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Abstract

The purpose of the paper is to develop a method for the statistical analysis of a Markov branching process. We first describe a Markovian model for a family tree and apply likelihood methods for the estimation of the parameters. Next we derive the maximum likelihood estimates of the moments of the population size and find their properties using the theory of counting processes and martingales.

0. Introduction

The present paper gives an example of an infinite dimensional (non-parametric) statistical problem, which is analysed using likelihood methods. In order to derive asymptotic properties we apply the theory of counting processes and martingales and the asymptotic theory for convergence of martingales to Gaussian processes.

The purpose of the paper is not so much to be directly useful in analysing family trees, but is considered a contribution to the application of the methodology of point processes and product integration to statistics.

1. The probability model for the family tree

We shall consider a set I of individuals η, each characterized by a finite sequence of integers $\eta = \langle i_1,\dots,i_k\rangle$, $k=1,2,\dots$, which for $k \geqq 2$ indicates that η is the i_k'th child of the mother $\tau(\eta) = \langle i_1,\dots,i_{k-1}\rangle$. See Harris (1963). We shall assume that if $\eta \in I$, then $\tau(\eta) \in I$. The individuals of the form $\eta = \langle i\rangle$ have no mother and constitute the original population at time 0.

We order the individuals lexicographically, that is $\langle i_1,\dots,i_m\rangle < \langle j_1,\dots,j_\ell\rangle$ if for some $k=1,2,\dots$ we have $i_1 = j_1,\dots,i_{k-1} = j_{k-1}$, $i_k \leqq j_k$. Note that $\tau(\eta) < \eta$ and that each individual has a finite number of ancestors.

To each individual we associate a death time u_η and a number of offspring produced at death y_η. Note that the birth time of an individual coincides with the death time of its mother. We also assume that $u_\eta \in [0,1]$, and that $y_\eta \in \{0,1,\ldots\}$.

For notational convenience we introduce a fictitious individual ϕ, the common ancestor for the original population, i.e. $\tau(<i>) = \phi$ and assume $U_\phi = 0$, that is we think of the original population at time $t=0$ as having been born at time 0. Thus there was a death at time 0 and an n-birth i.e. $Y_\phi = n$. We thus assume $\phi \in I$, and $\phi < \eta$, for all $\eta \in I$.

We want to build a stochastic model for the process $\{U_\eta, Y_\eta\}_{\eta \in I}$ by specifying the conditional distribution of (U_η, Y_η) given $\{U_\psi, Y_\psi\}_{\psi<\eta}$ and given that we start with n individuals at time 0.

For this purpose let $G(s,t)$ be defined for $0 \leqq s \leqq t \leqq 1$ with values in $[0,1]$ such that the following conditions are satisfied

(1.1) $$G(s,t) = G(s,u)\, G(u,t), \quad 0 \leqq s \leqq u \leqq t \leqq 1$$

(1.2) $G(s,\cdot)$ and $G(\cdot,t)$ are right continuous, $\quad 0 \leqq s \leqq t \leqq 1$

(1.3) $$G(s,s) = 1, \quad 0 \leqq s \leqq 1 .$$

We also consider $p_j(t)$, $j \in \{0,1,\ldots\}$, $t \in [0,1]$ such that

(1.4) $p_j(\cdot)$ measurable for all j

(1.5) $p_j(t) \geqq 0$, $\Sigma_j\, p_j(t) = 1$ for all j and t.

Finally we shall assume for convenience that $G(s,1) = 0$, $0 \leqq s < 1$ and that $p_j(1) = 1$ if $j=0$ and 0 otherwise. This corresponds to letting $t=1$ be a point where all individuals are forced to die and leave no offspring. Similarly we define $p_j(0) = \delta_{jn}$, that is, we start off with n individuals.

We shall now construct the distribution of $(U_\eta, Y_\eta)_{\eta \in I}$ from the parameters (G,p) and n.

We define

(1.6) $$P\{U_\eta > u \mid U_\psi = u_\psi, Y_\psi = j_\psi, \psi < \eta\} = G(u_{\tau(\eta)}, u), \quad u_{\tau(\eta)} \leqq u \leqq 1$$

(1.7) $$P\{Y_\eta = j \mid U_\psi = u_\psi, Y_\psi = j_\psi, \psi < \eta, U_\eta = u_\eta\} = p_j(u_\eta), \quad 0 < u_\eta \leqq 1, \; j = 0,1,\ldots$$

Thus we see that given the pedigree of an individual η, $G(u_{\tau(\eta)}, \cdot)$ describes the distribution of the lifelength of η and, if it dies at u_η, then $p_j(u_\eta)$ describes the probability that it is replaced by j offspring.

Note that we have fixed $X(0) = n$ and that lines of descent of different children from the same mother develop independently.

Due to the right continuity of G we see that $U_\eta > U_{\tau(\eta)}$. Note that the process is inhomogeneous in time due to the fact that $p_j(t)$ may depend on t. Since we have not assumed continuity of $G(s,\cdot)$ the process may have fixed points of discontinuity, where the probability is positive that the individual may die.

It is easy to check that if $G(u_{\tau(\eta)}, s) > 0$ then

$$P\{U_\eta > u \mid U_\eta > s, \; U_\psi = u_\psi, \; Y_\psi = j_\psi, \; \psi < \eta\} = G(s,u), \quad 0 \leqq s \leqq u \leqq 1$$

which means that $G(s,u)$ has the interpretation that it is the probability that an individual seen alive at time s, will also be alive at time u.

The function $G(0,\cdot)$ is decreasing and right continuous. There exists thus a point $t_0 \in]0,1]$ such that $G(0,t) > 0$, $t \in [0,t_0[$,

$G(0,t) = 0, \; t \in [t_0,1]$.

Now if u and $t \in [0,t_0[$ we find

$$G(u,t) = \frac{G(0,t)}{G(0,u)} .$$

Hence on $[0,t_0[$ the function $G(u,t)$ is completely determined by $G(0,t)$ which is the lifelength distribution of an individual alive at time 0.

The role of t_0 is best seen by considering

$$G(0,t_0) = G(0,s)\, G(s,t_0) , \qquad s \in [0,t_0[.$$

Since $G(0,s) > 0$ and $G(0,t_0) = 0$ we have $G(s,t_0) = 0$ which means that all particles alive at time s will certainly die before or at t_0. The population itself, however, may continue to exist, since new individuals may be born at t_0.

2. The statistical analysis of a family tree

Let us consider an observation of a family tree $F = (U_\eta, Y_\eta)_{\eta\in I}$ on a finite interval $[0,1[$. We shall assume that I is finite and that the number of individuals at time 0 is n.

From (1.6) and (1.7) we find the following expression for the probability of obtaining F.

$$L = \prod_{\eta\in I} \{G(U_{\tau(\eta)}, U_\eta -) - G(U_{\tau(\eta)}, U_\eta)\}\, p_{Y_\eta}(U_\eta) . \tag{2.1}$$

If $G(s,\cdot)$ is continuous this will be zero, but if atoms are allowed we sometimes get a positive value.

We shall derive the maximum likelihood estimate in the sense of Kiefer and Wolfowitz (1956), that is, we shall find $(\hat{G},\hat{p})$ such that for any (G,p) we have

$$L(\hat{G},\hat{p}) \geqq L(G,p) .$$

We clearly only need the expression for L if it is positive. We shall thus restrict attention to those functions G which have atoms at the points where the deaths have actually occured.

In order to simplify the expression for L we introduce the random variables $K,T_0,\ldots,T_K$ such that $K=1,2,\ldots$ and $0=T_0<T_1<\ldots<T_K<1$ which are the points where deaths have occured.

In general one may find many deaths at any given time and we therefore define

$$M_{ij} = \sum_{\eta} 1\{U_{\tau(\eta)}=T_i,\ U_\eta=T_j\}$$

that is, the number of individuals that are born at time T_i and die at time T_j. Let also

$$N_j(u) = \sum_{\eta} 1\{Y_\eta=j,\ 0<U_\eta\leqq u\} , \qquad 0\leqq u<1$$

be the number of j-births after time zero but before or at time u. Then $N(u) = \Sigma_j N_j(u)$ is the number of deaths before or at time u, and $X(u)=n+\Sigma_j (j-1)N_j(u)$ is the population size at time u. Note that N_j,N, and X are defined to be piecewise constant, right continuous and that N_j and N are increasing. The notation $\Delta N(u)$ is used for $N(u)-N(u^-)$ and we have the relation

$$\sum_{i\leqq k,\,j=k+1} M_{ij} = \Delta N(T_{k+1}) \tag{2.2}$$

since both sides indicate the number of individuals born before or at T_k who die at T_{k+1}.

Similarly

$$\sum_{i \leq k, j>k} M_{ij} = X(T^-_{k+1}) \tag{2.3}$$

since both sides count the number of individuals who are born before or at T_k and who die at T_{k+1} or later, that is, who are alive just before T_{k+1}. Note that $X(T^-_{k+1}) > 0$ if $k < K$.

With this notation the likelihood function (2.1) can be written

$$L = \prod_{i<j} \{G(T_i,T^-_j) - G(T_i,T_j)\}^{M_{ij}} \prod_{i,k} p_k(T_i)^{\Delta N_k(T_i)} .$$

The second factor is maximized by

$$\hat{p}_k(T_i) = \frac{\Delta N_k(T_i)}{\Delta N(T_i)} \tag{2.4}$$

or

$$\hat{p}_k(t) = \frac{dN_k}{dN}(t)$$

where the right hand side is the Radon - Nikodym derivative of the measure determined by N_k with respect to that determined by N.

The first factor is maximized as follows. Let

$$S_i = 1 - G(T_i,T^-_{i+1})$$

$$R_i = G(T_i,T^-_{i+1}) - G(T_i,T_{i+1})$$

$$V_i = G(T_i,T_{i+1})$$

denote the probabilities of an individual dying before T_{i+1}, at T_{i+1} or after T_{i+1} given it is born at T_i. We have $S_i + R_i + V_i = 1$ and (1.1) implies that

$$G(T_i,T_j^-) - G(T_i,T_j) = V_i V_{i+1} \cdot \ldots \cdot V_{j-2} R_{j-1}$$

which shows that

$$\prod_{i<j} \{G(T_i,T_j^-) - G(T_i,T_j)\}^{M_{ij}} = \prod_k V_k^{\sum_{i\leq k\leq j-2} M_{ij}} R_k^{\sum_{i\leq k, j=k+1} M_{ij}}$$

which is maximized for $S_k = 0$ and

$$\hat{R}_k = 1 - \hat{V}_k = \frac{\Delta N(T_{k+1})}{X(T_{k+1}^-)}$$

where we have used the relations (2.2) and (2.3). From this result we can write the estimate of G as

$$\hat{G}(s,t) = \prod_{]s,t]} (1 - \frac{dN(u)}{X(u^-)}), \tag{2.5}$$

using the product integral notation.

We have thus found $(\hat{G},\hat{p})$ such that L becomes positive. This estimate is the maximum likelihood estimate in the sense of Kiefer and Wolfowitz (1956). The estimates are the natural estimates, since $\hat{p}_k(t)$ is nothing but the fraction of births at time t which were k-births, and $\hat{G}(s,t)$, when s-t is small, is just the ratio of those surviving t to those surviving s. We have used the product integral notation, which seems to be the natural one for this type of estimator. The estimator is an analogue of the Kaplan Meier estimator, which has been treated from this point of view by Johansen (1978).

It is seen that the estimates are functions of the counting processes $\{N_j\}$. We can prove that $\{N_j\}$ is sufficient for the parameters (G,p). If we condition on $\{N_j(u),\ j=0,1,\ldots,\ u \in [0,1[\}$ then we know all the variables $K,T_1,\ldots,T_K$, where the processes jump and we also know how many die at u $(\Delta N(u))$ as well as how many j-birth we have $(\Delta N_j(u))$. The only variability in the outcome, when $\{N_j\}$ is given, is which individuals die and which of these give rise to j-births. Due to the interchangeability between individuals these probabilities equal $\binom{X(u^-)}{\Delta N(u)}^{-1}$ and $\binom{\Delta N(u)}{\Delta N_0(u),\ldots,\Delta N_j(u),\ldots}^{-1}$ respectively.

Thus the conditional distribution of the actual outcome given $\{N_j\}$ is a combinatorial coefficient, which proves sufficiency of $\{N_j\}$. Now from $\{\hat{p}\}$ and $\hat{G}$ it is easy to reconstruct $\{N_j\}$ and hence we have found sufficiency of the maximum likelihood estimator of (p,G).

3. The population size X

Often the observed quantity is the population size $X(u)$ and it is natural than to derive estimates of quantities derived from X, like the mean, the variance or even the distribution of X.

It should be emphasized, however, that in general X is not a sufficient statistic, since $\{N_j\}$ cannot be recovered from X. If the function G is continuous, however, such that two jumps cannot occur at the same time, then $\Delta N_j(t) = 1\{\Delta X(t) = j-1\}$, $j \neq 1$, which shows that at least $\{N_j\}$, $j \neq 1$ can be recovered from X in this case. Clearly N_1 can never be found since a death followed by a

1-birth will remain unnoticed by X, the population size.

Thus in general X is a reduction of the data which implies that certain parametric functions cannot be identified from the observation of X alone.

In order to get a relatively easy description of the process X we shall assume that

$$G(0,t) > 0 , \quad t < 1.$$

We shall let G denote the probability measure determined by $G(0,\cdot)$, thus $G(0,u) = G(]u,1]) = G]u,1]$. It was proven by Jacobsen (1972) that X is now a Markov jump process and that the transition probabilities are solutions of the forward and backward differential equations, even when G is allowed to have atoms.

If $P(s,t)$ denotes the transition probability matrix for X and ν denotes the matrix of integrated intensities, see below, then the Kolmogorov equations take the forms

$$(3.1) \qquad P(s,t) - I = \int_{]s,t]} \nu(du)\, P(u,t)$$

$$(3.2) \qquad P(s,t) - I = \int_{]s,t]} P(s,u^-)\, \nu(du)$$

where ν depends on G and p as follows:

Lemma 3.1 Let G denote the measure determined by $G(0,\cdot)$ then

$$(3.3) \quad \nu_{ij}(A) = \int_A \sum_{m=1}^{i} (p_{j-i+m}^{(m)}(t) - \delta_{ij}) \binom{i}{m} \frac{G[t]^{m-1}\, G]t,1]^{i-m}}{G[t,1]^i}\, G(dt) ,$$

$$i,j \in \{0,1,\dots\}, \quad A \subset [0,1].$$

Here $p_{\cdot}^{(m)}(t)$ denotes the m fold convolution of $\{p_k(t)\}$.

In particular

$$(3.4) \qquad \nu_{11}(A) = -\int_A (1-p_1(t))\,\frac{G(dt)}{G[t,1]}$$

and

$$(3.5) \qquad \frac{d\nu_{1j}}{d\nu_{11}} = -\frac{p_j(t)}{1-p_1(t)}\ ; \qquad j \neq 1.$$

Proof We shall first prove that

$$(3.6) \qquad \lim_{s\uparrow t} \frac{P\{X(t)=j \mid X(s)=i\} - \delta_{ij}}{G]s,t]}$$

$$= \sum_{m=1}^{i} (p^{(m)}_{j-i+m}(t) - \delta_{ij}) \binom{i}{m} \frac{G[t]^{m-1}\, G]t,1]^{i-m}}{G[t,1]^{i}} .$$

We shall think of s being close to t and that the i individuals alive at time s give rise to death times $T_1,\ldots,T_i$. For $j \neq i$ we want to find the probability

$$A_m(s,t) = P\{T_k \in\,]s,t],\ k=1,\ldots,m,\ \sum_{k=1}^{m}(Y_k-1)=j-1,\ T_k>t,\ k=m+1,\ldots,i\}$$

that is, the probability that m specified individuals die in]s,t], and that the increase in X is from i to j and that the remaining $i-m$ individuals survive t.

The probability can be found as follows

$A_m(s,t) =$

$$\sum_{j_1+\cdots+j_m=j-i} \int_{]s,t]^m} p_{j_1+1}(u_1)\cdot\ldots\cdot p_{j_m+1}(u_m)\frac{G(du_1)\ldots G(du_m)}{G]s,1]^m}\left(\frac{G]t,1]}{G]s,1]}\right)^{i-m}$$

hence

$$\lim_{s\uparrow t} \frac{A_m(s,t)}{G]s,t]} = \sum_{j_1+\dots+j_m=j-i} p_{j_1+1}(t)\cdot\ldots\cdot p_{j_m+1}(t) \frac{G]t,1]^{i-m}G[t]^{m-1}}{G[t,1]^i} \text{ a.s.}[G]$$

$$= p^{(m)}_{j-i+m}(t) \frac{G]t,1]^{i-m}G[t]^{m-1}}{G[t,1]^i}.$$

Now multiply by $\binom{i}{m}$ and sum over m. This gives the probability that the increment is $j-i$ but only caused by deaths among the i individuals alive at time s.

Clearly the offspring could also give rise to a second generation. This probability is negligable as the following argument shows.

Consider the probability that an individual dies and that at least one of its offspring also dies in]s,t]. This clearly equals

$$B(s,t) = \int_{]s,t[} \sum_{k\neq 0} \{1 - \left(\frac{G]t,1]}{G]u,1]}\right)^k\} p_k(u) \frac{G(du)}{G[u,1]}$$

but $B(s,t)/G]s,t] \to 0$, $s\uparrow t$. Note that the integration does not contain the point t, since if the first individual dies at t, the second will die later than t, due to condition (1.2) which ensures that lifelengths are strictly positive.

Combining the above results we have proved (3.6) when $i\neq j$, which shows that the intensity has the required form and hence that the integrated intensity ν_{ij} is given by (3.3). For $j=i$ we use the fact that $\nu_{ii} = -\sum_{j\neq i}\nu_{ij}$. The result (3.3) is more transparent if G is continuous, since then $G[t]^{m-1} = 0$ unless $m=1$. Hence we find

$$\nu_{ij}(A) = \int_A (p_{j-i+1}(t) - \delta_{ij})\, i \frac{G(dt)}{G[t,1]}$$

and hence the usual result that

$$\nu_{ij} = i\,\nu_{1(j-i+1)}$$

which shows that the j'th row of ν is determined by the first row. This also holds in general but is perhaps most easily seen from the generating functions below.

Let us introduce $h_t(z) = \Sigma_j\, p_j(t)\, z^j$, then we find from (3.3)

$$\nu_i(z,A) = \sum_j z^j \nu_{ij}(A) = \int_A [\{(h_t(z) - z)\frac{G[t]}{G[t,1]} + z\}^i - z^i]\,\frac{G(dt)}{G[t]}$$

where the integrand is interpreted as a limit if $G[t] = 0$.

In particular

$$\nu_1(z,A) = \int_A (h_t(z) - z)\,\frac{G(dt)}{G[t,1]}$$

and hence

$$\frac{d\nu_i(z,\cdot)}{d\nu_1(z,\cdot)}(t) = \frac{(z+\nu_1(z,[t]))^i - z^i}{\nu_1(z,[t])}$$

which shows that $\{\nu_{ij}\}$ is completely determined from $\{\nu_{1j}\}$.

<u>Corollary 3.2</u> From the observation of X alone only the parameters

$$p_j(t)/(1-p_1(t)), \quad j \neq 1 \quad \text{and} \quad \int_A (1-p_1(t))\,G(dt)/G[t,1]$$

are identifiable.

<u>Proof</u> When the above parameters are given one can construct ν_{ij} and hence ν and the distribution of X.

We shall now turn to the moments of X and their estimates. The usual way of obtaining moments is via the probability generating function which we shall therefore consider:

From the backward and forward equations we get for $f_{s,t}(z) = \Sigma_j z^j P_{1j}(s,t)$ that

$$(3.7) \qquad f_{s,t}(z) - z = \int_{]s,t]} \{h_u(f_{u,t}(z)) - f_{u,t}(z)\} \frac{G(du)}{G[u,1]}$$

and

$$(3.8) \qquad f_{s,t}(z) - z = \int_{]s,t]} \{f_{s,u-}(f_{u-,u}(z)) - f_{s,u-}(z)\} \frac{G(du)}{G[u]}$$

where again the integrand is interpreted as a limit when $G[u] = 0$.

It follows from (3.7) that for $s \uparrow t$ we have

$$f_{t-,t}(z) - z = (h_t(z) - z) \frac{G[t]}{G[t,1]} = \nu_1(z,[t]) .$$

<u>Theorem 3.3</u> Under the assumption that

$$\int_{[0,t]} \sum_j j^2 p_j(u) \frac{G(du)}{G[u,1]} < \infty$$

the variable $X(t)$ has finite mean and variance. Differential equations for these can be found by differentiation of (3.7) and (3.8) for $z = 1$.

The first part of this result follows from Lemma A.1 in the appendix. The proof of the second part will be omitted.

We want to derive expressions for the mean and variance of $X(t)$ given $X(0) = 1$. We define

$$m(s,t) = E\{X(t) \mid X(s) = 1\}$$

$$v(s,t) = V\{X(t) \mid X(s) = 1\}$$

and $m_2(s,t) = E\{X(t)(X(t)-1) \mid X(s)=1\}$.

From the forward equation (3.8) we find by differentiation for $z = 1$ the expressions

$$m(s,t) - 1 = \int_{]s,t]} m(s,u^-)\, \mu_1(du)$$

$$m_2(s,t) = \int_{]s,t]} m_2(s,u^-)\, \mu_3(du) + \int_{]s,t]} m(s,u^-)\, \mu_2(du)$$

and finally

$$\nu(s,t) = \int_{]s,t]} \nu(s,u^-)\, \mu_3(du) + \int_{]s,t]} m(s,u^-)\, \eta(du)$$

where

(3.9) $$\mu_1(A) = \int_A \sum_j (j-1)\, p_j(u) \frac{G(du)}{G[u,1]}$$

(3.10) $$\mu_2(A) = \int_A \sum_j j(j-1)\, p_j(u) \frac{G(du)}{G[u,1]}$$

(3.11) $$\mu_3(A) = \int_A [\{1 + \sum_j (j-1)\, p_j(u) \frac{G[u]}{G[u,1]}\}^2 - 1] \frac{G(du)}{G[u]}$$

(3.12) $$\eta(A) = \mu_2(A) - \mu_3(A) + \mu_1(A) .$$

One can show that the following relations hold

$$m(s,t) - 1 = \int_{]s,t]} m(s,u^-)\, \mu_1(du)$$

(3.13) $$m_2(s,t) = \int_{]s,t]} m(s,u^-)\, m(u,t)^2\, \mu_2(du)$$

and

$$m(s,t)^2 - 1 = \int_{]s,t]} m(s,u^-)^2\, \mu_3(du)$$

which give the relations between (m, m_2) and (μ_1, μ_2, μ_3).

We shall summarize the expressions for the moments as follows. Let us define

$$M = \begin{pmatrix} m & v \\ 0 & m^2 \end{pmatrix} \quad \text{and} \quad \mu = \begin{pmatrix} \mu_1 & \eta \\ 0 & \mu_3 \end{pmatrix}$$

then we have

$$M(s,t) - I = \int_{]s,t]} M(s,u^-)\,\mu(du) \tag{3.14}$$

or in product integral form:

$$M(s,t) = \prod_{]s,t]} (I + d\mu)\ . \tag{3.15}$$

Now these expressions for the mean and variance have been derived in the model where any G is allowed (provided $G(0,t) > 0,\ t < 1$). Thus in particular the results hold for the model with parameters $(\hat{p},\hat{G})$. Thus we can derive the maximum likelihood estimates of m and v (or M) by first estimating μ and then insert into (3.15).

If we let

$$\sigma(A) = \int_A \frac{G(du)}{G[u,1]} \quad \text{then} \quad G]s,t] = \prod_{]s,t]} (1 - d\sigma)$$

and we find from (2.5) the estimate

$$\hat{\sigma}(A) = \int_A \frac{dN(u)}{X(u^-)}$$

and

$$\hat{\mu}_1(A) = \int_A \sum_j (j-1)\,\frac{dN_j}{dN}\,\frac{dN}{X(u^-)} = \int_A \frac{dX}{X(u^-)} \tag{3.16}$$

$$\hat{\mu}_2(A) = \int_A \sum_j j(j-1)\,\frac{dN_j}{X(u^-)} \tag{3.17}$$

$$\hat{\mu}_3(A) = \int_A \left(\left(1 + \sum_j (j-1) \frac{dN_j}{X(u^-)}\right)^2 - 1\right) \tag{3.18}$$

$$\hat{\eta}(A) = \int_A \sum_j (j-1)^2 \frac{dN_j}{X(u^-)} - \int_A \left(\sum_j (j-1) \frac{dN_j}{X(u^-)}\right)^2 . \tag{3.19}$$

Hence we find

$$\hat{m}(s,t) = \prod_{]s,t]} (1 + d\hat{\mu}_1) = \prod_{]s,t]} \left(1 + \frac{dX}{X(u^-)}\right) = \frac{X(t)}{X(s)} \tag{3.20}$$

and

$$\hat{v}(s,t) = \frac{X(t)^2}{X(s)} \int_{]s,t]} \sum_j j(j-1) \frac{dN_j}{X(u^-)^2} - \left(\frac{X(t)}{X(s)}\right)^2 + \frac{X(t)}{X(s)} . \tag{3.21}$$

Note that the estimate of $E\{X(t) \mid X(s) = 1\}$ is just $X(t)/X(s)$. This is in agreement with the fact that $X(t)$ is a linear combination of the sufficient statistics and that in analogy with the result for finite dimensional exponential families one would expect that the estimate of a linear function of the sufficient statistic would be just the observed value.

If one wants moments of higher order the same approach would give a matrix of moments expressed as a product integral of simple functions of the measures $\int_A \Sigma_j j^p p_j(u) \frac{G(du)}{G[u,1]}$.

Note finally that the estimate of v is not a function of X alone, one needs more information to calculate $\Sigma_j j(j-1) dN_j$. If however, G is continuous then only one jump can occur at any given time and then $\Sigma_j j(j-1) dN_j = \int (dX)^2 + \int dX$, and then $\hat{v}$ can be calculated from X alone.

4. Exact properties of the moment estimators

We have derived the estimators of the moments m, v and M. We now want to find a stochastic integral equation which admits these estimators as solutions in order that we can derive some properties. The basic tools are the results about counting processes and martingales and in the next section we shall then apply the asymptotic theory of martingales to find the asymptotic properties.

We shall assume from now on that $G(0,\cdot)$ is absolutely continuous and we define the hazard or intensity by

$$\lambda(u) = \frac{dG}{du} / G]u,1]$$

such that

$$G]t,1] = \prod_{]0,t]} (1 - \lambda(u)\,du) = \exp\{-\int_0^t \lambda(u)\,du\}.$$

Since $G(0,t) > 0$, $t < \infty$ we find that

$$\int_0^t \lambda(u)\,du < \infty, \qquad t < 1.$$

We can now simplify the expression for μ and $\hat{\mu}$, since we now have only one jump at a time.

Let us define

$$(4.1) \qquad C_j(u) = \begin{pmatrix} 1 & (j-1)(1-1/X(u)) \\ 0 & 2+(j-1)/X(u) \end{pmatrix} 1\{X(u) > 0\}$$

and

$$(4.2) \qquad C_j = \begin{pmatrix} 1 & j-1 \\ 0 & 2 \end{pmatrix}.$$

Then we find from (3.9)-(3.12) and (3.16)-(3.19) that

(4.3) $$\mu(A) = \int_A \sum_j C_j\,(j-1)\,p_j(u)\,\lambda(u)\,du$$

and

(4.4) $$\hat{\mu}(A) = \int_A \sum_j C_j(u^-)\,(j-1)\,\frac{dN_j}{X(u^-)}$$

We shall first derive some properties of the counting processes $\{N_j\}$ and then derive the properties of $\hat{\mu}$ and finally $\hat{M}$.

Consider the processes $\{N_j\}$ as a marked point processes on $[0,1[$ with jump times

$$T_n = \inf\{t \mid N(t) \geqq n\}$$

indicating the occurence of the n'th death. The mark of this point is $Z_n = j$ if the n'th death gives rise to a j-birth. Since G is continuous we find $T_n < T_{n+1}$. Let $T_\infty = \lim_{n\to\infty} T_n$.

<u>Lemma 4.1</u> The processes $\{N_j\}$ are non-explosive in the sense that $T_\infty = 1$ a.s. if

$$\int_0^t \sum_j j\,p_j(u)\,\lambda(u)\,du < \infty, \qquad t<1.$$

Proof omitted.

Thus we only have a finite number of jumps on any compact interval of $[0,1[$.

We now want to find the compensator for N_j as discussed for instance by Brémaud and Jacod (1977). They give an explicit formula for the (predictable) compensator $\tilde{N}_j$.

Lemma 4.2 The compensator for N_j is given by $\tilde{N}_j(t) = \int_0^t p_j(u)\, X(u^-)\, \lambda(u)\, du$, that is $N_j = N_j - \tilde{N}_j$ is a local martingale and $\tilde{N}_j$ is increasing and predictable.

Proof omitted.

We shall also need some results about stochastic integrals and Stiltjes integrals with respect to counting processes.

The following result has been taken from Boel, Varaiya and Wong (1975).

Lemma 4.3 Let Y_{ij}, $i,j \in \mathbb{N}$ be predictable processes, such that

$$(4.5) \qquad E \sum_j \int_0^t Y_{ij}^2 \, d\tilde{N}_j < \infty, \qquad t < 1,\ i \in \mathbb{N}$$

than the Stiltjes integrals $M_i = \Sigma_j \int Y_{ij}\, d(N_j - \tilde{N}_j)$ are stochastic integrals and M_i is a local square integrable martingale with

$$(4.6) \qquad \langle M_i, M_k \rangle = \Sigma_j \int Y_{ij}\, Y_{kj}\, d\tilde{N}_j \, .$$

We shall now return to $\hat{\mu}$ which is a linear combination of integrals with respect to $\{N_j\}$. We first define

$$(4.7) \qquad \tilde{\mu}(A) = \sum_j \int_A C_j(u^-)\, \frac{j-1}{X(u^-)}\, d\tilde{N}_j = \sum_j \int_A C_j(u^-)\,(j-1)\, p_j(u)\, \lambda(u)\, du \, .$$

The main result about $\hat{\mu} - \tilde{\mu}$ can now be formulated as follows:

Theorem 4.4 If $\int_0^t \Sigma_j\, j^4\, p_j(u)\, \lambda(u)\, du < \infty$, $t < 1$, then the process $\hat{\mu} - \tilde{\mu}$ is a matrix valued local square integrable martingale and

$$(4.8) \qquad \langle \hat{\mu} - \tilde{\mu} \rangle = \Sigma_j \int C_j(u^-)^{2\otimes}\, \frac{(j-1)^2}{X(u^-)}\, p_j(u)\, \lambda(u)\, du \, ,$$

where $2\otimes$ denotes the Kronecker product of the matrix with itself.

Proof Let the norm $|A|$ of a matrix be $|A| = \sup_i \Sigma_j |a_{ij}|$. From

$$\hat{\mu} - \tilde{\mu} = \sum_j \int C_j(u^-) \frac{j-1}{X(u^-)} d(N_j - \tilde{N}_j)$$

we find that condition (4.5) is satisfied if

$$I = \sum_j E \int_0^t |C_j(u^-)|^2 \frac{(j-1)^2}{X(u^-)^2} d\tilde{N}_j < \infty .$$

Now $|C_j(u^-)|^2 \leqq (j+2)^2$ which shows that

$$I \leqq \sum_j \int_0^t (j+2)^4 p_j(u) \lambda(u) \, du < \infty .$$

Thus the elements of the matrix $\hat{\mu} - \tilde{\mu}$ are local square integrable martingales and from (4.6) we find

$$\langle \hat{\mu} - \tilde{\mu} \rangle = \sum_j \int C_j(u^-) \otimes C_j(u^-) \left(\frac{j-1}{X(u^-)}\right)^2 d\tilde{N}_j$$

$$= \sum_j \int C_j(u^-)^{2\otimes} \frac{(j-1)^2}{X(u^-)} p_j(u) \lambda(u) \, du .$$

This completes the proof of Theorem 4.4. Note that in particular $E\hat{\mu} = E\tilde{\mu}$, and in general $E\tilde{\mu} \neq \mu$, but the difference is small when $X(u^-)$ is large, since $C_j(u^-) \approx C_j$. This will be used in the next section where asymptotic results will be proved.

Before proceeding to the results about $\hat{M}$ we need a rather technical Lemma which will also be used in the next section.

We define $\tilde{M} = \Pi (I + d\tilde{\mu})$.

Lemma 4.5 If $\Sigma_j \int_0^t j^8 p_j(u) \lambda(u) \, du < \infty$, $t < 1$, then the variables

$$(4.9) \quad Z_{nj}(s) = n|\hat{M}(0,s^-)|^2 |\tilde{M}^{-1}(0,s)|^2 |C_j(s^-)|^2 \left(\frac{j-1}{X(s^-)}\right)^2 X(s^-) p_j(s) \lambda(s)$$

are integrable uniformly in n and $s \leqq t$ and we have

$$\sum_j \int_0^t \sup_n E|Z_{nj}(s)| \, ds < \infty, \qquad t < 1.$$

The proof of this result, contains the basic evaluations of moments of the branching process but since it is rather involved we shall leave the proof for the appendix.

We shall here apply the result to give some properties of $\hat{M}\tilde{M}^{-1}$.

Theorem 4.6 If $\int_0^t \Sigma_j j^8 p_j(u) \lambda(u) \, du < \infty$, $t < 1$ then $\hat{M}\tilde{M}^{-1} - I$ is a matrix valued local square integrable martingale with

$$(4.10) \quad \langle \hat{M}\tilde{M}^{-1} - I \rangle = \int \hat{M}(0,s^-)^{2\otimes} \, d\langle \hat{\mu} - \tilde{\mu} \rangle \, \tilde{M}(0,s)^{-2\otimes}$$

$$= \sum_j \int (\hat{M}(0,s^-) C_j(s^-) \tilde{M}^{-1}(0,s))^{2\otimes} \frac{(j-1)^2}{X(s^-)} p_j(s) \lambda(s) \, ds.$$

Thus in particular

$$E\hat{M}\tilde{M}^{-1} = I$$

and
$$V(\hat{M}\tilde{M}^{-1}) = E\langle \hat{M}\tilde{M}^{-1} - I \rangle .$$

Proof Fix a sample path $\{N_j\}$ and consider the matrix valued function $s \to \hat{M}(0,s)\,\tilde{M}(s,1)$. This function is differentiable with respect to a measure μ_0 that dominates $\{N_j\}$ as well as Lebesgue measure. We find

$$\frac{d}{d\mu_0} \hat{M}(0,s)\,\tilde{M}(s,1) = \hat{M}(0,s^-) \frac{d\hat{\mu}}{d\mu_0} \tilde{M}(s,1) + \hat{M}(0,s^-) \left(- \frac{d\tilde{\mu}}{d\mu_0}\right) \tilde{M}(s,1)$$

and hence

$$\hat{M}(0,s)\,\tilde{M}(s,1) - \tilde{M}(0,1) = \int_{[0,s]} \hat{M}(0,u^-)\, d(\hat{\mu} - \tilde{\mu})\, \tilde{M}(u,1)$$

or

$$(4.11)\qquad \hat{M}(0,s)\,\tilde{M}(0,s)^{-1} - I = \int_{[0,s]} \hat{M}(0,u^-)\, d(\hat{\mu} - \tilde{\mu})\, \tilde{M}^{-1}(0,u)\ .$$

This relation gives $\hat{M}\tilde{M}^{-1}$ as a Stiltjes integral with respect to the measure determined by $u \to (\hat{\mu} - \tilde{\mu})[0,u]$. In fact it is also a stochastic integral. We express the right hand side as

$$(4.12)\qquad \sum_j \int_{[0,s]} \hat{M}(0,u^-)\, C_j(u^-)\, \tilde{M}(0,u)\, \frac{j-1}{X(u^-)}\, d(N_j - \tilde{N}_j)$$

and find that Lemma 4.5 implies condition (4.5) of Lemma 4.3. Hence the conclusion of Theorem 4.6.

The result of Theorem 4.6 is rather complicated but we can find some consequences for the individual terms of the matrix.

Consider the upper left hand corner of $\hat{M}\tilde{M}^{-1} - I$ which is just $\hat{m}(0,s)/\tilde{m}(0,s) - 1$. On the interval $1\{X(s^-) > 0\}$ we have $\tilde{m} = m$ and hence we have the well known result that

$$\frac{X(s)}{n\,E\{X(s) \mid X(0) = 1\}} - 1$$

is a local square integrable martingale.

We can find the variance from

$$V\left(\frac{X(s)}{n\,m(0,s)}\right) = E\langle \hat{m}(0,s)\, \tilde{m}^{-1}(0,s) - 1\rangle = \int_0^s \sum_j (j-1)^2\, \frac{p_j(u)\,\lambda(u)}{n\,m(0,u)}\, du$$

This result could also be derived from the expressions for the moments in Section 3. Of more direct use are the results about the asymptotic properties in the next section.

5. Asymptotic results for the moment estimators

We shall consider the asymptotic properties of $\hat{\mu}$ and $\hat{M}$ when $X(0) = n \to \infty$.

The basic idea is that the counting processes, suitably normalized, converge weakly to Gaussian processes and that this holds for stochastic integrals with respect to these, as well.

Some asymptotic properties of the stochastic integrals $\int_0^t \frac{dN_j}{X(u-)}$ have been studied Harrington and Fleming (1978) in connection with the estimation of $\int_0^t p_j(u)\,\lambda(u)\,du$.

The results that we need are due to Aalen (1977) and Rebolledo (1977). We shall use the formulation of Rebolledo (1977), see also Aalen and Johansen (1978).

The basic results can be formulated in

Lemma 5.1 Let N_j have compensator $\tilde{N}_j(t) = \int_0^t \Lambda_{nj}(u)\,du$ and assume that H_{nj} is a predictable process with values in the space of $p \times p$ matrices.

We shall assume that

(5.1) $$H_{nj}(t) \xrightarrow{P} 0, \qquad n \to \infty,\ t < 1,\ j \in \mathbb{N}$$

(5.2) $$H_{nj}^{2\otimes}(t)\,\Lambda_{nj}(t) \xrightarrow{P} g_j(t)^{2\otimes}, \qquad n \to \infty,\ t < 1,\ j \in \mathbb{N}$$

where g_j is a deterministic function with values in the space of $p \times p$ matrices.

(5.3) $$H^{2\otimes}_{nj}(t)\,\Lambda_{nj}(t) \text{ is integrable uniformly in } n \text{ for each } t<1 \text{ and } j\in\mathbb{N}$$

(5.4) $$\sum_j \int_0^t \sup_n E\,|H_{nj}(u)|^2\,\Lambda_{nj}(u)\,du < \infty$$

(5.5) $$\sum_j \int_0^t |g_j(u)|^2\,du < \infty .$$

Then the Stiltjes integrals $Y_{nj} = \int H_{nj}\,d(N_j - \tilde{N}_j)$ are stochastic integrals and for each finite set $J\subset\mathbb{N}$ we have

$$\{Y_{nj}\}_{j\in J} \overset{w}{\Rightarrow} \{Y_j\}_{j\in J}$$

where the Y_j are independent Gaussian processes with $\langle Y_j\rangle(t) = \int_0^t g_j(u)^{2\otimes}\,du$.

Further $Y_n = \Sigma_j\,Y_{nj}$ and $Y = \Sigma_j\,Y_j$ exist

and $$Y_n \overset{w}{\Rightarrow} Y, \qquad n\to\infty$$

and $$V(Y_n) \to V(Y) = \sum_j \int g_j^{2\otimes}(u)\,du, \qquad n\to\infty.$$

<u>Proof</u> What we have to prove is that the results of Rebolledo for a finite number of martingales can be applied here to the countably many processes, since we have sufficiently small tail sums.

From assumption (5.4) we see that

$$\sum_j E \int_0^t |H_{nj}(u)|^2\,\Lambda_{nj}(u)\,du < \infty .$$

This shows that condition (4.5) is satisfied and that the stochastic integrals Y_{nj} and $Y_n = \Sigma_j\,Y_{nj}$ are local square integrable martingales and finally that

$$\langle Y_n \rangle = \sum_j \int H_{nj}^{2\otimes}(u)\, \Lambda_{nj}(u)\, du .$$

We also find

$$(5.6)\qquad |V(\sum_{j>M} Y_{nj})(t)| = |\sum_{j>M} E\langle Y_{nj}\rangle(t)| \leq \sum_{j>M} \int_0^t \sup_n E|H_{nj}(u)|^2 \Lambda_{nj}(u)\, du$$

$$(5.7)\qquad |V(\sum_{j>M} Y_j)(t)| \leq \sum_{j>M} \int_0^t |g_j(u)|^2\, du .$$

As $M \to \infty$ these variances tend to zero uniformly in n and it follows from Kolmogorov's inequality for martingales that

$$\sup_n P\{\sup_{u\leq t} |\sum_{j>M} Y_{nj}(u)| \geq \varepsilon\} \to 0 , \qquad M \to \infty$$

and

$$P\{\sup_{u\leq t} |\sum_{j>M} Y_j(u)| \geq \varepsilon\} \to 0 , \qquad M \to \infty.$$

Next we shall prove that

$$\sum_{j\leq M} Y_{nj} \overset{W}{\Rightarrow} \sum_{j\leq M} Y_j , \qquad n \to \infty \text{ and } M \text{ fixed.}$$

This follows from the general conditions of Rebolledo (1977) if we can prove that

$$(5.8)\qquad E \int_0^t |H_{nj}(u)|^2\, 1\{|H_{nj}(u)| \geq \varepsilon\}\, \Lambda_{nj}(u)\, du \to 0$$

$$(5.9)\qquad \int_0^t H_{nj}(u)^{2\otimes} \Lambda_{nj}(u)\, du \overset{P}{\to} \int_0^t g_j(u)^{2\otimes}\, du , \qquad n \to \infty.$$

We have assumed that $H_{nj}^{2\otimes}\Lambda_{nj}$ is uniformly integrable and that $H_{nj}(n) \overset{P}{\to} 0$, this implies (5.8).

Next consider

$$R_{nj}(u) = |H_{nj}(u)^{2\otimes} \Lambda_{nj}(u) - g_j(u)^{2\otimes}|$$

We have assumed that $R_{nj}(u) \overset{P}{\to} 0$, hence since $R_{nj}(u)$ is also uniformly integrable this implies that for each u and j $E|R_{nj}(u)| \to 0$.

We also assumed, (5.4) and (5.5), that $\sup_n E|R_{nj}(u)|$ is integrable with respect to u on [0,t]. Hence by dominated convergence we find that $\int_0^t E|R_{nj}(u)|\, du \to 0$ which again shows that $\int_0^t R_{nj}(u)\, du \overset{P}{\to} 0$.

This establishes the regularity conditions of Rebolledo and we find that any finite set of the Y_{nj} converges to the set of Y_j, which have the stated variance.

Finally (5.6) and (5.7) imply that

$$V\{\Sigma_j Y_{nj}\} \to V\{\Sigma_j Y_j\}, \qquad n \to \infty.$$

The main result about the estimate $\hat{\mu}$ can now be formulated in

<u>Theorem 5.2</u> If $\Sigma_j \int_0^t j^4 p_j(u)\, \lambda(u)\, du < \infty$, $t < 1$, then

$$\sqrt{n}\,(\hat{\mu} - \mu) \overset{w}{\Rightarrow} \sum_j \int (j-1)\, C_j\, dU_j = W$$

where $C_j = \begin{pmatrix} 1 & j-1 \\ 0 & 2 \end{pmatrix}$ and $\{U_j\}$ is a sequence of independent Gaussian processes with $\langle U_j \rangle = \int (p_j(u)\, \lambda(u)) / m(0,u)\, du$.

Further

$$\sup_{u \leq t} |(\hat{\mu} - \mu)(0,u)| \overset{P}{\to} 0, \qquad n \to \infty$$

and

$$V(\sqrt{n}\,(\hat{\mu} - \mu)) \to \Sigma_j \int (j-1)^2\, C_j^{2\otimes}\, \frac{p_j(u)\, \lambda(u)}{m(0,u)}\, du = V(W), \qquad n \to \infty.$$

Proof We have the expressions

$$\hat{\mu}(0,t) = \sum_j \int_0^t C_j(u^-) \frac{j-1}{X(u^-)} dN_j$$

$$\tilde{\mu}(0,t) = \sum_j \int_0^t C_j(u^-)\,(j-1)\, p_j(u)\, \lambda(u)\, du$$

and

$$\mu(0,t) = \sum_j \int_0^t C_j\,(j-1)\, p_j(u)\, \lambda(u)\, du\,.$$

Let us first evaluate

$$\sqrt{n}\,(\tilde{\mu} - \mu)(0,t) = \sum_j \int_0^t \sqrt{n}\,(C_j(u^-) - C_j)(j-1)\, p_j(u)\, \lambda(u)\, du\,.$$

Now $\sqrt{n}\, |C_j(u) - C_j| \leqq \sqrt{n}\, \frac{|j-1|}{X(u^-)}\, 1\{X(u^-) > 0\} + \sqrt{n}\,(j+2)\, 1\{X(u^-) = 0\}$

and hence

$$E\, \sqrt{n} \sup_{s \leqq t} |\tilde{\mu} - \mu|(0,s)$$

$$\leqq \int_0^t \left(\frac{1}{\sqrt{n}}\, E\, \frac{n\, 1\{X(u^-) > 0\}}{X(u^-)} + \sqrt{n}\, P\{X(u^-) = 0\}\right) \sum_j (j+2)^2\, p_j(u)\, \lambda(u)\, du.$$

We show in the appendix that $E\, \frac{n\, 1\{X(u^-) > 0\}}{X(u^-)}$ is uniformly bounded and hence the first part of the integral goes to zero. The second part is evaluated as follows

$$\sqrt{n}\, P\{X(u^-) = 0\} \leqq (1 - G(0,u))^n \sqrt{n} \leqq (1 - G(0,t))^n \sqrt{n}$$

which shows that the second part of the integral goes to zero.

Thus we can throughout replace μ by $\tilde{\mu}$ in the results to be proven.

We shall now discuss the process $\sqrt{n}\,(\hat{\mu} - \tilde{\mu})$. We define $\Lambda_{nj}(u) = p_j(u)\, X(u^-)\, \lambda(u)$, $H_{nj}(u) = C_j(u^-) \frac{j-1}{X(u^-)} \sqrt{n}$, $Y_{nj}(t) = \int_0^t \sqrt{n}\, C_j(u^-) \frac{j-1}{X(u^-)}\, d(N_j - \tilde{N}_j)$, $g_j(u) = C_j\,(j-1)\sqrt{p_j(u)\,\lambda(u)\,/\,m(0,u)}$, then

$$\sqrt{n}\,(\hat{\mu} - \tilde{\mu}) = \Sigma_j\, Y_{nj}\,.$$

To prove convergence we must now check the conditions from Lemma 5.1.

From $X(u)\,/\,n \xrightarrow{P} m(0,u)$ we find immediately that $H_{nj}(u) \xrightarrow{P} 0$ and $H_{nj}(u)^{2\otimes}\, \Lambda_{nj}(u) \xrightarrow{P} g_j(u)^{2\otimes}$.

Next evaluate $H_{nj}^{2\otimes}\, \Lambda_{nj}$ as follows

$$|H_{nj}(u)|^2\, \Lambda_{nj}(u) \leqq |C_j(u^-)|^2\, (j-1)^2\, \frac{n\, 1\{X(u^-) > 0\}}{X(u^-)}\, p_j(u)\, \lambda(u)$$

$$\leqq (j+2)^4\, \frac{n\, 1\{X(u^-) > 0\}}{X(u^-)}\, p_j(u)\, \lambda(u)$$

which shows that $H_{nj}^{2\otimes}\, \Lambda_{nj}$ is uniformly integrable and that

$$\sum_j \int_0^t \sup_n E\, |H_{nj}(u)|^2\, \Lambda_{nj}(u)\, dt$$

$$\leqq \sum_j \int_0^t (j+2)^4\, p_j(u)\, \lambda(u)\, du \sup_{n,u \leqq t} E\, \frac{n\, 1\{X(u^-) > 0\}}{X(u^-)}$$

which is finite since $\frac{n\, 1\{X(u^-) > 0\}}{X(u^-)}$ is integrable uniformly in n and $u \leqq t$.

Finally

$$\sum_j \int_0^t |g_j(u)|^2\, du \leqq \sum_j \int_0^t (j+2)^4\, \frac{p_j(u)\, \lambda(u)}{m(0,u)}\, du$$

$$\leq \sum_j \int_0^t (j+2)^4 \, p_j(u) \, \lambda(u) \, du \, / \, m(0,t) < \infty .$$

Thus Lemma 5.1 applies and we find the results of Theorem 5.2.

We can now state our main results about the moment estimator.

Theorem 5.3 If $\Sigma_j \int_0^t j^8 \, p_j(u) \, \lambda(u) \, du < \infty$, $t < 1$ then

$$\sqrt{n} \, (\hat{M}(0,t) - M(0,t)) \overset{w}{\Rightarrow} \int_0^t M(0,u) \, dW(u) \, M(u,t)$$

$$= \sum_j (j-1) \int_0^t M(0,u) \, C_j \, M(u,t) \, dU_j(u)$$

and further

$$\sup_{u \leq t} |\hat{M}(0,t) - M(0,t)| \overset{P}{\to} 0 \qquad n \to \infty$$

and

$$V\{\sqrt{n} \, (\hat{M} - M)\} \to \sum_j \int (M(0,u) \, C_j \, M(u,t))^{2\otimes} (j-1)^2 \frac{p_j(u) \, \lambda(u)}{m(0,u)} du, \; n \to \infty.$$

Proof Consider first

$$\sqrt{n} \, (\tilde{M} M^{-1} - I)(0,t) = \int_0^t \tilde{M}(0,s) \, d\sqrt{n} \, (\tilde{\mu} - \mu) \, M^{-1}(0,s) .$$

Now $|\tilde{M}(0,s)|$ as well as $|M^{-1}(0,s)|$ are uniformly bounded in $(0,t)$ and hence

$$\sup_{u \leq t} |\sqrt{n} \, (\tilde{M} M^{-1} - I)(0,u)| \overset{P}{\to} 0$$

and we shall hence only consider the process

$$\sqrt{n}\,(\hat{M}\tilde{M}^{-1} - I)(0,t) = \int_0^t \hat{M}(0,s-)\; d(\hat{\mu} - \tilde{\mu})\; \sqrt{n}\,\tilde{M}^{-1}(0,s)\,.$$

From Theorem 4.6 we have that it is a local square integrable martingale and that its variance is given by

$$nE\langle\hat{M}\tilde{M}^{-1}-I\rangle = E\sum_j \int \left(\hat{M}(0,s-)C_j(s-)\tilde{M}^{-1}(0,s)\right)^{2\otimes} \frac{(j-1)^2 n 1\{X(s-)>0\}}{X(s-)} p_j(s)\lambda(s)ds$$

which by Lemma 4.5 is bounded uniformly in n. Hence $V\{\hat{M}\tilde{M}^{-1}\} \to 0$, $n \to \infty$ which shows that $\sup_{u\leq t} |\hat{M}\tilde{M}^{-1}(0,u) - I| \xrightarrow{P} 0$ which again shows that $\hat{M}$ is a consistent estimator of $\tilde{M}$ and hence of M.

M.

To prove the asymptotic normality of $\sqrt{n}\,(\hat{M}\tilde{M}^{-1} - I)$ we define

$$H_{nj}(u) = \hat{M}(0,u-)C_j(u-)\tilde{M}^{-1}(0,u)\,\frac{j-1}{X(u-)}\,\sqrt{n}$$

$$\Lambda_{nj}(u) = X(u-)\,p_j(u)\,\lambda(u)$$

$$g_j(u) = M(0,u)\,C_j\,M^{-1}(0,u)\,\sqrt{\frac{p_j(u)\,\lambda(u)}{m(0,u)}}\,(j-1)$$

From the consistency of $\hat{M}$ and $\tilde{M}$ we find that $H_{nj}(u) \xrightarrow{P} 0$, $H_{nj}(u)^{2\otimes}\,\Lambda_{nj}(u) \xrightarrow{P} g_j(u)^{2\otimes}$.

Next evaluate

$$|H_{nj}(u)^{2\otimes}\Lambda_{nj}(u)| \leq |\hat{M}(0,u-)|^2 |C_j(u-)|^2 |\tilde{M}^{-1}(0,u)|^2 \frac{n 1\{X(u-)>0\}}{X(u-)} p_j(u)\lambda(u)$$

this variable is by Lemma 4.5 uniformly integrable and satisfies

$$\sum_j \int_0^t \sup_n E\,|H_{nj}(u)|^2\,\Lambda_{nj}(u)\,du < \infty$$

which shows that Lemma 5.1 can be applied and this then completes the proof.

<u>Theorem 5.4</u> If $\Sigma_j \int_0^t j^{13} p_j(u)\,\lambda(u)\,du < \infty$, $t<1$ then a consistent estimate of the asymptotic variance of $\sqrt{n}\,(\hat{M}-M)$ is given by the Stiltjes integral

$$\hat{V}(0,t) = \sum_j \int_0^t \left(\hat{M}(0,s^-)\, C_j\, \hat{M}(s^-,t)\right)^{2\otimes} \frac{n\,(j-1)^2}{X(s^-)^2}\, dN_j$$

Infact

$$\sup_{s\leq t} |\hat{V}(0,s) - V(0,s)| \overset{P}{\rightarrow} 0 .$$

<u>Proof</u> The variance of $\sqrt{n}\,(\hat{M}\tilde{M}^{-1} - I)$ is given by

$$E\,\sqrt{n} <\hat{M}\tilde{M}^{-1} - I>(t) = \sum_j E \int_0^t \left(\hat{M}(0,s^-) C_j (s^-) \tilde{M}^{-1}(0,s)\right)^{2\otimes} \frac{n\,(j-1)^2}{X(s^-)^2}\, d\tilde{N}_j$$

We replace $\tilde{M}^{-1}(0,s)$ by $\hat{M}^{-1}(0,s^-)$ and $\tilde{N}_j$ by N_j and derive the stochastic process

$$Z_n(t) = \sum_j \int_0^t \left(\hat{M}(0,s^-)\, C_j\, \hat{M}^{-1}(0,s^-)\right)^{2\otimes} \frac{n\,(j-1)^2}{X(s^-)^2}\, dN_j$$

Let $\tilde{Z}_n(t)$ by the same expression with N_j replaced by $\tilde{N}_j$.

The process $Z_n - \tilde{Z}_n$ is a square integrable local martingale if we can prove that the integral

$$I_n = \sum_j \int_0^t E\, |\hat{M}(0,s^-)\, C_j\, \hat{M}^{-1}(0,s^-)|^4 \frac{n^2\,(j-1)^4}{X(s^-)^3}\, 1\{X(s^-)>0\}\, p_j(u)\,\lambda(u)\, du$$

is finite.

The proof is analogous to that of Lemma 4.5 as given in the appendix. The same application of Hölder's inequality will show that $n\,I_n$ is bounded, and hence that $I_n \to 0$, $n \to \infty$.

Thus $Z_n - \tilde{Z}_n$ is a local square integrable martingale, with a variance which tends to zero as $n \to \infty$. Hence $\sup_{s \leq t} |Z_n(s) - \tilde{Z}_n(s)| \overset{P}{\to} 0$, $n \to \infty$.

We shall now find the limit of $\tilde{Z}_n$.

From

$$\hat{M}\, C_j\, \hat{M}^{-1} = \begin{pmatrix} 1 & (j-1)\hat{m}^{-1} + \hat{v}\hat{m}^{-2} \\ 0 & 2 \end{pmatrix}$$

it follows that

$$|\hat{M}\, C_j\, \hat{M}^{-1} - M\, C_j\, M^{-1}| \,/\, (j-1) \overset{P}{\to} 0$$

uniformly in $[0,t]$. Since also $\left|\frac{n\,1\{X(s-)>0\}}{X(s-)} - \frac{1}{m(0,s)}\right| \overset{P}{\to} 0$ uniformly in $[0,t]$ it follows that

$$\tilde{Z}_n(s) \xrightarrow{P} \sum_j \int_0^s \left(M(0,u)\, C_j\, M^{-1}(0,u)\right)^{2\otimes} \frac{(j-1)^2\, p_j(u)\, \lambda(u)\, du}{m(0,u)}$$

uniformly in $[0,t]$.

Combining these results we find that

$$\hat{V}(0,t) = Z_n(t)\, \hat{M}(0,t)^{2\otimes} = \sum_j \int_0^t \left(\hat{M}(0,s-)\, C_j\, \hat{M}(s-,t)\right)^{2\otimes} \frac{n\,(j-1)^2}{X_n(s-)^2}\, dN_j$$

converges in probability and uniformly in $[0,t]$ to the expression for the asymptotic variance of $\sqrt{n}\,(\hat{M} - M)$.

6. Appendix

We shall first prove some inequalities for Markov branching processes and then apply them to the proof of Lemma 4.5 and Theorem 5.4.

Lemma A.1 Let $Z(t) = \Sigma_j a_j N_j(t)$, $a_j \geqq 0$, $j = 0,1,\ldots$ and

$$\phi_p(A) = \int_A 2^{p-1} \sum_j (a_j^p + j^p)\, p_j(u) \frac{G(du)}{G[u,1]} ,$$

then

$$E\{Z(t)^p \mid X(0) = 1\} \leqq \exp(\phi_p]0,t]) - 1 .$$

Proof We define

$$m_p^{(n)}(s,t) = E\{(Z(t) - Z(s))^p 1\{0 \leqq N(t) - N(s) \leqq n\} \mid X(s) = 1\} .$$

Now $Z(t) - Z(s) = \Sigma_j a_j (N_j(t) - N_j(s))$ and hence $N(t) - N(s) = 0 \Rightarrow$ $Z(t) - Z(s) = 0$ which shows that

$$m_p^{(n)}(s,t) = E\{(Z(t) - Z(s))^p 1\{1 \leqq N(t) - N(s) \leqq n\} \mid X(s) = 1\} .$$

Since there is at least one death in $]s,t]$ if $N(t) - N(s) \geqq 1$ we decompose after the first death time U and the birth size Y as follows

$$m_p^{(n)}(s,t) =$$

$$\sum_k \int_{]s,t]} E\left[(Z(t)-Z(s))^p 1\{1 \leqq N(t)-N(s) \leqq n\} \mid X(s)=1, U=u, Y=k\right] p_k(u) \frac{G(du)}{G[u,1]} .$$

Under the condition in the expectation we have $\Delta N(u) = 1$ and

$$Z(t) - Z(s) = Z(u) - Z(s) + (Z(t) - Z(u)) = a_k + \Sigma_1^k (Z_i(t) - Z_i(u))$$

where $(Z_i(t) - Z_i(u))$, $i = 1,\dots,k$ are i.i.d. and each gives the increment of a Z'process starting with 1 individual at time u. Let $N^{(i)}$ denote the number of deaths in this process, then

$$1\{1 \leqq N(t) - N(s) \leqq n\} \leqq 1\{0 \leqq N^{(i)}(t) - N^{(i)}(u) \leqq n-1\}.$$

We can then evaluate the integrand as follows:

$$E\left[(Z(t)-Z(s))^p 1\{1\leqq N(t)-N(s)\leqq n\} \mid X(s)=1, U=u, Y=k\right]$$

$$\leqq E\left\{\left[\sum_{i=1}^{k}\left(\frac{a_k}{k}+Z_i(t)-Z_i(u)\right)1\{0\leqq N^{(i)}(t)-N^{(i)}(u)\leqq n-1\}\right]^p \middle| X(s)=1, U=u, Y=k\right\}$$

$$\leqq k^p E\left\{\left(\frac{a_k}{k} + Z_1(t)-Z_1(u)\right)^p 1\{0\leqq N^{(i)}(t)-N^{(i)}(u)\leqq n-1\} \mid X^{(1)}(u)=1\right\}$$

$$\leqq 2^{p-1}\left(a_k^p + k^p m_p^{(n-1)}(u,t)\right).$$

This then gives the recursion

$$m_p^{(n)}(s,t) \leqq \phi_p(s,t) + \int_{]s,t]} m_p^{(n-1)}(u,t)\,\phi_p(du).$$

From $m_p^{(0)}(s,t) \leqq 1$ we find by induction that

$$m_p^{(n)}(s,t) \leqq \sum_{k=1}^{n} \int_{s<u_1<\dots<u_k\leqq t} \phi_p(du_1)\dots\phi_p(du_k) \leqq \sum_{k=1}^{n} (\phi_p]s,t])^k/k!$$

$$\leqq \exp(\phi_p]s,t]) - 1$$

Now we let $n \to \infty$ and obtain the result of Lemma A.1.

<u>Lemma A.2</u> Let B be binomially distributed with parameters (n,p) then

$$E\left(\frac{n}{B}\right)^k 1\{B > 0\} \leqq \frac{n^k (k+1)!}{(n+k)^{(k)} p^k}$$

and

$$E\left(\frac{n}{1+B}\right)^k \leqq \frac{n^k k!}{(n+k)^{(k)} p^k} .$$

Proof From the inequality

$$\frac{1}{Z^k} \leqq \frac{(k+1)!}{(Z+1)\dots(Z+k)}, \qquad Z \geqq 1$$

it follows that

$$E\left(\frac{n}{B}\right)^k 1\{B > 0\} \leqq \frac{(k+1)!\, n^k}{(n+k)^{(k)} p^k} \sum_{y=k+1}^{n+k} \binom{n+k}{y} p^y q^{n-y}$$

which proves the first result. The other one is proved similarly.

Corollary A.3 The variables

$$\left(\frac{n}{X(s)}\right)^p 1\{X(s) > 0\}$$

are uniformly integrable for all p.

Proof It is clearly enough to prove that

$$E\left\{\left(\frac{n}{X(s)}\right)^p 1\{X(s) > 0\} \mid X(0) = n\right\}$$

is bounded uniformly in n, and $s \leqq t$ and any $p \geqq 1$.

Let $B(t) = \Sigma_1^n 1\{X_\nu(t) > 0\}$, where $X_1,\dots,X_n$ are independent identically distributed Markov branching processes each starting with 1 individual at time 0. Note that B(t) is binomially distributed (n,p(t)), with

$$p(t) = P\{X_1(t) > 0\} > G(0,t) .$$

We now evaluate as follows:

$$\left(\frac{n}{X(s)}\right)^p 1\{X(s)>0\} \leqq \left(\frac{n}{B(s)}\right)^p 1\{B(s)>0\} + n^p\, 1\{B(s)=0\}$$

since clearly $X(s) \geqq B(s)$. The first term has a bounded mean by Lemma A.2 and the second since $P[\{B(s)=0\} \mid X(0)=n] \leqq (1-G(0,s))^n \leqq (1-G(0,t))^n$ decreases exponentially fast. Note that the evaluation is independent of $s \in [0,t]$, since $p(s)$ is bounded below by $G(0,t)>0$.

<u>Lemma A.4</u> Let $Z = \Sigma\, a_j N_j$, $a_j \geqq 0$, then for any p and $q \geqq 1$ we have that if $\phi_p(0,t) < \infty$, $t<1$ then

$$E\left\{\left(\frac{n}{B(s)}\, 1\{B(s)>0\}\right)^q \left(\frac{Z(s)}{n}\right)^p \;\middle|\; X(0)=n\right\}$$

and

$$n^q\, E\left\{\left(\frac{Z(s)}{n}\right)^p 1\{B(s)=0\} \;\middle|\; X(0)=n\right\}$$

are bounded uniformly in n and $s \leqq t$.

<u>Proof</u> If $X(0)=n$, then we write $Z(t) = \Sigma_1^n\, Z_i(t)$, where $Z_1,\ldots,Z_n$ are independent identically distributed processes starting with 1 individual at time 0. Then $\left(\frac{1}{n}\Sigma_1^n\, Z_i(t)\right)^p \leqq \frac{1}{n}\Sigma_1^n\, Z_i^p(t)$, and hence

$$I(s) = E\left\{1\{B(s)>0\}\left(\frac{n}{B(s)}\right)^q \left(\frac{Z(s)}{n}\right)^p \;\middle|\; X(0)=n\right\}$$

$$\leqq E\left\{1\{B(s)>0\}\left(\frac{n}{B(s)}\right)^q Z_1(s)^p \;\middle|\; X(0)=n\right\}$$

$$= E[1\{B'(s)>0\}\left(\frac{n}{B'(s)}\right)^q \mid X(0)=n]\; E(Z_1(s)^p \mid X(0)=n)$$

$$+ E[1\{B'(s)+1>0\}\left(\frac{n}{B'(s)+1}\right)^q \mid X(0)=n]\; E(Z_1(s)^p \mid X(0)=n)$$

where we have decomposed the integral according as $X_1(s) = 0$ or $X_1(s) > 0$. We have used the notation $B'(s) = \sum_{i=2}^{n} 1\{X_i(s) > 0\}$. It now follows from Lemma A.1 and A.2 that $I(s)$ is bounded uniformly in $s \leqq t$ and n.

Next consider

$$n^q \, E\left[\left(\frac{Z(s)}{n}\right)^p 1\{B(s) = 0\} \;\middle|\; X(0) = n\right]$$

$$\leqq n^q \, E[Z_1(s)^p \, 1\{B'(s) = 0\} \mid X(0) = n]$$

$$= n^q \, E\{Z_1(s)^p \mid X_1(0) = 1\} \; P\{B'(s) = 0 \mid X(0) = n\}$$

$$= n^q \, E\{Z_1(s)^p \mid X_1(0) = 1\} \; (1 - G(0,s))^{n-1}$$

$$\leqq n^q \, (1 - G(0,t))^{n-1} \, (\exp(\phi_p]0,t]) - 1)$$

which is bounded uniformly in n and $s \leqq t$.

We can now prove Lemma 4.5.

We have

$$Z_{nj}(s) = n|\hat{M}(0,s^-)|^2 |\tilde{M}^{-1}(0,s)|^2 |C_j(s^-)|^2 \left(\frac{j-1}{X(s^-)}\right)^2 X(s^-) p_j(s) \lambda(s).$$

We first evaluate $|C_j(s^-)|^2 \leqq (j+2)^2 1\{X(s^-) > 0\}$. Next we get for any matrix $M = \Pi(I + d\mu)$ that

$$|M(0,s)| \leqq \max(m(0,s) + v(0,s), m^2(0,s)) \leqq 3m_2(0,s) + 2 .$$

and hence

$$|\tilde{M}^{-1}(0,s)| \leqq \frac{\max(\tilde{m}^2(0,s) + \tilde{v}(0,s), \tilde{m}(0,s))}{\tilde{m}^3(0,s)} \leqq \frac{3\tilde{m}_2(0,s)+2}{\tilde{m}^3(0,s)} .$$

Now $\tilde{m}(0,s) = m(0,s)$ on the interval where $X(s-) > 0$ and hence bounded below since

$$m(0,s) \geqq G(0,s) \geqq G(0,t) > 0 .$$

Similarly one proves that $\tilde{\mu}_2 \leqq \mu_2$ and $\tilde{m}_2 \leqq m_2$, and by Lemma A.1, $m_2(0,s) = E\{X(s)(X(s)-1) \mid X(0)=1\}$ is bounded uniformly on $[0,t]$. Thus the coefficient $|\tilde{M}^{-1}(0,s)|^2$ is bounded by some constant $A(t)$, say.

Next consider

$$|\hat{M}(0,s-)| \leqq 3 \frac{X(s-)^2}{n} \int_{[0,s[} \frac{\Sigma_j\, j(j-1)\, dN_j}{X(u-)^2} + 2 .$$

Let $Y = \Sigma_j\, j\,(j-1)\, N_j$ and $B(u) = \Sigma_1^n\, 1\{X_\nu(u) > 0\}$ then

$$|\hat{M}(0,s-)| \leqq 2 + 3 \frac{X(s-)^2}{n} \frac{Y(s-)}{B(s-)^2}\, 1\{B(s-)>0\} + 3 \frac{X(s-)^2}{n} Y(s-)\, 1\{B(s-)=0\}$$

and

$$|Z_{nj}(s)| \leqq A(t) \left(2 + 3 \frac{X(s-)^2 Y(s-)}{n\, B(s-)^2}\, 1\{B(s-)>0\}\right.$$

$$\left. + 3 \frac{X(s-)^2}{n} Y(s-)\, 1\{B(s-)=0\}\right)^2 n \frac{1\{X(s-)>0\}}{X(s-)} (j+2)^4\, p_j(s)\, \lambda(s) .$$

Thus $Z_{nj}(s)$ is uniformly integrable if each of the following variables are

$$\frac{n 1\{X(s-)>0\}}{X(s-)} , \quad \frac{Y(s-)^2 X(s-)^3}{n\, B(s-)^4}\, 1\{B(s-)>0\} , \quad \frac{Y(s-)^2 X(s-)^3}{n}\, 1\{B(s-)=0\} .$$

This is just the corollary A.3 for the first variable.

From Hölder's inequality we find

$$E\left(\frac{Y(s-)}{n}^2 \frac{X(s-)^3}{B(s-)^4} 1\{B(s-)>0\}\right)^{8/7} \leq E^{4/7}\left(\frac{Y(s-)}{n}\right)^4 E^{3/7}\left(\frac{X(s-)^8 n^{8/3}}{B(s-)^{32/3}}\right).$$

By Lemma A.4 these integrals are uniformly bounded under the condition $\int_0^t \Sigma j^8 p_j(u)\lambda(u)\,du < \infty$, $t<1$, hence the second variable is uniformly integrable.

We have left out the conditioning event $\{X(0)=n\}$ in the notation.

Similarly we get

$$E\left(\frac{Y(s-)^2 X(s-)^3}{n} 1\{B(s-)=0\}\right)^{\frac{8}{7}} \leq n^{-\frac{8}{7}} E^{\frac{4}{7}}\, Y(s-)^4\, E^{\frac{3}{7}}\, X(s-)^8\, 1\{B(s-)=0\}$$

which by Lemma A.4 is uniformly bounded in n and $s \leq t$.

Thus we have proved uniform integrability using the fact that if $E|V_n|^{1+\varepsilon}$ is uniformly bounded then V_n is uniformly integrable.

This proves the uniform integrability of $|Z_{nj}(s)|$ and we find that

$$\sup_n E|Z_{nj}(s)| \leq c_2(t)\,(j+2)^4\, p_j(s)\,\lambda(s)$$

which shows that

$$\int_0^t \sum_j \sup_n E|Z_{nj}(s)|\,ds < \infty .$$

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