infty)\delta^{-B_0^*(\infty)} < \infty$ <u>and</u> if $E\delta^{Q(0)} < \infty$, <u>then</u> ${}^aE\delta^{Q(0)-B_0^*(N)} \to E\delta^{Q(0)}\, {}^aE\delta^{-B_0^*(\infty)}$, <u>while if in addition</u> $EQ(0)\delta^{Q(0)} < \infty$, <u>then also</u>

$${}^aE\delta^{Q(0)-B_0^*(N)} B_i^*(N) \to E\delta^{Q(0)}\, {}^aEB_i^*(\infty)\delta^{-B_0^*(\infty)} .$$

Lemma 4.2 No matter initial conditions, it holds that (i) $\Theta^*(N)/N \xrightarrow{^aP} \mu$; (ii) $V^*(N) = (\Theta^*(N) - \mu N)/\sigma N^{\frac{1}{2}} \Rightarrow V^*(\infty)$ (mixing) w.r.t. ^aP with $V^*(\infty)$ standard normal.

In the proof of Lemma 4.1, we need

Lemma 4.3 If $S_1, S_2, \ldots$ is a random walk adapted to $\{F_n\}$ with $ES_1^2 < \infty$ and $ES_1 \neq 0$, and ν is a stopping time w.r.t. $\{F_n\}$ with $E\nu < \infty$, then $ES_\nu^2 < \infty$ if and only if $E_\nu^2 < \infty$

which follows from $L_2 = \{X : EX^2 < \infty\}$ being a linear space in conjunction with $S_\nu - \nu ES_1 \in L_2$ (Neveu [19] IV-4-21).

Proof of Lemma 4.1. Let $^a\tilde{P}$, $^a\tilde{E}$ refer to initial conditions with $Q^*(0)$ distributed as $-X_n^{(2)}$, recall that $\tau^*(\nu^*(1))$ is the time of the first downwards jump after $\{1,2,\ldots\}$ has been hit, and define

$$N_1 = \sup\{Q^*(t) : 0 \leq t \leq \tau^*(\nu^*(1))\} = Q^*(\tau^*(\nu^*(1)) - 0)$$

Then the $^a\tilde{P}$-distribution of $N_1 - Q^*(\tau^*(\nu^*(1)))$ is that of $-X_n^{(2)}$ and it is easily seen that $(\underline{B}^*(N))_{N \geq 0}$ regenerates itself at N_1 and that the d.f. of N_1 is aperiodic. Thus the first part of the lemma, with the distribution of the limit given by

$$^aEf(\underline{B}^*(\infty)) = \frac{1}{^aEN_1} \, ^aE \sum_{N=1}^{N_1} f(\underline{B}^*(N)) \tag{4.3}$$

will follow from the theory of discrete time regenerative processes (Feller [11],[12] Ch.IX) if we can show $^a\tilde{E}N_1 < \infty$.

With the convention $^a\tilde{P}(Q^*(0-0) = 0) = 1$, $\{Q^*(\tau^*(n) - 0)\}$ is a random walk with finite variance, N_1 is the (strict) ascending ladder variable and $\nu^*(1)$ the corresponding ladder epoch. Thus from well-known facts on random walks, we may even infer $^a\tilde{E}N_1^2 < \infty$ and, appealing to the 'only if' part of Lemma 4.3, $^a\tilde{E}\nu^*(1)^2 < \infty$ which then implies $^a\tilde{E}\tau^*(\nu^*(1))^2 < \infty$ by using instead the 'if' part. The finiteness of the expectations follows now easily from (4.3) and the independence of $X_{\nu^*(1)}^{(2)}$ of the process before

$\tau^*(\nu^*(1))$. In fact, for $N=1,\dots,N_1$ we have $B_0^*(N) \geqslant - X^{(2)}_{\nu^*(1)}$ and hence

$${}^{a}\tilde{E}N_1\;{}^{a}EB_1^*(\infty)\delta^{-B_0^*(\infty)} = {}^{a}\tilde{E}\sum_{N=1}^{N_1} B_1^*(N)\delta^{-B_0^*(N)} \leqslant$$

$${}^{a}\tilde{E}N_1^2\;\delta^{X^{(2)}_{\nu^*(1)}} = {}^{a}\tilde{E}N_1^2\;{}^{a}E\delta^{X_n^{(2)}} = {}^{a}\tilde{E}N_1^2/\hat{g}(\delta^{-1}) < \infty ,$$

$${}^{a}\tilde{E}N_1\;{}^{a}EB_2^*(\infty)\delta^{B_0^*(\infty)} = {}^{a}\tilde{E}\sum_{N=1}^{N_1} B_2^*(N)\delta^{-B_0^*(N)} \leqslant$$

$${}^{a}\tilde{E}N_1\tau^*(\nu^*(1))\delta^{-B_0^*(N)} \leqslant [{}^{a}\tilde{E}N_1^2\;{}^{a}\tilde{E}\tau^*(\nu^*(1))^2]^{\frac{1}{2}}\;{}^{a}E\delta^{X_n^{(2)}} < \infty ,$$

using Cauchy-Schwarz' inequality.

By general results on regenerative processes,

$${}^{a}\tilde{E}B_i^*(N)\delta^{-B_0^*(N)} \to {}^{a}EB_i^*(\infty)\delta^{-B_0^*(\infty)} .$$ For an arbitrary initial distribution with $EQ(0)\delta^{Q(0)} < \infty$, write ${}^{a}EB_i^*(N)\delta^{Q(0)-B_0^*(N)}$ as

$${}^{a}EB_i^*(N)\delta^{Q(0)-B_0^*(N)}\,I(Q(0)+X_1^{(1)}\geqslant N)$$

$$+\quad {}^{a}E\sum_{n=0}^{\infty} I(Q(0)+X_1^{(1)} = n,\ N > n)\delta^{Q(0)}\;{}^{a}\tilde{E}B_i^*(N-n)\delta^{-B_0^*(N-n)} .$$

Here the second term has the asserted limit by dominated convergence, whereas in the first the integrand $\overset{a_P}{\to}0$ and is bounded by the r.v.

$$(Q(0)+X_1^{(1)})\delta^{Q(0)+X_1^{(2)}} \quad i = 1, \quad U_1\delta^{Q(0)+X_1^{(2)}} \quad i=2$$

with finite expectation. The proof of ${}^{a}E\delta^{Q(0)-B_0^*(N)} \to E\delta^{Q(0)}\;{}^{a}E\delta^{-B_0^*(\infty)}$ subject to $E\delta^{Q(0)} < \infty$ is similar (though simpler).

$\square$

Define $A(t) = (Q^*(t) - \mu^{-1}t)/t^{\frac{1}{2}}$ and $A_+(t), A_-(t)$ in a similar manner, cf. (2.5). Then $A_+(t)$, $A_-(t)$ are well-known to be asymptotically normal w.r.t. ${}^aP$ with variances ω_+^2, ω_-^2 and hence $A(t) \Rightarrow \omega V^*(\infty)$ with $V^*(\infty)$ standard normal. The condition

$$\lim_{\delta\downarrow 0} \overline{\lim_{T\to\infty}} \; {}^aP(\sup\{|A(t)-A(T)|: T\le t\le T(1+\delta)\} > \varepsilon) = 0 \quad \text{for all } \varepsilon > 0$$

introduced by Anscombe [2] is well-known to be of basic importance in central limit theorems with random indexing and has been extensively studied, cf. e.g. Aldous [1] and his references. Since we could not find a reference covering $\{A_-(t)\}$, we need to prove

__Lemma 4.4__ __If__ $\{Q_-^*(t)\}$ __is a compound renewal process with finite variances and__ μ_-^{-1} __the linear growth rate of__ $EQ_-^*(t)$, __then__ $\{(Q_-^*(t) - \mu_-^{-1}t)/t^{\frac{1}{2}}\}$ __satisfies Anscombe's condition.__

__Proof.__ For ease of notation, we suppress $^*,-$ and it can also be assumed without loss of generality that the increments are non-negative. Define $A(t) = (Q(t)-\mu^{-1}t)/t^{\frac{1}{2}}$ and

$$M^1_{T,\delta} = \sup_{T\leqslant t\leqslant T(1+\delta)} \frac{Q(t)-Q(T)-\mu^{-1}(t-T)}{t^{\frac{1}{2}}}, \quad M^2_{T,\delta} = -\inf_{T\leqslant t\leqslant T(1+\delta)} \frac{Q(t)-Q(T)-\mu^{-1}(t-T)}{t^{\frac{1}{2}}}$$

$$M^3_{T,\delta} = \sup_{T\leqslant t\leqslant T(1+\delta)} \frac{t^{\frac{1}{2}}-T^{\frac{1}{2}}}{t^{\frac{1}{2}}} |A_T|, \quad M_{T,\delta} = \sup_{T\leqslant t\leqslant T(1+\delta)} |A(t)-A(T)|$$

Then $M_{T,\delta} \leqslant M^1_{T,\delta} + M^2_{T,\delta} + M^3_{T,\delta}$ so that it suffices to show $\overline{\lim_{T\to\infty}} P(M^k_{T,\delta} > \varepsilon) \downarrow 0$ as $\delta \to 0$ for $k = 1,2,3$. The case $k=3$ is immediate from the asymptotic normality of $A(T)$. For $k=1$, define

$$\underline{\sigma} = \inf\{t: T\le t\le T(1+\delta),\; Q(t)-Q(T)-\mu^{-1}(t-T) > \varepsilon t^{\frac{1}{2}}\},$$

$= \infty$ if no such t exists. If $\underline{\sigma} < \infty$, then $\underline{\sigma}$ is necessarily an arrival instant. If $q(t)$ is the probability of the event $\{Q(t) \geqslant \mu^{-1}t\}$ subject to initial conditions with the first arrival governed by the interarrival distribution, it follows from the asymptotic normality that $q(t) \to \frac{1}{2}$ so that $q(t) \geqslant \frac{1}{4}$ for $t \geqslant a$ say. Define $\hat{M}^1_{T,\delta} = (Q(T(1+\delta)) - Q(T) - \mu^{-1}(T\delta - a))/T^{\frac{1}{2}}$. Then $\hat{M}^1_{T,\delta} > \varepsilon$ on $\{T(1+\delta) - a \leqslant \underline{\sigma} \leqslant T(1+\delta)\}$ whereas on $\{T \leqslant \underline{\sigma} \leqslant T(1+\delta) - a\}$ the conditional probability of $(Q(T(1+\delta)) - Q(T) - \mu^{-1}T\delta)/T^{\frac{1}{2}}$ and hence $\hat{M}^1_{T,\delta}$ to exceed ε is at least $\frac{1}{4}$.

It follows that $P(M^1_{T,\delta} > \varepsilon) \leqslant 4P(\hat{M}^1_{T,\delta} > \varepsilon)$ and since $\hat{M}^1_{T,\delta}$ is easily seen to be asymptotically normal with variance $\omega^2\delta$, the claim follows for $M^1_{T,\delta}$. The case $k=2$ follows rather similar lines.

□

<u>Proof of Lemma 4.2.</u> Since $Q^*(t)/t \to \mu^{-1}$ a.s.w.r.t. ${}^aP$, (i) follows from

$$\frac{N}{\Theta^*(N)} = \frac{Q^*(\Theta^*(N)) - B_1^*(N)}{\Theta^*(N)} \overset{{}^aP}{\to} \mu^{-1} - 0 = \mu^{-1} ,$$

using $B_1^*(N) \Rightarrow B_1^*(\infty)$ and $\Theta^*(N) \to \infty$. Similarly

$$A(\Theta^*(N)) \cong \frac{N - \mu^{-1}\Theta^*(N)}{\mu^{\frac{1}{2}}N} = -\omega V^*(N)$$

so that it suffices to show $A(\Theta^*(N)) \Rightarrow \omega V^*(\infty)$ (mixing). Now it is well-known that $\{A_+(t)\}$ satisfies Anscombe's condition [this is also a special case of Lemma 4.4] and that the normal convergence is mixing [these facts follow, e.g., from $\{Q_+^*(t)\}$ having stationary independent increments either by copying the proofs for sums of i.i.d. r.v. or by the method of discrete skeletons]. The mixing combined with the asymptotic normality of $A_-(t)$ is easily seen to imply $A(t) = A_+(t) - A_-(t) \Rightarrow \omega V^*(\infty)$ (mixing). Since Anscombe's condition holds for both of $A_+(t), A_-(t)$, it holds for $A(t)$, and Th. 8 of Csörgö and Fischler [9] completes the proof.

□

Proof of Lemma 3.4. By an argument just along the lines of the proof of [4] relation (5.3), we may deduce from Lemmata 4.1, 4.2 that

$$(\underline{B}^*(N), V^*(N), I(\Theta^*(N) \leqslant \tau(n-1))) \Rightarrow (\underline{B}^*(\infty), V^*(\infty), I(\underline{n}=\infty)) \tag{4.4}$$

w.r.t. ${}^aP$, where $\underline{B}^*(\infty)$, $V^*(\infty)$ are mutually independent and independent of $(Q(0), I(\underline{n}=\infty))$ [mixing is a convenient but not crucial way to get the independence of $V^*(\infty)$ and $(Q(0), I(\underline{n}=\infty))$: a direct proof is not difficult]. Now let $g \in C_b(\mathbb{R}^3)$ and suppose $EQ(0)\delta^{Q(0)} < \infty$. Then by Proposition 2.1

$$\delta^N \int_0^\infty \int_0^\infty \int_0^\infty (x_0+x_1 b_1+x_2 b_2) g(u,b_1,b_2) K_N(du,db_1,db_2) =$$

$$\delta^N E(x_0+x_1 B_1^*(N)+x_2 B_2^*(N)) g(V^*(N), B_1^*(N), B_2^*(N)) I(\Theta^*(N) \leqslant \tau(\underline{n}-1)) =$$

$${}^aE\; \delta^{Q(0)-B_0^*(N)} (x_0+x_1 B_1^*(N)+x_2 B_2^*(N)) g(V^*(N), B_1^*(N), B_2^*(N)) I(\Theta^*(N) \leqslant \tau(n-1)) \tag{4.5}$$

It follows from Billingsley [6] Th. 5.4 and the last part of Lemma 4.1 that the r.v.

$$\{\delta^{Q(0)-B_0^*(N)} (x_0+x_1 B_1^*(N) + x_2 B_2^*(N))\}$$

are uniformly integrable. Hence the integrand in (4.5) is so. Furthermore, it converges in distribution, cf.(4.4). Thus using [6] Th.5.4 once more, it follows that the limit of (4.5) exists and is

$${}^aE[\delta^{Q(0)}; \underline{n}=\infty]\; {}^aE\delta^{-B_0^*(\infty)} (x_0+x_1 B_1^*(\infty) + x_2 B_2^*(\infty)) g(V^*(\infty), B_1^*(\infty), B_2^*(\infty)) =$$

$$C(P) \int_{-\infty}^\infty \int_0^\infty \int_0^\infty (x_0+x_1 b_1+x_2 b_2) g(u,b_1,b_2) K(du,db_1,db_2) \quad \text{where}$$

$$K(u,b_1,b_2) = P(V^*(\infty) \leqslant u, B_1^*(\infty) \leqslant b_1, B_2^*(\infty) \leqslant b_2) =$$

$${}^aE[\delta^{-B_0^*(\infty)}; V^*(\infty) \leqslant u,\; B_1^*(\infty) \leqslant b_1, B_2^*(\infty) \leqslant b_2] / {}^aE\delta^{-B_0^*(\infty)} ,$$

$C(P) = {}^aE[\delta^{Q(0)}; \underline{n}=\infty]\ {}^aE\delta^{-B_0^*(\infty)}$. That $V^*(\infty)$ is normal and independent of $(B_1^*(\infty), B_2^*(\infty))$ w.r.t. P follows from $V^*(\infty)$ being normal and independent of $\underline{B}^*(\infty)$ w.r.t. ${}^aP$. Similarly $EB_1^*(\infty) < \infty$, $EB_2^*(\infty)$ is a consequence of the last part of Lemma 4.1. Thus we have proved the last convergence statement in Lemma 3.4, and the proof of the first is similar, deleting factors like $\varkappa_0 + \varkappa_1 B_1^*(N) + \varkappa_2 B_2^*(N)$.

It thus only remains to prove $C(P_j) > 0$, which will follow from ${}^aP_j(\underline{n}=\infty) > 0$. In the model $\mathcal{B}$, ${}^aP_j(\underline{n}=\infty) = {}^aP(S_n > 0 \text{ all } n) > 0$ because of ${}^aEX_n > 0$. In $\mathcal{A},\mathcal{C},\mathcal{D}$ it follows from finite means that $S_n/n \to {}^aEX_n > 0$ and that $X_n^{(2)}/n \to 0$ a.s.w.r.t. ${}^aP_j$. Hence $T = \inf\{S_n - X_{n+1}^{(2)} : n \geqslant 0\} > -\infty$ and thus since $Y_n \geqslant Y_0 + S_n$,

$$ {}^aP(\underline{n} = \infty \mid Y_0 = i) = {}^aP(Y_n \geqslant X_{n+1}^{(2)} \quad \text{all} \quad n \geqslant 0 \mid Y_0 = i) \geqslant {}^aP(T > i) > 0 $$

for i large enough, say $i \geqslant i_0$. But in all cases Y_0 is stochastically larger than $X_n^{(1)}$, which has unbounded support. Thus ${}^aP_j(Y_0 \geqslant i_0) > 0$, completing the proof.

□

It only remains to give the

Proof of Lemma 3.1. The first part is an easy consequence of

$\Sigma_1^p \int_0^\infty u H_i(du) = E\tau(\underline{n})$ combined with the easily checked relations

$$ E[\tau(\underline{n}) - \tau(\underline{n}-1)] < \infty, \ E\tau(\underline{n}-1) \leqslant E \inf\{t \geqslant 0 : Q^*(t) \leqslant 0\} = O(EQ(0)) . $$

For the second part, note that the estimate $v_N^k(T) \leqslant P(M^k \geqslant N) = o(\delta^N)$ in the proof of Lemma 3.3 implies that it suffices to show $\delta^N u_N(T) \to 0$ uniformly in T. We use (3.5). Given $\varepsilon > 0$, it follows from Lemma 3.4 that we can find b_1^0, b_2^0 with

$$\delta^N \int_0^\infty \int_{b_1^0}^\infty \int_0^\infty \tilde{K}_N(du,db_1,db_2) \leqslant \varepsilon, \delta^N \int_0^\infty \int_0^\infty \int_{b_2^0}^\infty \tilde{K}_N(du,db_1,db_2) \leqslant \varepsilon$$

for all N, and it is easy to see that for some t^0 $k_{N,b_1,b_2}(t) \leqslant \varepsilon$

whenever $t \geqslant t^0, b_1 \leqslant b_1^0, b_2 \leqslant b_2^0$.

Hence since $k_{N,b_1,b_2}(t) \leqslant 1$

$$u_N(T) \leqslant \varepsilon \tilde{K}_N(\infty,\infty,\infty) + \int_0^T \int_0^\infty \int_0^\infty I(T-u \leqslant t^0) \tilde{K}_N(du,db_1,db_2) + 2\delta^{-N}\varepsilon ,$$

$$\overline{\lim_{N\to\infty}} \sup_{0 \leqslant T \leqslant \infty} \delta^N u_N(T) \leqslant$$

$$\varepsilon D(P) + \overline{\lim_{N\to\infty}} \sup_{0 \leqslant T < \infty} \delta^N P(T-t^0 \leqslant \Theta^*(N) \leqslant T) + 2\varepsilon = \varepsilon D(P) + 2\varepsilon ,$$

using Lemma 3.4 once more for the last identity. Let $\varepsilon \downarrow 0$.

□

5. The imbedded Markov chain, the steady state and the evaluation of C.

We start by pointing out

Proposition 5.1. *Suppose that Conditions 1.1, 2.1 are in force. Then in all of the models $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ there exists a constant $\tilde{C}$ such that whenever $E\delta^{Y_0} < \infty$, then*

$$\lim_{N\to\infty} \sup_{0 \leqslant K \leqslant \infty} \left| \delta^N P(Y_K \geqslant N) - \tilde{C}\Phi\left(\frac{K-\tilde{\mu}N}{\tilde{\sigma}N^{\frac{1}{2}}}\right) \right| = 0 \tag{5.1}$$

with δ as in Sections 2-4 and

$$\tilde{\mu}^{-1} = {}^aEX_n^{(1)} - {}^aEX_n^{(2)}, \quad \tilde{\sigma}^2 = \tilde{\mu}^3 \{ {}^aVarX_n^{(1)} + {}^aVar\ X_n^{(2)} \} \tag{5.2}$$

In fact, the proof is just a discrete time analogue of the proof of Theorem 3.1. In some cases, however, Proposition 5.1 and in particular the steady state case

$$P(Y_\infty \geqslant N) \cong \tilde{C}\delta^{-N} \tag{5.3}$$

comes out more directly by reference to random walks. Define $M_K = \max\{S_n : 0 \leqslant n \leqslant K\}$, $M = \max\{S_n : n \geqslant 0\}$, let $\{\phi_n\}$ be the distribution of the strictly ascending ladder variable, $\overline{\phi} = \Sigma_1^\infty n\,\phi_n$ and recall the estimates

$$P(M \geqslant N) \cong D\delta^{-N} \tag{5.4}$$

$$D = {}^a\overline{\phi}^{-1} \sum_{n=0}^{\infty} \delta^{-n} \sum_{k=n+1}^{\infty} {}^a\phi_k = \frac{\delta(1-\phi_1-\phi_2-\ldots)}{(\delta-1)\,{}^a\overline{\phi}} \tag{5.5}$$

$$= \frac{\delta}{(\delta-1)\,{}^aEX_n} \exp\left[-\sum_{n=1}^{\infty} \frac{1}{n} \{P(S_n>0) + {}^aP(S_n \leqslant 0)\}\right]$$

$$\lim_{N\to\infty} \sup_{0 \leqslant K \leqslant \infty} \left|\delta^N P(M_K \geqslant N) - D\Phi\left(\frac{K-\tilde{\mu}N}{\tilde{\sigma}N^{\frac{1}{2}}}\right)\right| = 0 \tag{5.6}$$

(see e.g. [4] for references). Then, letting $\overset{d}{=}$ denote equality in distribution:

<u>Proposition 5.2</u> (i) <u>The relation</u> $Y_n = (Y_{n-1}+X_n)^+$ <u>in</u> $\mathcal{B}$ <u>implies that</u> $Y_n \overset{d}{=} \max\{M_n, Y_0 + S_n\}$, $Y_\infty \overset{d}{=} M$. <u>Hence</u> $\tilde{C} = D$; (ii) <u>The relation</u> $Y_n = (Y_{n-1} - X_n^{(2)})^+ + X_n^{(1)}$ <u>in</u> $\mathcal{A}$ <u>and in</u> $\mathcal{D}$ <u>with single arrivals</u> $(f_1=1)$ <u>implies that</u> $Y_n \overset{d}{=} \max\{M_{n-1} + X_0^{(1)}, (Y_0-X_0^{(2)})^+ + S_{n-1} + X_0^{(1)}\}$ $(n \geqslant 1)$ <u>with the</u> $X_0^{(i)}$ <u>mutually independent, independent of the</u> $X_n^{(i)}$ $(n \geqslant 1)$ <u>and</u> $X_0^{(i)}$ <u>distributed as</u> $X_n^{(i)}$. <u>Hence</u> $Y_\infty \overset{d}{=} M + X_0^{(1)} \overset{d}{=} \max\{S_n : n \geqslant 1\} + X_1^{(2)}$ <u>and</u> $\tilde{C} = \hat{G}(\gamma)D$.

Proof: (i) is well-known, cf. e.g. [13] VI. 9. For (ii), define $X'_n = X_n^{(1)} - X_{n+1}^{(2)}$, $Z_n = Y_{n+1} - X_{n+1}^{(1)}$. Then $Z_n = (Z_{n-1} + X'_n)^+$ n=1,2,... so that from (i)

$$Z_n \overset{d}{=} \max\{M'_n, Z_0 + S'_n\} = \max\{M'_n, (Y_0 - X_1^{(2)})^+ + S'_n\} ,$$

$$Y_n = Z_{n-1} - X_n^{(1)} \overset{d}{=} Z_{n-1} + X_0^{(1)} \overset{d}{=} \max\{M_{n-1} + X_0^{(1)}, (Y_0 - X_0^{(2)})^+ + S_{n-1} + X_0^{(1)}\} .$$

Since $S_n \to -\infty$, $M_n \overset{d}{\Rightarrow} M$, it follows by letting $n \to \infty$ that

$$Y_\infty \overset{d}{=} M + X_0^{(1)} \overset{d}{=} \max\{X_1^{(1)}, X_1^{(1)} + S_2 - S_1, X_1^{(1)} + S_3 - S_1, \ldots\}$$

$$= \max\{S_n : n \geqslant 1\} + X_1^{(2)} ,$$

$$P(Y_\infty \geqslant N) = P(M + X_0^{(1)} \geqslant N) = \sum_{n=0}^{\infty} P(X_0^{(1)} = n)\, P(M \geqslant N-n) \cong$$

$$D\delta^{-N} \sum_{n=0}^{\infty} \delta^n P(X_0^{(1)} = n) = \hat{G}(\gamma) D\delta^{-N}.$$

□

Cf. also Prabhu [22]p.127. In cases where the proposition applies, (5.1) follows, of course, easily from (5.6). The $E_p/G/1$ case is covered by $\mathcal{D}$ with single arrivals, but general bulks in $\mathcal{D}$ seem to present somewhat more complicated problems. In C, $\tilde{C}$ could be determined from the expression [8] p.382 for $\hat{\psi}(s) = Es^{Y_\infty}$ (here and in the following $\psi_n = P(Y_\infty = n)$) in conjunction with the standard Abelian theorem

$$\psi_N \cong \tilde{C}\delta^{-N} \Rightarrow \tilde{C} = \lim_{s \uparrow 1}(1-s)\hat{\psi}(\delta s) = \lim_{s \uparrow \delta}(1-s\delta^{-1})\hat{\psi}(s) \qquad (5.7)$$

with the limit evaluated by l'Hospital's rule after heavy calculations. Note in this connection that, given an explicit form of $\hat{\psi}(s)$, the

Tauberian argument produces only a weaker form of (5.3), viz. an estimate of $\Sigma_0^N \delta^n \psi_n$. However, (5.3) could be derived by imposing analyticity conditions somewhat stronger than Condition 2.1, using poles and residues (cf. Gaver [14] and Le Gall [17]).

We next turn to the continuous time case, viz. the study of C rather than $\tilde{C}$. The problem is closely connected to the discrete time case in view of the following proposition.

Define $\underline{m} = \inf\{n \geq 1 : Y_n = 0\}$, $\underline{c} = \tau(\underline{m})$, cf. (3.1), and let P_0, E_0 refer to the case $Q(0) = Y_0 = 0$.

Proposition 5.3 . Under the assumptions of Theorem 3.1 it holds that

$$C = \tilde{C}(\hat{G}(\gamma)-1)/\psi_0 \gamma E_0 \underline{c} \tag{5.8}$$

Proof. It is more convenient to relate $P(Y_\infty = N)$ to $P(Q(\infty) = N)$ than to consider the tails. Let $F(t)$ denote the event that the next departure instant after t is one of the semi-regeneration points considered in Sections 3-4. Then, using (3.4) and Lemma 3.3,

$$P(Q(\infty) \geq N;\ F(\infty)) = \frac{1}{\underline{\pi}\ \underline{\xi}} \sum_{j=1}^{p} \pi_j \int_0^\infty v_{j,N}(t)\,dt = o(\delta^{-n}) \quad \text{so that}$$

$$P(Q(\infty) = N) \cong P(Q(\infty) = N;\ F(\infty)^c) = \frac{1}{E_0\underline{c}} E_0 \int_0^{\underline{c}} I(Q(t)=N);\ F(t)^c)\,dt \tag{5.9}$$

$$= \frac{1}{E_0\underline{c}} E_0 \sum_{n=1}^{\infty} J_n^N \quad \text{where} \quad J_n^N = \int_{\tau(n-1)}^{\tau(n)} I(Q(t) = N)\,dt\, I(Y_{n-1} \geq X_n^{(2)}, n \leq \underline{m}) ,$$

using instead the returns to zero as regeneration points. Now let $p_k(t) = e^{-\alpha t}(\alpha t)^k/k!$, $\xi_k = \int_0^\infty p_k(t)(1-G(t))\,dt$. Then, letting $f^{(k)}$ be the kth convolution power of f, it holds on $\{Y_{n-1} = i\}$ that

$$E(J_n^N \mid Y_{n-1}=i) = E(J_1^N \mid Y_0 = i) = P(X_1^{(2)} \leqslant i)\, E_0 \int_0^{U_1} I(Q(t) = N-i)dt =$$

$$P(X_1^{(2)} \leqslant i) \int_0^\infty \sum_{k=0}^\infty p_k(t)\; f_{N-i}^{(k)}\; (1-G(t))dt =$$

$$P(X_1^{(2)} \leqslant i) \sum_{k=0}^\infty f_{N-1}^{(k)}\; \xi_k \quad .$$

It is a standard fact from Markov chain theory that $E_0 \#\{n: 1 \leqslant n \leqslant \underline{m},\ Y_n = i\} = \psi_i/\psi_0$.
Hence

$$P(Q(\infty) = N;\ \ F(\infty)^c) = \frac{1}{E_0 \underline{c}} \sum_{i=0}^N \frac{\psi_i}{\psi_0} P(X_1^{(2)} \leqslant i) \sum_{k=0}^\infty f_{N-i}^{(k)}\; \xi_k \quad =$$

$$\frac{1}{E_0 \underline{c}} \sum_{k=0}^\infty \xi_k \sum_{i=0}^N \frac{\psi_i}{\psi_0} P(X_1^{(2)} \leqslant i)\; f_{N-i}^{(k)}$$

In view of $f_{N-i}^{(k)} = o(\delta^{-N})$, any finite number of terms $i=0,\ldots,i_0$ can be neglected. For $i > i_0$ with i_0 large, we have
$P(X_1^{(2)} \leqslant i) \cong 1, \quad \psi_i \cong \tilde{C}(1-\delta^{-1})\delta^{-i}$.
Hence

$$P(Q(\infty) = N) \cong \frac{\tilde{C}(1-\delta^{-1})}{\psi_0 E_0 \underline{c}} \sum_{k=0}^\infty \xi_k \sum_{i=0}^N \delta^{-i}\; f_{N-i}^{(k)} \qquad \cong$$

$$\frac{\tilde{C}(1-\delta^{-1})}{\psi_0 E_0 \underline{c}} \sum_{k=0}^\infty \xi_k \delta^{-N}\, \hat{f}^{(k)}(\delta) \;=\; \delta^{-N} \frac{\tilde{C}(1-\delta^{-1})}{\psi_0 E_0 \underline{c}}\, \hat{\xi}(\hat{f}(\delta)) =$$

$$\frac{\delta^{-N}\tilde{C}(1-\delta^{-1})(\hat{G}(\gamma)-1)}{\psi_0 E_0\, \underline{c}\, \gamma}$$

which completes the proof. We omit the elementary calculations needed to check $\hat{\xi}(s) = (1 - \hat{G}(\alpha[s-1]))/\alpha(1-s)$.

□

We thus need to compute $E_0\underline{c}$. The models $\mathcal{A}$, $\mathcal{B}$ are easy. Here by Wald's identity $E_0\underline{c} = E\underline{m} \cdot EU_n = \lambda/\psi_0$ and thus

$$C_{\mathcal{A}} = \frac{D\hat{G}(\gamma)(\hat{G}(\gamma)-1)}{\gamma\lambda} \tag{5.10A}$$

$$C_{\mathcal{B}} = \frac{D(\hat{G}(\gamma)-1)}{\gamma\lambda} \tag{5.10B}$$

Note that (5.10$\mathcal{A}$)> (5.10$\mathcal{B}$) as was to be expected from the description of the models. In $\mathcal{C}$, the expected length of $[\tau(n-1),\tau(n))$ is $\alpha^{-1} + \lambda$ for $n = 1$, λ for $1 < n \leqslant \underline{m}$. Hence $E_0\underline{c} = \alpha^{-1} + E\underline{m}\,\lambda = \alpha^{-1} + \lambda/\psi_0$ so that

$$C_{\mathcal{C}} = \frac{\tilde{C}_{\mathcal{C}}(\hat{G}(\gamma)-1)}{\gamma(\alpha^{-1}\psi_0 + \lambda)} \tag{5.10C}$$

with $\tilde{C}_{\mathcal{C}}$ determined, e.g., by the Abelian argument (5.7). The case $\mathcal{D}$ is more involved so we consider only single arrivals where we found $\tilde{C} = \hat{G}(\gamma)D$ above. The expected length of $[\tau(n-1),\tau(n))$ given $\{Y_{n-1} = i\}$ is $\alpha^{-1}(p-i)+\lambda$ for $i \leqslant p$, λ for $i \geqslant p$ and thus we arrive at the expression (to be somewhat simplified below)

$$C_{\mathcal{D}} = \frac{D\hat{G}(\gamma)(\hat{G}(\gamma)-1)}{\gamma(\alpha^{-1}\sum_{i=1}^{p-1}(p-1)\psi_i+\lambda)} \quad \text{if } f_1 = 1 \tag{5.10D}$$

The expression (5.5) for D clearly requires some reduction to be ameneable to numerical computations. A rather simple approach would be to compute the $P(S_n > 0)$, ${}^aP(S_n \leqslant 0)$ for small values of n and use asymptotic expressions like those of Bahadur and Ranga Rao [5] for large n.

The ladder variables can only be explicitly found in very few cases, in the present context mainly if g is the geometric distribution or has bounded support. To elaborate upon the last case, we conclude the discussion by

Example 5.1 Consider a general random walk $\{S_n\}$ on the integer lattice, suppose that $EX_n < 0$, $P(X_n < -p) = 0$, $P(X_n = -p) > 0$ for some $p \geqslant 1$, and write $Es^{X_n} = s^{-p}\hat{h}(s)$ with $\hat{h}$ the p.g.f. of the non-negative r.v. X_n+p. Then the equation

$$1 = Ew^{X_n} = w^{-p}\hat{h}(w) \tag{5.11}$$

has $p-1$ roots $w_1,\dots,w_{p-1}$ in the open complex unit circle and with $\{\phi_n^-\}$ the weak descending ladder height distribution we have

$$\hat{\phi}^-(s) = 1 - s^{-p}(s-1)\prod_{j=1}^{p-1}(s-w_j) \tag{5.12}$$

$$\hat{\phi}(s) = \frac{s^{-p}\,\hat{h}(s) - \hat{\phi}^-(s)}{1-\hat{\phi}^-(s)} \tag{5.13}$$

$$Es^M = \frac{1-\hat{\phi}(1)}{1-\hat{\phi}(s)} = \frac{(p-\hat{h}'(1))(s-1)\prod_{j=1}^{p-1}\frac{s-w_j}{1-w_j}}{s^p - \hat{h}(s)} \tag{5.14}$$

$$D = \frac{(p-\hat{h}'(1))(\delta-1)\prod_{j=1}^{p-1}\frac{\delta-w_j}{1-w_j}}{\delta\hat{h}'(\delta) - p\delta^p} \tag{5.15}$$

In fact, (5.13) and the first identity in (5.14) are general random walk results ([13]) so that elementary calculations show the equivalence of (5.12) and (5.14). For (5.12), (5.13), see Kemperman [16] Lemma 13.4. Alternatively, (5.17) is essentially proved in [23] within the framework of $E_p/G/1$ with $\hat{h}(s) = \hat{G}(\alpha(s-1))$ mixed Poisson, cf. the relation (104) in [23] for $\hat{\psi}(s) = Es^M\,\hat{h}(s)$. See also [13] p.427 where (5.12) is derived subject to the additional assumption of h having bounded support making the application of Rouché's theorem somewhat more direct. Finally (5.15) follows from (5.14), (5.7) and l'Hospital's rule.

We shall treat one additional model (though related to $E_p/D/1$). Consider the fixed cycle traffic light, with the cycle of (say) unit length divided into a period of length $|G|$ where customers (say cars or pedestrians) can pass and one of length $|R| = 1 - |G|$ where they cannot. We term these the green and red period and let ${}^{G}Y_n$ be the number of customers at the start of the nth green period and ${}^{G}X_n^{(1)}$ the number of customers arriving during that time. Similar notations apply for ${}^{R}Y_n$, ${}^{R}X_n^{(1)}$. It seems reasonable to take ${}^{G}X_n^{(1)}$, ${}^{R}X_n^{(1)}$ Poisson distributed with means $\alpha|G|$, resp. $\alpha|R|$, and the batch capacity, viz. the number of customers which can pass during a green period, equal to some fixed number $p = X_n^{(2)} > \alpha$. Then with $X_n^{(1)} = {}^{G}X_n^{(1)} + {}^{R}X_{n-1}^{(1)}$

$$ {}^{G}Y_n = ({}^{G}Y_{n-1} + {}^{G}X_n^{(1)} - p)^+ + {}^{R}X_n^{(1)} \tag{5.16} $$

$$ {}^{R}Y_n = ({}^{R}Y_{n-1} + X_n^{(1)} - p)^+ \tag{5.17} $$

The solution of these relations are described by parts (ii) and (i) of Proposition 5.2 respectively. Note in particular the intuitively obvious fact that ${}^{G}Y_n \overset{d}{=} {}^{R}Y_{n-1} + {}^{R}X_n^{(1)}$. The equations (2.1), (5.14) reduce to $\delta^p = e^{\alpha(\delta-1)}$, resp. $w^p = e^{\alpha(w-1)}$ and we may immediately deduce that (5.1) holds for ${}^{G}Y_n$, ${}^{R}Y_n$ with $\tilde{\mu}^{-1} = \delta^p\alpha - p$, $\tilde{\sigma}^2 = \tilde{\mu}^3\delta^p\alpha$,

$$ {}^{R}\tilde{C} = D = \frac{(p-\alpha)(\delta-1)\prod_{j=1}^{p-1}\dfrac{\delta-w_j}{1-w_j}}{\delta^p(\alpha\delta-p)}, \quad {}^{G}\tilde{C} = E\delta^{{}^{R}X_n^{(1)}} D = e^{\alpha|R|(\delta-1)} D . $$

6. The GI/M/m queue

The model is treated in a number of textbooks, e.g. Takács [24]Ch.2. We let $V_1, V_2, \ldots$ denote the interarrival times, $F(x) = P(V_n \leqslant x)$, $\alpha^{-1} = EV_n$ and β the service intensity. The number of customers in the system at time t is still denoted by $Q(t)$, but $\tau(n) = V_1 + \ldots + V_n$ is now the time of arrival of customer n $(n = 0,1,2,\ldots)$ and the imbedded Markov chain is given by $Y_n = Q(\tau(n)-0)$. Thus if $X_1^{(2)}, X_2^{(2)}, \ldots$ are independent with p.g.f.

$$\hat{g}(s) = \int_0^\infty e^{m\beta t(s-1)} \, dF(t) = \hat{F}(m\beta[s-1]), \quad \text{we have}$$

$$Y_n = Y_{n-1} + 1 - X_n^{(2)} \quad \text{on} \quad \{Y_{n-1} + 1 - X_n^{(2)} \geqslant m\} \qquad (6.1)$$

Corresponding relations on $\{Y_{n-1} + 1 - X_n^{(2)} < m\}$ are discussed in detail in (e.g.) Kleinrock[17] Ch.6 in the formulation of the transition function, but need not concern us. Subject to

<u>Condition 6.1</u> $\alpha^{-1}\beta m > 1$

<u>Condition 6.2</u> F <u>is non-lattice</u>

the existence of the limiting steady state is well-known and the form of the solution is geometric modified in a finite number of terms, viz.

$$P(Y_\infty \geqslant N) = \alpha^{-1}\beta m P(Q(\infty) \geqslant N+1) = B\delta^{-N} \qquad N \geqslant m-1 \qquad (6.2)$$

where $\delta > 1$ is determined by the equation

$$1 = E\delta^{1-X_n^{(2)}} = \delta\hat{F}(m\beta[\delta^{-1} - 1]) \qquad (6.3)$$

describing the associated random walk and $B = A\delta/(\delta-1)$ with A the constant explicitly determined in [24] p.148-149. Note that the analogue of Condition 2.1 is automatic in the present setting.

It is now readily checked that the associated parameters compatible with ${}^{a}P(1-X_n^{(2)} = i) = \delta^i P(1-X_n^{(2)} = i)$ are given by ${}^{a}\beta = \beta\delta^{-1}$, ${}^{a}F(dx) = e^{-\gamma x}F(dx)/\hat{F}(-\gamma)$ where $\gamma = m\beta(1-\delta^{-1})$. The Q*-process evolves now as the difference between a renewal process governed by F and a Poisson process with intensity $m\beta$. Hence μ^{-1}, ω^2 (defined in analogy with (2.5)) are given by

$$\mu^{-1} = {}^{a}\alpha - m\,{}^{a}\beta \ , \quad \omega^2 = {}^{a}\alpha^3\, {}^{a}\mathrm{Var}V_n - m\,{}^{a}\beta \tag{6.4}$$

where, e.g., ${}^{a}\alpha^{-1} = {}^{a}EV_n = {}^{a}\hat{F}'(0) = \hat{F}'(-\gamma)/\hat{F}(-\gamma)$ and we have

<u>Theorem 6.1</u> <u>Suppose that Conditions 6.1, 6.2 are in force and define</u> $C = \alpha B/\beta m$. <u>Then whenever</u> $E\delta^{Q(0)} < \infty$, (1.4) <u>holds with</u> $\sigma^2 = \mu^3\omega^2$.

The proof follows just the same lines as in Sections 3-4, but is in fact rather much simpler. As the regeneration instants (of one type $i=1$ only) we take the instants where all servers become busy, viz. where an arriving customer meets $m-1$ customers in the system. The behaviour is different in the Q and Q^* systems in periods where not all servers are busy. That is, if

$$\underline{m} = \inf\{n \geq 0 : Y_n + 1 - X_{n+1}^{(2)} < m\} \ , \quad \underline{n} = \inf\{n \geq \underline{m} : Y_n = m-1\} \ ,$$

then $Q(t) = Q^*(t)$ for $t \in [0, \tau(\underline{m}))$ so that $\underline{m}$ takes the role of $\underline{n}-1$. E.g. in analogy with Section 3 we define

$$v_N(t) = P(Q(t) \geq N, \ \tau(\underline{m}) \leq t < \tau(\underline{n}))$$

and Lemma 3.3 becomes a triviality because of $Q(t) < m$ $t \in [\tau(\underline{m}), \tau(\underline{n}))$.

Also the first passage problem reduces greatly since the paths are upwards skipfree. Thus we need not invoke overshot variables like the $B_i^*(N)$ to describe the post-$\theta^*(N)$ process, no analogue of Lemma 4.1 is required and also the proof of Lemma 3.4 admits for a number of simplifications. Finally the proofs of Lemmata 2.1 and 3.1 are just the same and the problem of identifying C is, as mentioned above, treated in the literature.

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