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Abstract

Let $S_n = X_1 + \ldots + X_n$ be a random walk with negative drift $\mu < 0$, let $F(x) = P(X_k \le x)$, $\nu(u) = \inf\{n : S_n > u\}$ and assume that for some $\gamma > 0$ $d\bar{F}(x) = e^{\gamma x} dF(x)$ is a proper distribution with finite mean $\bar{\mu}$. Various limit theorems for functionals of $X_1, \ldots, X_{\nu(u)}$ are derived subject to conditioning upon $\{\nu(u) < \infty\}$ with u large, showing similar behaviour as if the X_i were i.i.d. with distribution $\bar{F}$. For example, the deviation of the empirical distribution function from $\bar{F}$, properly normalised, is shown to have a limit in D, and an approximation for $(u^{-\frac{1}{2}}[S_{[\nu(u)t]} - tu])_{0 \le t \le 1}$ by means of Brownian bridge is derived. Similar results holds for risk reserve processes in the time up to ruin and the GI/G/1 queue considered either within a busy cycle or in the steady state. The methods produce an alternate approach to known asymptotic formulae for ruin probabilities as well as related waiting time approximations for the GI/G/1 queue. E.g. $\lim_{u\to\infty} e^{\gamma u} P(W_N > u) = C\Phi((N - u/\bar{\mu})/(\bar{\sigma}^2 u/\bar{\mu}^3)^{\frac{1}{2}})$ uniformly in N, with W_N the waiting time of the N^{th} customer.

Keywords and Phrases. Random walk, risk reserve process, GI/G/1 queue, conditioned limit theorem, first passage time, associated random walk, empirical distribution function, Brownian bridge, ruin probability, waiting time.

1. Introduction

Let $X_1, X_2, \ldots$ be i.i.d. random variables with common distribution function (d.f.) $F(x) = P(X_n \le x)$ and define $S_0 = 0$, $S_n = X_1 + \ldots + X_n$, $\mu = EX_n = \int_{-\infty}^{\infty} x dF(x)$. It is throughout assumed that $-\infty < \mu < 0$ so that, as is well-known, $M = \sup\{S_n : n \ge 0\} < \infty$ a.s. and the first passage time $\nu(u) = \inf\{n : S_n > u\}$ is defective for $u > 0$, $P(\nu(u) < \infty) = P(M > u) < 1$.

Two main examples where these contexts come up are in the theory of collective risk, Seal (1969) Ch.4, Sparre Andersen (1957), and queueing theory, Feller (1971), Cohen (1969). In both cases we have the additional structure $X_n = X_n^{(1)} - cX_n^{(2)}$, where $\{X_n^{(1)}\}$, $\{X_n^{(2)}\}$ are independent sequences of i.i.d. random variables (obvious notations like $F^{(i)}$ or $S_n^{(i)} = X_1^{(i)} + \ldots + X_n^{(i)}$ are used for quantities defined relative to $\{X_n^{(i)}\}$ rather than $\{X_n\}$). In risk theory, the $X_n^{(1)}$ are the amounts of claims, the $X_n^{(2)}$ the inter-occurence times between claims and $c > 0$ the gross risk premium intensity. With the first claim arriving at time $X_1^{(2)}$, the risk reserve at time t is

$$R(t) = R(0) + ct - \sum_{n=1}^{\nu^{(2)}(t)-1} X_n^{(1)}$$

so that $S_n = R(0) - R(S_n^{(2)})$. One of the main objectives of study of risk theory are the probabilities of ruin,

$$\psi(t,u) = P(\inf_{0 \le s \le t} R(s) < 0 \mid R(0) = u) ,$$

$$\psi(u) = \lim_{t\to\infty} \psi(t,u) = P(\inf_{0 \le s \le \infty} R(s) < 0 \mid R(0) = u)$$

and in view of $c > 0$ ruin can only occur at the times $S_n^{(2)}$ of claims so that $\psi(u) = P(M > u)$. In the GI/G/1 queue we have $c = 1$, the $X_n^{(1)}$ are the service times and the $X_n^{(2)}$ the interarrival times. If we number the customers $0,1,2,\ldots,$ then $\underline{n} = \inf\{n \ge 1 : S_n \le 0\}$ is the number of customers served during the first busy period, for

$n < \underline{n}$ $W_n = S_n$ is the (actual) waiting time of the n^{th} customer and $\underline{c} = S^{(2)}_{\underline{n}}$ the duration of the first busy cycle. The assumption $\mu < 0$ amounts to the traffic intensity $\rho = \mu^{(1)}/\mu^{(2)}$ being less than one so that $\underline{n} < \infty$, $\underline{c} < \infty$ a.s. and the limiting steady state (denoted by index ∞) exists. It is well-known that $P(W_\infty > u) = P(M > u)$ and, more generally, that $P(W_n > u) = P(\max\{S_0,\ldots,S_n\} > u) = P(\nu(u) \leq n)$.

In view of these facts and the difficulties in obtaining explicit tractable expressions, much attention has been given to the approximation

$$(1.1) \qquad P(M > u) \tilde{=} Ce^{-\gamma u} \quad \text{as} \quad u \to \infty .$$

For a simple proof under general assumptions, see Feller (1971) Ch.XII. The precise conditions for (1.1), which will be assumed for the rest of the paper, are the following:

Condition 1.1 F is non-lattice.

Condition 1.2 $\gamma > 0$ satisfies $Ee^{\gamma X_1} = 1$, $E|X_1|e^{\gamma X_1} < \infty$.

[various expressions for C are cited in Section 2]. Condition 1.2 describes the associated random walk, cf. Feller (1971) pg. 406. In fact, it follows that $\bar{F}(x) = \int_{-\infty}^x e^{\gamma y} dF(y)$ is a proper d.f. with mean $0 < \bar{\mu} < \infty$ and we let $\bar{P}$ denote a probability measure making $X_1, X_2, \ldots$ i.i.d. with common d.f. $\bar{F}$. If as above $X_n = X_n^{(1)} - cX_n^{(2)}$, then it is easy to see that $\bar{P}$ can be chosen so as to make also the sequences $\{X_n^{(1)}\}$, $\{X_n^{(2)}\}$ i.i.d. with d.f.

$$\bar{F}^{(1)}(x) = \int_{-\infty}^{x} e^{\gamma y} dF^{(1)}(y)/\hat{F}^{(1)}(\gamma) \text{ , respectively}$$

$$\bar{F}^{(2)}(x) = \int_{-\infty}^{x} e^{-c\gamma y} dF^{(2)}(y)/\hat{F}^{(2)}(-c\gamma) \text{ ,}$$

$\hat{F}^{(i)}$ being the moment generating function of $F^{(i)}$ so that $\hat{F}^{(1)}(\gamma)\hat{F}^{(2)}(-c\gamma) = 1$.

The present paper grew out of the wish to exploit somewhat further the role of the associated random walk by means of a number of conditioned limit theorems. In the random walk and risk theory situations, we are concerned with $P_u = P(\cdot|\nu(u) < \infty)$ as $u \to \infty$, and our objective is loosely speaking to show that the $X_1, \ldots, X_{\nu(u)}$ behave in the same way w.r.t. P_u and $\bar{P}$.

The simplest such result is the convergence of the P_u-distribution of the whole sequence $(X_1, X_2, \ldots)$ to its $\bar{P}$-distribution. This follows easily from weak convergence in sequence space just meaning convergence of any finite number of coordinates and

$$P_u(X_1 \le x_1, \ldots, X_n \le x_n) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} P(M > u - y_1 - \ldots - y_n) dF(y_1) \ldots dF(y_n) / P(M > u) \cong$$

$$\int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} e^{\gamma(y_1 + \ldots + y_n)} dF(y_1) \ldots dF(y_n) = \bar{F}(x_1) \ldots \bar{F}(x_n) ,$$

using (1.1) and dominated convergence. A similar, though somewhat more complicated, behaviour is exhibited for $(\ldots, X_{\nu(u)-1}, X_{\nu(u)})$ in Section 8. However, from the main point of view of this paper the role of such individual limit results is largely as motivation and complement. In fact, we are concerned with functionals of $(X_1, \ldots, X_{\nu(u)})$ for which the contribution from any finite number of the X_n vanishes in the limit.

For example, if

$$F_n(x) = \frac{1}{n} \sum_{k=1}^{n} I(X_k \le x)$$

is the empirical d.f. and $\|\cdot\|$ denotes the supremum norm, we show that $P_u(\|F_{\nu(u)} - \bar{F}\| > \varepsilon) \to 0$ for any $\varepsilon > 0$ as well as we obtain the corresponding expected convergence of the empirical process to a certain Gaussian process. Another main result is an approximation of $(u^{-\frac{1}{2}}[S_{[t\nu(u)]} - tu])_{0 \le t \le 1}$ by Brownian bridge. These results are stated and shown in Section 3, while some preliminaries (most of which are well known) are collected in Section 2. Section 4 then deals with risk reserve processes. The results and proofs mainly parallel those for random walks. However, as a byproduct we obtain a reasonably simple derivation of some approximations for the probabilities $\psi(t,u)$ of ruin, proved in their greatest generality by von Bahr (1974) (however, Seal (1969) Ch.4 should be consulted for more complete references to earlier work; see in particular pg.104 and pg.111-116).

For the GI/G/1 queue treated in Sections 5-7, the conditioning upon $\{\nu(u) < \infty\}$ lacks intuitive appeal. Instead, conditioning the random walk upon hitting (u,∞) before $(-\infty,0]$ yields the behaviour within the busy cycle, as studied in Section 5, and which by means of the regenerative structure of the process then in Sections 6-7 is converted into other types of limit statements. In Section 6, we obtain approximations for quantities like $P(W_N > u)$ and $P(V(T) > u)$, with $V(T)$ the virtual waiting time at time T. E.g. with Φ the standard normal d.f.

$$\lim_{u\to\infty} \sup_{0\leq N\leq\infty} \left| e^{\gamma u} P(W_N > u) - C\Phi\left(\frac{N-u/\bar{\mu}}{(\bar{\sigma}^2 u/\bar{\mu}^3)^{\frac{1}{2}}}\right)\right| = 0 \ ,$$

a relation similar in form to the ruin probability formulae referred to above. In Section 7 then a conditioned limit theorem for the steady state is given, describing the past prior to a large value and complementing individual limit results of a similar type treated in Asmussen (1980). Section 8 gives the results for $(\ldots,X_{\nu(u)-1},X_{\nu(u)})$ referred to above. Finally in Section 9 we take the opportunity to make some remarks on the relation between the queueing and actuarial literature. Conditioned limit theorems for random walks in more or less related settings have received considerable attention, see e.g. the survey paper by Iglehart (1975) and for later contributions Kaigh (1976), Kao (1978) and Durret (1980). Iglehart propagates the point of view that such problems "open up the door to ... an enormous array of interesting problems", but probably the practical worker in the field of insurance or operations research would regard most results of the present paper as curiosities. Nevertheless, some of the approximations referred to above are not entirely academic and might motivate the study of the mathematics of the area. For other applications of weak convergence to risk theory, see Iglehart (1969) and Grandell (1977, 1978). Related fields of applications, which will not be pursued here, are transient renewal processes and dam and storage models.

2. Preliminaries for the random walk setting.

We first introduce some notation. Let $\mathcal{F}_n = \sigma(X_1,\dots,X_n)$ and $\mathcal{G}_u = \mathcal{F}_{\nu(u)}$, with the usual definition of a stopping time σ-algebra. Also define $B(u) = S_{\nu(u)}-u$ as the overshot of u [in such statements, the qualifier "on $\{\nu(u) < \infty\}$" is frequently omitted. Note that $\nu(u) < \infty$ a.s.w.r.t. $\bar{P}$] . Weak convergence is denoted by $\Rightarrow$. Thus if ξ_u,ξ are random elements of a metric space $\mathcal{T}$, $\xi_u \overset{d}{\Rightarrow} \xi$ or $\xi_u \overset{d_u}{\Rightarrow} \xi$ means that for any $f \in C_b(\mathcal{T})$ (the bounded continuous functions on $\mathcal{T}$), it holds that $Ef(\xi_u) \to Ef(\xi)$, resp. $E_u f(\xi_u) \to Ef(\xi)$, as $u \to \infty$. Similar conventions are used for $\overset{\bar{d}}{\Rightarrow}$ and convergence in probability, $\overset{P}{\to}$, $\overset{\bar{P}}{\to}$ and $\overset{P_u}{\to}$. Apart from Euclidean spaces, the main examples of $\mathcal{T}$ are the function spaces $D[a,b]$, $D[0,\infty)$, $D(-\infty,\infty)$ discussed in Billingsley (1968), Stone (1963), Whitt (1970) and Lindvall (1973). The sample paths of the processes considered are always assumed to be in D . A basic fact used at a number of occasions is that uniform convergence on compact sets always entails D-convergence.

We first give some basic, though elementary, formulae connecting P, $\bar{P}$, P_u . Cf. Iglehart (1972) pg.629-630 and von Bahr (1974) pg.193.

Lemma 2.1 *For any Borel measurable function* g *of* n *variables such that the expectations exist,*

$$(2.1)\qquad Eg(X_1,\dots,X_n) = \bar{E}e^{-\gamma S_n}g(X_1,\dots,X_n)\ ,\quad \bar{E}g(X_1,\dots,X_n) = Ee^{\gamma S_n}g(X_1,\dots,X_n)\ .$$

Proof $Eg(X_1,\dots,X_n) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} g(x_1,\dots,x_n)dF(x_1)\dots dF(x_n) =$

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} g(x_1,\dots,x_n)e^{-\gamma(x_1+\dots+x_n)}d\bar{F}(x_1)\dots d\bar{F}(x_n) = \bar{E}e^{-\gamma S_n}g(X_1,\dots X_n)\ . \ \square$$

Lemma 2.2 *For any events* $G_u \in \mathcal{G}_u$,

$$(2.2)\qquad P(G_u,\ \nu(u) < \infty) = e^{-\gamma u}\bar{E}e^{-\gamma B(u)}I(G_u)\ ,$$

$$(2.3)\qquad P_u G_u = \frac{\bar{E}e^{-\gamma B(u)} I(G_u)}{\bar{E}e^{-\gamma B(u)}} .$$

Proof (2.3) is a consequence of (2.2). The definitions of F_n and G_u imply the existence of n-dimensional Borel sets $H_{u,n}$ such that $G_u \cap \{\nu(u) = n\} = \{(X_1,\ldots,X_n) \in H_{u,n}\}$. But $S_n = u + B(u)$ on $\{\nu(u) = n\}$ so that by (2.1)

$$P(G_u, \nu(u) < \infty) = \sum_{n=1}^{\infty} P((X_1,\ldots,X_n) \in H_{n,u}) =$$

$$\sum_{n=1}^{\infty} \bar{E}e^{-\gamma S_n} I((X_1,\ldots,X_n) \in H_{n,u}) =$$

$$e^{-\gamma u}\bar{E}e^{-\gamma B(u)} \sum_{n=1}^{\infty} I((X_1,\ldots,X_n) \in H_{n,u}) = e^{-\gamma u}\bar{E}e^{-\gamma B(u)} I(G_u) . \square$$

Let $H, \bar{H}$ be the d.f. of the ascending ladder height w.r.t. P ,resp. $\bar{P}$, and $h, \bar{h}$ the corresponding means. Then

Lemma 2.3 $B(u) \overset{\bar{d}}{\Rightarrow} B(\infty)$, *where* $\bar{P}(B(\infty) \le \xi) = \frac{1}{\bar{h}}\int_0^{\xi} (1-\bar{H}(t))dt$.

cf. Feller (1971) pg.371. In particular, combining with (2.2) yields the two first identities in the expression

$$(2.4)\qquad C = \bar{E}e^{-\gamma B(\infty)} = \frac{1}{\bar{h}}\int_0^{\infty} e^{-\gamma\xi}(1-\bar{H}(\xi))d\xi = \frac{1-H(\infty)}{\gamma\bar{h}}$$

$$= \frac{1}{\gamma\bar{\mu}} \exp[-\sum_{n=1}^{\infty} \frac{1}{n}\{P(S_n > 0) + \bar{P}(S_n \le 0)\}]$$

for the constant C in (1.1). For the third equality, perform an integration by parts and note that $d\bar{H}(x) = e^{\gamma x}dH(x)$ [Feller derives this fact from a set of integral equations, but it is in fact an easy consequence of Lemma 2.2 with $u = 0$]. Finally the last expression follows from Wald's identity and Feller (1971) equations (2.5) and (7.16). Cf. also Lemma 1 of Iglehart (1972) and the following discussion, and Chung (1974) Ch.8.

Lemma 2.4 If Y_u *is* G_u*-measurable and* $Y_u \overset{\bar{P}}{\to} 0$ *then* $Y_u \overset{P_u}{\to} 0$ *as well.*

Proof From (2.3) and (2.4),

$$\overline{\lim_{u\to\infty}}\, P_u(|Y_u| > \varepsilon) \le C^{-1}\overline{\lim_{u\to\infty}}\, \bar{P}(|Y_u| > \varepsilon) = 0 \ . \ \square$$

Lemma 2.5 *(i)* $\nu(u)/u \overset{\bar{P}}{\to} \bar{\mu}^{-1}$; *(ii)* $\bar{E}\nu(u)/u \to \bar{\mu}^{-1}$;
(iii) $\nu(u)/u \overset{P_u}{\to} \bar{\mu}^{-1}$.

Proof Part (i), even with convergence $\bar{P}$ - a.s., follows from the law of large numbers and $S_{\nu(u)-1} \le u < S_{\nu(u)}$. The proof of (ii) is the same as for the elementary renewal theorem, e.g. Karlin and Taylor (1975) pg.188-189. Finally (iii) is a consequence of (i) and Lemma 2.4. $\square$

3. Conditioned limit theorems for random walks

We let throughout $\xi = (\xi(t))_{0 \le t \le 1}$ be standard Brownian motion in $D[0,1]$ and

$$\mathring{\xi} = (\mathring{\xi}(t))_{0 \le t \le 1} = (\xi(t) - t\xi(1))_{0 \le t \le 1}$$

the Brownian bridge derived from ξ . Also define

$$\zeta = \mathring{\xi} \circ \bar{F} = (\mathring{\xi}(\bar{F}(t)))_{-\infty < t < \infty}$$

so that ζ is a Gaussian random element of $D(-\infty,\infty)$ with $E\zeta(t) = 0$, $\mathrm{Cov}(\zeta(s),\zeta(t)) = \bar{F}(s)(1-\bar{F}(t))$ $s \le t$. Recall the definition of the empirical d.f. F_n from Section 1.

Theorem 3.1 (i) $\|F_{\nu(u)}-\bar{F}\| \overset{P_u}{\to} 0$; (ii) $(u/\bar{\mu})^{\frac{1}{2}}(F_{\nu(u)}-\bar{F}) \overset{d_u}{\Rightarrow} \zeta$.

Part (i) follows immediately from Lemma 2.4 and the Glivenko-Cantelli theorem, implying $\|F_n - \bar{F}\| \to 0$ $\bar{P}$ - a.s. and hence $\|F_{\nu(u)} - \bar{F}\| \to 0$ $\bar{P}$ - a.s., $\|F_{\nu(u)} - \bar{F}\| \overset{\bar{P}}{\to} 0$. The proof of (ii) is slightly more involved. The idea is the one obvious from (2.3), to convert standard results from the i.i.d. $\bar{P}$-setting into statements dealing with random indexing by $\nu(u)$ and establish asymptotic independence of $B(u)$. With an application in Section 5 in mind, the first of these steps is carried out in a slightly more general setting than needed at present in the following lemma. Define $T_n : \mathbb{R}^n \to D(-\infty,\infty)$ by

$$T_n(x_1,\ldots,x_n) = (n^{\frac{1}{2}}[\frac{1}{n}\sum_{k=1}^{n} I(x_k \le t) - \bar{F}(t)])_{-\infty<t<\infty}$$

and let $\zeta_n = T_n(X_1,\ldots,X_n)$ be the empirical process.

Lemma 3.1 $\zeta_{\nu(u)} \overset{\bar{d}}{\Rightarrow} \zeta$ *(mixing)* .

Proof By Th.16.4 of Billingsley(1968), adapted to $D(-\infty,\infty)$, $\zeta_n \overset{\bar{d}}{\to} \zeta$.

Since $\nu(u)/u \overset{\bar{P}}{\to} \bar{\mu}^{-1}$, it follows from Pyke (1968) that $\zeta_{\nu(u)} \overset{\bar{d}}{\Rightarrow} \zeta$ and Pyke also proves that $\{\zeta_n\}$ satisfies Anscombe's (1952) condition (cf. also Aldous (1978) Prop.1). Thus the lemma follows from Th.8 of Csörgö and Fischler (1973) and the mixing property of $\{\zeta_n\}$, which ought to be well-known, but can easily be established by means of Th.5 of Eagleson (1976): Note that the tail σ-field of the X_n is $\bar{P}$-trivial and that

$$(3.1)\quad \|T_n(X_1,\ldots,X_n) - T_{n-\ell}(X_{\ell+1},\ldots,X_n)\| =$$

$$\|[n^{\frac{1}{2}} - (n-\ell)^{\frac{1}{2}}]\bar{F}(\cdot) + [n^{-\frac{1}{2}} - (n-\ell)^{-\frac{1}{2}}]\sum_{k=\ell+1}^{n} I(X_k \le \cdot) + n^{-\frac{1}{2}}\sum_{k=1}^{\ell} I(X_k \le \cdot)\| =$$

$$O(n^{\frac{1}{2}} - (n-\ell)^{\frac{1}{2}}) + O(n[n^{-\frac{1}{2}} - (n-\ell)^{-\frac{1}{2}}]) + O(n^{-\frac{1}{2}}) = O(n^{-\frac{1}{2}})$$

so that for any metric d defining the topology on $D(-\infty,\infty)$

$$d(T_n(X_1,\ldots,X_n)\ ,\ T_{n-\ell}(X_{\ell+1},\ldots,X_n)) \overset{\bar{P}}{\to} 0\ .\ \square$$

Define $u' = u - u^{\frac{1}{4}}$.

<u>*Lemma 3.2*</u> *For any* $g \in C_b[0,\infty)$, $\bar{E}(g(B(u))|\mathcal{G}_{u'}) \overset{\bar{P}}{\to} \bar{E}g(B(\infty))$.

<u>*Proof*</u> Define $h(x) = \bar{E}g(B(x))$ so that $h(x) \to h(\infty) = \bar{E}g(B(\infty))$ as $x \to \infty$ and

$$\bar{E}(g(B(u))|\mathcal{G}_{u'}) = h(u^{\frac{1}{4}}-B(u'))I(B(u') \le u^{\frac{1}{4}}) + g(B(u')-u^{\frac{1}{4}})I(B(u') > u^{\frac{1}{4}})\ .$$

Hence if K , x_0 are chosen such that $P(B(u) > K) < \varepsilon$ for all u , $|h(x) - h(\infty)| < \varepsilon\ x\ge x_0$, then the probability of the r.h.s. to deviate more than ε from $h(\infty)$ is at most ε for $u^{\frac{1}{4}}\ge K + x_0$. $\square$

<u>*Proof of Theorem 3.1 (ii)*</u> We first show that $(\zeta_{\nu(u)},B(u)) \overset{\bar{d}}{\Rightarrow} (\zeta,B)$, where B is independent of ζ and distributed as $B(\infty)$ in Lemma 2.3. By Lemma 2.5 (ii)

$$\bar{E}[\nu(u) - \nu(u')] = \int_0^{u^{1/4}} \bar{E}\nu(u^{1/4}-b)P(B(u')\in db) \le \bar{E}\nu(u^{1/4}) = o(u^{1/4})$$

and using Taylor's formula it follows as in (3.1) that

$$\|\zeta_{\nu(u)} - \zeta_{\nu(u')}\| \le o(\nu(u)^{1/2} - \nu(u')^{1/2}) + o(\nu(u)[\nu(u)^{-1/2} - \nu(u')^{-1/2}]) + o(\nu(u)^{-1/2})[\nu(u) - \nu(u')]) \overset{\bar{P}}{\to} 0$$

so that it suffices to show $(\zeta_{\nu(u')},B) \overset{\bar{d}}{\Rightarrow} (\zeta,B)$. But let $f \in C_b(D(-\infty,\infty))$, $g \in C_b[0,\infty)$. Then by Lemma 3.1 and 3.2

$$\bar{E}f(\zeta_{\nu(u')})g(B(u)) = \bar{E}f(\zeta_{\nu(u')})\bar{E}(g(B(u))|G_{u'}) \cong$$

$$\bar{E}f(\zeta_{\nu(u')})\bar{E}g(B(\infty)) \to Ef(\zeta)\bar{E}g(B(\infty)) .$$

Finally $\zeta_{\nu(u)} \overset{d_u}{\Rightarrow} \zeta$ follows from (2.3) since for $f \in C_b(D(-\infty,\infty))$ we have

$$E_u f(\zeta_{\nu(u)}) = \frac{\bar{E}e^{-\gamma B(u)}f(\zeta_{\nu(u)})}{\bar{E}e^{-\gamma B(u)}} \to \frac{\bar{E}e^{-\gamma B}\bar{E}f(\zeta)}{\bar{E}e^{-\gamma B}} = \bar{E}f(\zeta) .$$

Note that $\zeta_{\nu(u)} \overset{d_u}{\Rightarrow} \zeta$ is equivalent to (ii) in view of Lemma 2.5 (iii). □

We next turn to a conditioned analogue of Donsker's theorem, which for the present purposes it seems more natural to study in D[0,1] than D[0,∞) (a similar remark applies to the first passage time process in Theorem 3.2 below). Thus we assume that $0 \le \bar{\sigma}^2 = \overline{\mathrm{Var}}\, X_n < \infty$ and study the random element

$$\xi_n = (n^{-1/2} \sum_{k=1}^{[nt]} (X_k - \bar{\mu}))_{0 \le t \le 1}$$

of D[0,1] . As is well-known, $\xi_n \overset{\bar{d}}{\Rightarrow} \bar{\sigma}\xi$.

Proposition 3.1 $\xi_{\nu(u)} \overset{d_u}{\Rightarrow} \bar{\sigma}\xi$.

Proof The argument is so close to the proof of Theorem 3.1 that we just carry out in detail the proof of $\|\xi_{\nu(u)} - \xi_{\nu(u')}\| \overset{\bar{P}}{\to} 0$. Define $w(y,\delta) = \sup_{|s-t|\le\delta} |y(s)-y(t)|$. Then

(3.2) $\|\xi_{\nu(u)} - \xi_{\nu(u')}\| =$

$$\sup_{0\le t\le 1} |\xi_{\nu(u)}(t) - \frac{\nu(u)^{\frac{1}{2}}}{\nu(u')^{\frac{1}{2}}} \xi_{\nu(u)}(t\nu(u')/\nu(u))| \le$$

$$(\frac{\nu(u)^{\frac{1}{2}}}{\nu(u')^{\frac{1}{2}}} - 1)\|\xi_{\nu(u)}\| + w(\xi_{\nu(u)} , 1 - \frac{\nu(u')}{\nu(u)}) .$$

To complete the proof, note that $\xi_{\nu(u)} \overset{\bar{d}}{\Rightarrow} \bar{\sigma}\xi$, the distribution of ξ being concentrated on the subset C of continuous functions and the continuity of $\|\cdot\|$ and $w(\cdot,\delta)$ on C together imply $\bar{P}$-tightness of $\|\xi_{\nu(u)}\|$ as well as

$$\lim_{\delta\downarrow 0} \overline{\lim_{u\to\infty}} \bar{P}(w(\xi_{\nu(u)},\delta) > \varepsilon) = 0 .$$

Combine with $\nu(u)/\nu(u') \overset{\bar{P}}{\to} 1$. □

The following corollary is not much more than a reformulation of Proposition 3.1, but might be somewhat more intuitively appealing.

Corollary 3.2

$$((\frac{S_{[t\nu(u)]}-tu}{(\bar{\sigma}^2 u/\bar{\mu})^{\frac{1}{2}}})_{0\le t\le 1} , (\frac{\nu(tu)-tu/\bar{\mu}}{(\bar{\sigma}^2 u/\bar{\mu}^3)^{\frac{1}{2}}})_{0\le t\le 1}) \overset{d_u}{\Rightarrow} (\overset{\circ}{\xi},-\xi) .$$

Note in particular the asymptotic normality of $\nu(u)$, cf. e.g. von Bahr (1974) pg.203-204 and Nagaev (1973).

Proof Denote the random element of $D[0,1] \times D[0,1]$ under investigation by (η_u^1,η_u^2) and let $\eta_u^0 = \bar{\sigma}^{-1}\xi_{\nu(u)}$. Then by standard weak convergence arguments (e.g. Billingsley (1968) Th.4.1, somewhat adapted) it suffices to prove $(\eta_u^0,\eta_u^2) \overset{d_u}{\Rightarrow} (\xi,-\xi)$ and $(\eta_n^0,\eta_n^1) \overset{d_u}{\Rightarrow} (\xi,\overset{\circ}{\xi})$. For the first assertion, a number of criteria can be found in the literature, e.g. Billingsley (1968) Th.17.3, Iglehart and Whitt (1971), Whitt (1971) and Verwaat (1972). In order to apply the last reference, we must extend from $D[0,1]$ to $D[0,\infty)$. To this end, define $b_u = (\bar{\sigma}^2/\nu(u)\bar{\mu}^2)^{\frac{1}{2}}$,

$$H_u(t) = \begin{cases} \frac{1}{\nu(u)\bar{\mu}} \sum_{k=1}^{[\nu(u)t]} X_k & t \leq 1 \\ \frac{1}{\nu(u)\bar{\mu}} \left(\sum_{k=1}^{\nu(u)} X_k + (t-1)\nu(u)\bar{\mu} \right) & t > 1 , \end{cases}$$

$$H_u^{\uparrow}(t) = \sup_{0\leq s\leq t} H_u(s) , \quad H_u^{\uparrow -1}(t) = \inf\{s : H_u^{\uparrow}(s) > t\}$$

$$I(t) = t , \quad I_u(t) = \frac{t\nu(u)\bar{\mu}}{u} \quad 0 \leq t < \infty , \quad \xi(t) = \xi(1) \quad t > 1 .$$

Then Proposition 3.1 implies that $(H_u - I)/b_u \overset{d_u}{\Rightarrow} \xi$ in $D[0,\infty)$ and applying first Theorem 2 and next Theorem 1 of Verwaat (1972) we get

$$\left(\frac{H_u - I}{b_u} , \frac{H_u^{\uparrow} - I}{b_u}\right) \overset{d_u}{\Rightarrow} (\xi,\xi) , \quad \left(\frac{H_u - I}{b_u} , \frac{H_u^{\uparrow} - I}{b_u} , \frac{H_u^{\uparrow -1} - I}{b_u}\right) \overset{d_u}{\Rightarrow} (\xi,\xi,-\xi)$$

[in fact, Verwaat considers only the marginals but his proof is easily seen to apply to the joint distributions as well. Cf. his Lemmata 1 and 2 and also Iglehart and Whitt (1971) Th.1]. Hence since $I_u^{-1} \overset{P_u}{\to} I$ in the sense of uniform convergence on compact sets,

$$\left(\frac{H_u - I}{b_u} , \frac{H_u^{\uparrow -1} \circ I_u^{-1} - I_u^{-1}}{b_u}\right) \overset{d_u}{\Rightarrow} (\xi,-\xi)$$

in $D[0,\infty)$. Since ξ is continuous at 1, the restrictions to $[0,1]$ converge also in $D[0,1]$. But in view of $b_u \cong (\bar{\sigma}^2/u\bar{\mu})^{\frac{1}{2}}$ and

$$H_u^{\uparrow -1}(t) = \frac{\nu(t\nu(u)\bar{\mu})}{\nu(u)} = \frac{\nu(I_u(t)u)}{\nu(u)} \quad 0 \leq t \leq 1$$

this amounts exactly to $(\eta_u^0 , \eta_u^2) \overset{d_u}{\Rightarrow} (\xi,-\xi)$.

For $(\eta_u^0 , \eta_u^1) \overset{d_u}{\Rightarrow} (\xi,\mathring{\xi})$, note that

$$\left(\frac{\bar{\mu}\nu(u)}{u}\right)^{\frac{1}{2}} \eta_u^0(t) = \frac{S_{[t\nu(u)]} - t\nu(u)\bar{\mu}}{(\bar{\sigma}^2 u/\bar{\mu})^{\frac{1}{2}}} = \eta_u^1(t) - t\eta_u^2(1) .$$

But since $\eta_u^0 \overset{d_u}{\Rightarrow} \xi \in C$ and $\nu(u)/u \overset{P_u}{\to} 1/\bar{\mu}$, it follows that

$\|(\eta_u^1(t) - t\eta_u^2(1) - \eta_u^0(t))_{0 \le t \le 1}\| \overset{P_u}{\to} 0$ and since

$$(\eta_u^0, (\eta_u^0(t) + t\eta_u^2(1))_{0 \le t \le 1}) \overset{d_u}{\Rightarrow} (\xi, \mathring{\xi})$$

from above, the proof is complete. □

It is of some interest to note that $\mathring{\xi}$ and $\xi(1)$ are independent. This might be rephrased to the statement that loosely the fluctuations of $(S_n)_{0 \le n \le \nu(u)}$ are independent of $\nu(u)$. Similar remarks apply to results in the following.

4. Risk reserve processes

Throughout we let $R(0) = u$, so that the conditioning event $\{\nu(u) < \infty\}$ is the event of ruin and we let $\Theta(u) = S^{(2)}_{\nu(u)}$ be the time to ruin. We recall the notational convention of Section 1, according to which e.g.

$$S_n = S_n^{(1)} - cS_n^{(2)} \ , \ \mu = \mu^{(1)} - c\mu^{(2)} \ , \ \bar{\sigma}^2 = \bar{\sigma}^{(1)^2} + c^2\bar{\sigma}^{(2)^2} \quad \text{etc.}$$

Let $\xi^{(0)}, \xi^{(1)}, \xi^{(2)}$ be standard Brownian motions with $\xi^{(1)}, \xi^{(2)}$ independent and $\xi^{(0)} = \bar{\kappa}^{-1}\bar{\mu}^{-3/2}(\bar{\mu}^{(1)}\bar{\sigma}^{(2)}\xi^{(2)} - \bar{\mu}^{(2)}\bar{\sigma}^{(1)}\xi^{(1)})$, cf. (4.1) below. We let $\zeta^{(i)} = \overset{\circ}{\xi}{}^{(i)} \circ \bar{F}^{(i)}$. The two-dimensional empirical d.f. is defined by

$$H_n(t_1,t_2) = \frac{1}{n}\sum_{k=1}^{n} I(X_k^{(1)} \le t_1 \ , \ X_k^{(2)} \le t_2)$$

and $F^{(1)} \otimes F^{(2)}$ denotes the d.f. of $(X_n^{(1)}, X_n^{(2)})$, $F^{(1)} \otimes F^{(2)}(t_1,t_2) = F^{(1)}(t_1)F^{(2)}(t_2)$. Finally $\mathcal{F}_n$ is extended from $\sigma(X_k^{(1)} - cX_k^{(2)} \ ; \ k \le n)$ to $\sigma(X_k^{(1)}, X_k^{(2)} \ ; \ k \le n)$.

Repetitions or minor extensions of arguments from Sections 2-3 produce the following analogues of Theorem 3.1 and Proposition 3.1.

Theorem 4.1 (i) $\|H_{\nu(u)} - \bar{F}^{(1)} \otimes \bar{F}^{(2)}\| \overset{P_u}{\to} 0$;

(ii) $(\zeta^{(1)}_{\nu(u)} \ , \ \zeta^{2}_{\nu(u)}) \overset{d_u}{\Rightarrow} (\zeta^{(1)} \ , \ \zeta^{(2)})$ *in* $D(-\infty,\infty) \times D(-\infty,\infty)$.

Proposition 4.1 $(\xi^{(1)}_{\nu(u)} \ , \ \xi^{(2)}_{\nu(u)}) \overset{d_u}{\Rightarrow} (\bar{\sigma}^{(1)}\xi^{(1)} \ , \ \bar{\sigma}^{(2)}\xi^{(2)})$ *in* $D[0,1] \times D[0,1]$.

Thus Theorem 4.1 states that if ruin occurs with a large initial risk reserve u, the waiting times between claims may be modelled as governed by $\bar{F}^{(2)}$ rather than $F^{(2)}$. For the most widely studied case where the claims arrive according to a Poisson process with intensity α , this corresponds to arrival intensity $\alpha + c\gamma$. Similarly, the claim sizes may be modelled as governed by $\bar{F}^{(1)}$ rather than $F^{(1)}$. Note that always $\bar{F}^{(1)} \geq F^{(1)}$ and $\bar{F}^{(2)} \leq F^{(2)}$ in the sense of stochastic ordering. It would of course, have been more natural to formulate (ii) in terms of $(u/\bar{\mu})^{\frac{1}{2}}(H_{\nu(u)} - \bar{F}^{(1)} \otimes \bar{F}^{(2)})$, but in view of the conceptual difficulties in the study of multivariate empirical processes (e.g. Dudley (1978) and the references therein) we shall not insist on that.

We next proceed to derive some consequences of Proposition 4.1 which seem more appealing from the point of view of risk theory than the proposition itself. Define

$$(4.1) \qquad \bar{\lambda} = \frac{\bar{\mu}^{(2)}}{\bar{\mu}} \ , \ \bar{\kappa}^2 = \bar{\mu}^{-3}\{\bar{\mu}^{(1)^2}\bar{\sigma}^{(2)^2} + \bar{\mu}^{(2)^2}\bar{\sigma}^{(1)^2}\} \ .$$

Corollary 4.1 (i) $u^{-\frac{1}{2}}\{\Theta(u) - \bar{\lambda}u\} \overset{d_u}{\Rightarrow} \bar{\kappa}\xi^{(0)}$; (ii)

$$(4.2) \qquad \lim_{u\to\infty} \sup_{0\leq t\leq\infty} |e^{\gamma u}\psi(t,u) - C\Phi(\frac{t-\bar{\lambda}u}{\bar{\kappa}u^{\frac{1}{2}}})| = 0 \ .$$

Proof. Since $S_{\nu(u)} = u + B(u)$, we have

$$u^{-\frac{1}{2}}\{\Theta(u) - \bar{\lambda}u\} = u^{-\frac{1}{2}}\{S^{(2)}_{\nu(u)} - \bar{\lambda}S_{\nu(u)} + \bar{\lambda}B(u)\} =$$

$$\frac{\nu(u)^{\frac{1}{2}}}{u^{\frac{1}{2}}}\{(1+c\bar{\lambda})\xi^{(2)}_{\nu(u)}(1) - \bar{\lambda}\xi^{(1)}_{\nu(u)}(1)\} + u^{-\frac{1}{2}}\bar{\lambda}B(u) \overset{d_u}{\Rightarrow}$$

$$\bar{\mu}^{-\frac{1}{2}}\{(1+c\bar{\lambda})\bar{\sigma}^{(2)}\xi^{(2)}(1) - \bar{\lambda}\bar{\sigma}^{(1)}\xi^{(1)}(1)\} = \bar{\kappa}\xi^{(0)}(1) \ ,$$

using $B(u)/u^{\frac{1}{2}} \overset{\bar{P}}{\to} 0$, $B(u)/u^{\frac{1}{2}} \overset{P_u}{\to} 0$, cf. Lemma 2.3 and 2.4. For (ii), note that $\psi(t,u) = P(\nu(u) < \infty \ , \ \Theta(u) \leq t)$ so that

$$\sup_{0\leq t\leq\infty} |e^{\gamma u}\psi(t,u) - CP_u(\Theta(u) \leq t)| = \sup_{0\leq t\leq\infty} e^{\gamma u}\psi(t,u)|1 - \frac{C}{\bar{E}e^{-\gamma B(u)}}| \to 0$$

in view of $\psi(t,u) \leq \psi(u)$, $e^{\gamma u}\psi(u) \to C$ and $\bar{E}e^{-\gamma B(u)} \to C$.

(4.2) follows from (i) by noting that weak convergence to a continuous limit implies uniform convergence of the d.f. □

In the present generality, (4.2) was proved by von Bahr (1974). See Seal (1969) for references to earlier work on Poisson arrival of claims. For (4.2) alone, the present proof can be reduced to a reasonably simple argument. E.g., the asymptotic normality of $S^{(2)}_{\nu(u)} - \bar{\lambda}S_{\nu(u)}$ w.r.t. $\bar{P}$ follows from Lemma 2.5 (i) and Anscombe's (1952) theorem.

We next turn to the risk reserve process $(R(t))_{t\geq 0}$ and the first passage times

$$\Theta(v) = \inf\{t\geq 0 : R(t) < R(0) - v\} = \inf\{S^{(2)}_n : S_n > v\} = S^{(2)}_{\nu(v)} .$$

Corollary 4.2 $$\left(\left(\frac{R(t\Theta(u)) - (1-t)u}{u^{\frac{1}{2}}}\right)_{0\leq t\leq 1}, \left(\frac{\Theta(tu) - tu\bar{\lambda}}{u^{\frac{1}{2}}}\right)_{0\leq t\leq 1}\right) \overset{d_u}{\Rightarrow} \left(\frac{\bar{\kappa}}{\bar{\lambda}}\overset{\circ}{\xi}{}^{(0)}, \bar{\kappa}\xi^{(0)}\right) .$$

Proof. Let first $H_u(t) = S^{(2)}_{[t\nu(u)]}/\nu(u)\bar{\mu}^{(2)}$, $b_u = (\nu(u)\bar{\mu}^{(2)^2})^{-\frac{1}{2}}$, $J_u(t) = t\nu(u)\bar{\mu}^{(2)}/\Theta(u)$. Then $(H_u - I)/b_u \overset{d_u}{\Rightarrow} \bar{\sigma}^{(2)}\xi^{(2)}$,

$$H_u^{-1}(t) = \nu^{(2)}(\nu(u)\bar{\mu}^{(2)})/\nu(u) = \nu^2(J_u(t)\Theta(u))/\nu(u)$$

and in a similar manner as in the proof of Corollary 3.1 we may deduce that

$$\left(\frac{\nu^{(2)}(t\Theta(u))/\nu(u) - J_u^{-1}(t)}{b_u}\right)_{0\leq t\leq 1} \overset{d_u}{\Rightarrow} -\bar{\sigma}^{(2)}\xi^{(2)}(t) ,$$

$$(4.3) \qquad \left(\frac{\nu^{(2)}(t\Theta(u)) - t\Theta(u)/\bar{\mu}^{(2)}}{u^{\frac{1}{2}}}\right)_{0\leq t\leq 1} \overset{d_u}{\Rightarrow} -\bar{\mu}^{-\frac{1}{2}}\bar{\mu}^{(2)^{-1}}\bar{\sigma}^{(2)}\xi^{(2)} .$$

Clearly, the convergences in Proposition 4.1, Corollary 4.1 (i) and (4.3) hold simultaneously.

We now approximate $R(t)$ by the value after the next claim. Define $h(t) = R(t) - R(S^{(2)}_{\nu^{(2)}(t)})$. Then

$$u^{-\frac{1}{2}} \sup_{0\le t\le 1} |h(t\Theta(u))| \le u^{-\frac{1}{2}} \max_{k=1,\dots,\nu(u)} \{|X_k^{(1)}| + cX_k^{(2)}\} \overset{P_u}{\to} 0 \ ,$$

using $\max\limits_{k=1,\dots,n} n^{-\frac{1}{2}}|X_k^{(i)}| \overset{\bar{P}}{\to} 0$ (as follows from $EX_k^{(i)^2} < \infty$) and $\nu(u)/u \overset{P_u}{\to} \bar{\mu}^{-1}$. Thus, denoting the random processes under study by (η_u^1,η_u^2) and letting $I_u(t) = \nu^{(2)}(t\Theta(u))/\nu(u)$, we may conclude that

$$\eta_u^1(t) = u^{-\frac{1}{2}}[h(t\Theta(u)) + R(S^{(2)}_{\nu^{(2)}(t\Theta(u))}) - (1-t)u] \cong$$

$$u^{-\frac{1}{2}}[R(S^{(2)}_{\nu^{(2)}(t\Theta(u))}) - (1-t)u] = -u^{-\frac{1}{2}}[S_{\nu^{(2)}(t\Theta(u))} - tu] =$$

$$(\frac{\nu(u)}{u})^{\frac{1}{2}}[c\xi^{(2)}_{\nu(u)}(I_u(t)) - \xi^{(1)}_{\nu(u)}(I_u(t))] - u^{-\frac{1}{2}}[\bar{\mu}\nu^{(2)}(t\Theta(u)) - tu] \cong$$

$$\bar{\mu}^{-\frac{1}{2}}[c\bar{\sigma}^{(2)}\xi^{(2)}(t) - \bar{\sigma}^{(1)}\xi^{(1)}(t)] + \frac{\bar{\mu}^{-\frac{1}{2}}}{\bar{\lambda}}\bar{\sigma}^{(2)}\xi^{(2)} - tu^{-\frac{1}{2}}[\frac{\Theta(u)}{\bar{\lambda}} - u] \cong$$

$$\bar{\lambda}^{-1}\bar{\mu}^{-\frac{1}{2}}[(1+c\bar{\lambda})\bar{\sigma}^{(2)}\xi^{(2)}(t) - \bar{\lambda}\bar{\sigma}^{(1)}\xi^{(1)}(t)] - t\frac{\bar{\kappa}}{\bar{\lambda}}\xi^{(0)}(1) = \frac{\bar{\kappa}}{\bar{\lambda}}\overset{\circ}{\xi}{}^{(0)}(t)$$

in the sense of convergence not only for t fixed but in $D[0,1]$, $\eta_u^1 \overset{d_u}{\Rightarrow} \bar{\kappa}/\bar{\lambda}\overset{\circ}{\xi}{}^{(0)}$. Here we have used $\|I_u - I\| \overset{P_u}{\to} 0$, as follows easily from $\bar{P}$-a.s. properties of $\nu^{(2)}(t),\nu(u),\Theta(u)$, and Corollary 4.1 (i) and (4.3).

The rest of the proof is now easy. We have $-\eta_u^1(t) = u^{\frac{1}{2}}(H_u(t) - t)$ with $H_u(t) = u^{-1}[u - R(t\Theta(u))]$ and noting that $H_u^{\uparrow-1}(t) = \Theta(tu)/\Theta(u)$, it follows as in the proof of Corollary 3.1 that

$$(\eta_u^1 , (\frac{\Theta(tu)/\Theta(u) - t}{u^{-\frac{1}{2}}})_{0\le t\le 1}) \overset{d_u}{\Rightarrow} (\frac{\bar{\kappa}}{\bar{\lambda}}\overset{\circ}{\xi}{}^{(0)} , \frac{\bar{\kappa}}{\bar{\lambda}}\overset{\circ}{\xi}{}^{(0)}) \ ,$$

$$\frac{\Theta(tu) - tu\bar{\lambda}}{u^{\frac{1}{2}}} = \frac{\Theta(tu) - t\Theta(u)}{u^{\frac{1}{2}}} + t\frac{\Theta(u) - u\bar{\lambda}}{u^{\frac{1}{2}}} \cong$$
$$\bar{\kappa}\overset{\circ}{\xi}{}^{(0)}(t) + t\bar{\kappa}\xi^{(0)}(1) = \bar{\kappa}\xi^{(0)}(t)$$

in the sense of D-convergence simultaneously with η_u^1 . □

5. Conditioned limit theorems for the GI/G/1 queue within a busy cycle.

The notation introduced in Section 4 is used without change. Recall that $W_n = S_n$, $0 \leq n < \underline{n}$. Clearly, the waiting time of one of the customers $0,1,\dots,\underline{n}-1$ exceeds u if and only if $\nu(u) < \underline{n}$ and we let $P_u^+(\cdot) = P(\cdot|\nu(u) < \underline{n})$. Our first result is the analogue of Theorem 4.1.

Theorem 5.1 (i) $\|H_{\nu(u)} - \bar{F}^{(1)} \otimes \bar{F}^{(2)}\| \overset{P_u^+}{\to} 0$;

(ii) $(\zeta_{\nu(u)}^{(1)}, \zeta_{\nu(u)}^{(2)}) \overset{d_u^+}{\Rightarrow} (\zeta^{(1)}, \zeta^{(2)})$ *in* $D(-\infty,\infty) \times D(-\infty,\infty)$.

In the proof, we need several variants of the material of Section 2, e.g.

$$(5.1) \qquad P_u^+ G_u = \frac{\bar{E}e^{-\gamma B(u)} I(G_u, \nu(u) < \underline{n})}{\bar{E}e^{-\gamma B(u)} I(\nu(u) < \underline{n})}, \quad G_u \in \mathcal{G}_u,$$

which follows from (2.2) by noting that $G_u \cap \{\nu(u) < \underline{n}\} \in \mathcal{G}_u$, as well as

Lemma 5.1 (i) $e^{\gamma u} P(\nu(u) < \underline{n}) = \bar{E}e^{-\gamma B(u)} I(\nu(u) < \underline{n}) \to C^+ = C\bar{P}(\underline{n} = \infty)$;

(ii) $B(u) \overset{d_u^+}{\Rightarrow} B^+$, *where* $Eg(B^+) = \bar{E}e^{-\gamma B(\infty)} g(B(\infty))/C$;

(iii) $E_u^+ B(u) \to EB^+$.

Note that $\bar{P}(\underline{n} = \infty) > 0$ because of $\bar{\mu} > 0$. More precisely,

$$(5.2) \qquad \bar{P}(\underline{n} = \infty) = \bar{\mu}/\bar{h} = \exp\{-\sum_{n=1}^{\infty} \frac{1}{n} \bar{P}(S_n \leq 0)\},$$

cf. (2.4) and the following discussion.

Proof. For (i), note that $\{\nu(u) < \underline{n}\} \downarrow \{\underline{n} = \infty\}$ so that $P(\nu(u') < \underline{n} \leq \nu(u)) \to 0$ $(u' = u - u^{1/4})$. Hence by Lemma 3.2,

$$\bar{E}e^{-\gamma B(u)} I(\nu(u) < \underline{n}) \cong \bar{E}e^{-\gamma B(u)} I(\nu(u') < \underline{n}) =$$

$$\bar{E}I(\nu(u') < \underline{n}) \; \bar{E}(e^{-\gamma B(u)}|\mathcal{G}_{u'}) \cong \bar{P}(\nu(u') < \underline{n}) \; \bar{E}e^{-\gamma B(\infty)} \cong$$

$$\bar{P}(\underline{n} = \infty) C = C^+ .$$

The proofs of (ii), (iii) are contained in the proof of Theorem 5.1. □

Proof of Theorem 5.1. Part (i) is an immediate consequence of (5.1) with $G_u = \{\|H_{\nu(u)} - \bar{F}^{(1)} \otimes \bar{F}^{(2)}\| > \varepsilon\}$ and $\bar{P}G_u \to 0$. With a later application in mind, we prove (ii) in the form

$$(5.3) \qquad (\zeta^{(1)}_{\nu(u)}, \zeta^{(2)}_{\nu(u)}, B(u)) \overset{d^+_u}{\Rightarrow} (\zeta^{(1)}, \zeta^{(2)}, B^+)$$

with B independent of $(\zeta^{(1)}, \zeta^{(2)})$ and distributed as in Lemma 5.1 (ii). Note first that it follows as in the proofs of Lemma 3.1 and Theorem 3.1 that

$$(\zeta^{(1)}_{\nu(u)}, \zeta^{(2)}_{\nu(u)}) \overset{\bar{d}}{\Rightarrow} (\zeta^{(1)}, \zeta^{(2)}) \quad (\text{mixing}),$$

$$\|\zeta^{(i)}_{\nu(u)} - \zeta^{(i)}_{\nu(u')}\| \overset{\bar{P}}{\to} 0, \quad i = 1,2 \quad (u' = u - u^{\frac{1}{4}}).$$

Thus for any $f^{(1)}, f^{(2)} \in C_b(D(-\infty,\infty))$, $g \in C_b[0,\infty)$, $\tilde{g}(b) = e^{-\gamma b}g(b)$,

$$\lim_{u\to\infty} E^+_u f^{(1)}(\zeta^{(1)}_{\nu(u)})f^{(2)}(\zeta^{(2)}_{\nu(u)})g(B(u)) =$$

$$\lim_{u\to\infty} \bar{E}f^{(1)}(\zeta^{(1)}_{\nu(u)})f^{(2)}(\zeta^{(2)}_{\nu(u)})\ \tilde{g}(B(u))\, I(\nu(u) < \underline{n})/C^+ =$$

$$\lim_{u\to\infty} \bar{E}f^{(1)}(\zeta^{(1)}_{\nu(u')})f^{(2)}(\zeta^{(2)}_{\nu(u')})\ \tilde{g}(B(u))\ I(\nu(u') < \underline{n})/C^+ =$$

$$\lim_{u\to\infty} \bar{E}f^{(1)}(\zeta^{(1)}_{\nu(u')})f^{(2)}(\zeta^{(2)}_{\nu(u')})\ \bar{E}\tilde{g}(B(\infty))\ I(\underline{n} = \infty)/C^+ =$$

$$Ef^{(1)}(\zeta^{(1)})\ Ef^{(2)}(\zeta^{(2)})\ \bar{E}\tilde{g}(B(\infty))\ \bar{P}(\underline{n} = \infty)/C^+ =$$

$$Ef^{(1)}(\zeta^{(1)})\ Ef^{(2)}(\zeta^{(2)})\ Eg(B^+),$$

using Lemma 3.2 for the third equality and the mixing property for the fourth. This proves Theorem 5.1 as well as Lemma 5.1 (ii). For (iii), take $f^{(1)} \equiv 1$, $f^{(2)} \equiv 1$, $g(x) = x$ and note that the argument is still valid in view of $\tilde{g} \in C_b$. □

Similar arguing produces

Theorem 5.2

$$\left(\left(\frac{W_{[t\nu(u)]}-tu}{(\bar{\sigma}^2 u/\bar{\mu})^{\frac{1}{2}}}\right)_{0\le t\le 1}, \left(\frac{\nu(tu)-tu/\bar{\mu}}{(\bar{\sigma}^2 u/\bar{\mu}^3)^{\frac{1}{2}}}\right)_{0\le t\le 1}\right) \overset{d_u^+}{\Rightarrow} (\mathring{\xi},-\xi) .$$

This is the analogue of Corollary 3.1. The analogue of Corollary 4.2 describes the virtual waiting time $V(t)$. Note that in view of the connection between the $\{X_n^{(i)}\}$, $\{R(t)\}$ and $\{V(t)\}$ the sign changes and that now $\Theta(v) = \inf\{t : V(t) > v\} = S^{(2)}_{\nu(v)}$.

Theorem 5.3

$$\left(\left(\frac{V(t\Theta(u))-tu}{u^{\frac{1}{2}}}\right)_{0\le t\le 1}, \left(\frac{\Theta(tu)-tu\bar{\lambda}}{u^{\frac{1}{2}}}\right)_{0\le t\le 1}\right) \overset{d_u^+}{\Rightarrow} \left(\frac{\bar{\kappa}}{\bar{\lambda}}\mathring{\xi}^{(0)} , \bar{\kappa}\xi^{(0)}\right)$$

Remark 5.1 In just the same manner as in (5.3), one can prove that the convergence in Theorems 5.2, 5.3 hold simultaneously with $B(u) \overset{d_u^+}{\Rightarrow} B^+$ and with the limits independent of B^+ .

Note that the study of the $\bar{P}$-distribution of $\{S_n\}$ on $\{\underline{n} = \infty\}$ has already been touched upon in Iglehart (1974).

6. Waiting time approximations

We shall now apply the preceding analysis to a somewhat more practical question, viz. to establish

Theorem 6.1 $\lim_{u\to\infty} \sup_{0\le N\le\infty} |e^{\gamma u}P(W_N > u) - C\Phi(\frac{N-u/\bar{\mu}}{(\bar{\sigma}^2u/\bar{\mu}^3)^{\frac{1}{2}}})| = 0$

Theorem 6.2 *Suppose that* $F^{(2)}$ *is non-lattice. Then*

$$\lim_{u\to\infty} \sup_{0\le T\le\infty} |e^{\gamma u}P(V(T) > u)) - C_1\Phi(\frac{T-\bar{\lambda}u}{\bar{\kappa}u^{\frac{1}{2}}})| = 0 ,$$

where $C_1 = C(\hat{F}^{(1)}(\gamma) - 1)/\gamma\mu^{(2)}$ *and* $\bar{\lambda}$, $\bar{\kappa}^2$ *are defined by* (4.1).

Here Theorem 6.1 follows immediately. In fact,

$$P(W_N > u) = P(\max_{0\le n\le N} S_n > u) = P(\nu(u) \le N)$$

and one can apply the asymptotic normality of $\nu(u)$, cf. Corollary 3.1, in the same way as in the proof of (4.2). Also some particular cases of Theorem 6.2 are immediate. In the M/G/1 case, note that $C = C_1$, that the virtual waiting time d.f. is connected to the ruin probabilities by means of $P(V(T) > u) = \psi(T,u)$, cf. Seal (1972), and apply (4.2). In the equilibrium GI/G/1 case $T = \infty$, use the well-known connection between the d.f. of the actual and virtual waiting time, Cohen (1969) pg. 297, to write

$$(6.1) \qquad P(V(\infty) > u) = \frac{\rho}{\mu^{(1)}} \{\int_u^\infty (1-F^{(1)}(t))dt + \int_0^u P(W_\infty > u - t)(1-F^{(1)}(t))dt =$$

$$o(e^{-\gamma u}) + \frac{1}{\mu^{(2)}} \int_0^u Ce^{-\gamma(u-t)}(1-F^{(1)}(t))dt \cong$$

$$e^{-\gamma u}C/\mu^{(2)} \int_0^\infty e^{\gamma t}(1-F^{(1)}(t))dt = e^{-\gamma u}C_1 .$$

The case $T < \infty$ seems substantially more technical. Let U be the renewal measure associated with the d.f. of the busy cycle $\underline{c}$. Then (Cohen (1976) Ch.I)

$$(6.2)\qquad P(V(T) > u) = \int_0^T P(V(T-y) > u\ ,\ T-y < \underline{c})\, dU(y)\ .$$

The probability under the integral sign is now evaluated by conditioning upon G_u . Introduce the auxiliary process

$$A(t) = A(0) - t + \sum_{k=1}^{\nu^{(2)}(t)-1} X_k^{(1)} \qquad \text{and define}$$

$$k_{u,b}(t) = P(A(t) > 0\ ,\ \inf_{0\le s\le t} A(s) > -u | A(0) = b)\ ,$$

$$k_b(t) = \lim_{u\to\infty} k_{u,b}(t) = P(A(t) > 0 | A(0) = b)\ ,\ \tilde{k}_b = \frac{1}{E\underline{c}}\int_0^\infty k_b(t)dt$$

$$K_u^*(b,t) = P(B(u) \le b\ ,\ \Theta(u) \le t\ ,\ \nu(u) < \underline{n})\ ,$$

$$K_u(b,t) = K_u^*(b, \bar{\lambda}u + \bar{\kappa}u^{\frac{1}{2}}t)\ .$$

Then, by Theorem 5.3, Remark 5.1 and Lemma 5.1 (iii) it holds that

$$(6.3)\qquad e^{\gamma u}K_u(b,t) \to K(b,t) = C^+ P(B^+ \le b)\Phi(t)\ ,$$

$$(6.4)\qquad e^{\gamma u}\int_0^\infty\int_0^\infty b\, dK_u(b,t) = e^{\gamma u}P(\nu(u) < \underline{n})E_u^+ B(u) \to C^+ EB^+$$

and we have

$$P(V(t) > u\ ,\ t < \underline{c}) = \int_0^\infty\int_0^t k_{u,b}(t-s)dK_u^*(b,s)\ ,$$

$$(6.5)\qquad P(V(T) > u) = \int_0^\infty\int_0^T\int_0^{T-y} k_{u,b}(T-y-s)dK_u^*(b,s)dU(y) =$$

$$\int_0^\infty\int_0^T U*k_{u,b}(T-s)dK_u^*(b,s) =$$

$$\int_0^\infty \int_{-\infty}^{\frac{T-\bar{\lambda}u}{\bar{\kappa}u^{\frac{1}{2}}}} U*k_{u,b}(T - \bar{\lambda}u - \bar{\kappa}u^{\frac{1}{2}}s)dK_u(b,s)\ .$$

<u>*Lemma 6.1*</u> *There exists constants* $c_1, c_2 < \infty$ *such that for all* t $U*k_{u,b}(t) \le c_1 + c_2 b$.

<u>*Proof*</u> Let $0 < \varepsilon < \gamma_1 < \gamma$. Then $\hat{F}^{(1)}(\gamma_1)\hat{F}^{(2)}(-\gamma_1) < 1$ so that for ε small enough $\hat{F}^{(1)}(\gamma_1)\hat{F}^{(2)}(\varepsilon - \gamma_1) < 1$.

Define $Z(t) = e^{\varepsilon t}E(e^{\gamma_1 A(t)} | A(0) = 0)$. Then

$$Z(t) = e^{\varepsilon t}\{e^{-\gamma_1 t}(1-F^{(2)}(t)) + \int_0^t\int_0^\infty e^{\gamma_1(y_1-y_2)}e^{-\varepsilon(t-y_2)}Z(t-y_2)dF^{(1)}(y_1)dF^{(2)}(y_2)\} =$$

$$e^{-(\gamma_1-\varepsilon)t}(1-F^{(2)}(t)) + \int_0^t Z(t-y_2)e^{(\varepsilon-\gamma_1)y_2}dF^{(2)}(y_2)/\hat{F}^{(1)}(\gamma_1)$$

and reference to transient renewal equations yields $Z(t) \to 0$, in particular $E(e^{\gamma_1 A(t)} | A(0) = 0) \le c_3 e^{-\varepsilon t}$.

Thus

$$(6.6) \quad k_{u,b}(t) \le k_b(t) \le I(t \le \gamma_1 b/\varepsilon) + E(e^{\gamma_1 A(t)} | A(0) = b)I(t > \gamma_1 b/\varepsilon) \le$$

$$I(t < \gamma_1 b/\varepsilon) + c_3 e^{-\varepsilon(t-\gamma_1 b/\varepsilon)} I(t > \gamma_1 b/\varepsilon) .$$

The lemma follows from $U(x + 1) - U(x)$ being bounded in x and arguments similar to those employed by Feller (1971) in the proof of the key renewal theorem. □

Lemma 6.2 $\lim_{u\to\infty} \sup_{t \ge Ku^{1/4}} |U*k_{u,b}(t) - \tilde{k}_b| = 0$ *uniformly in* b *on bounded intervals.*

Proof. By (6.6), k_b is dominated by a directly Riemann integrable function and is thus itself d.R.i. in view of its continuity a.e. as follows from A having D-paths, cf. Miller (1972). Since $F^{(2)}$ non-lattice implies $\underline{c}$ non-lattice, it follows from the key renewal theorem that $U*k_{u,b}(t) \to \tilde{k}_{u,b}$ as $t \to \infty$ with u fixed. The $k_{u,b}(t)$, $\tilde{k}_b$ etc. are non-decreasing in u,b and right-continuous in b with left-hand limits $k_{u,b-0}(t)$, $\tilde{k}_{b-0}$ etc. Similar considerations as above yield $U*k_{u,b-0}(t) \to \tilde{k}_{u,b-0}$ and it follows that given $\varepsilon > 0$ and $0 \le B < \infty$, we can choose first $0 = b(0) < b(1) < \dots < b(n) = B$ and next v , t_0 such that

$$\tilde{k}_{b(i)-0} - \tilde{k}_{b(i-1)} < \varepsilon \ , \ \tilde{k}_{b(i)} - \tilde{k}_{v,b(i)} < \varepsilon$$

$$|U*k_{b(i)-0}(t) - \tilde{k}_{b(i)-0}| < \varepsilon \ , \ |U*k_{v,b(i)}(t) - \tilde{k}_{v,b(i)}| < \varepsilon$$

for $i = 1,\ldots,n$ and $t \geq t_0$. Hence if $b(i-1) \leq b < b(i)$ and $u \geq v$, $t \geq \bar{\kappa}u^{\frac{1}{4}} \geq t_0$

$$U*k_{u,b}(t) - \tilde{k}_b \leq U*k_{b(i)-0}(t) - \tilde{k}_{b(i-1)} < 2\varepsilon$$

$$U*k_{u,b}(t) - \tilde{k}_b \geq U*k_{v,b(i-1)}(t) - \tilde{k}_{b(i)-0} > -3\varepsilon$$

and the uniformity on $[0,B)$ follows. □

Proof of Theorem 6.2 Define $q_1(T,u) = e^{\gamma u}P(V(T) > u)$,

$$q_2(T,u) = e^{\gamma u}\int_0^{\infty}\int_{-\infty}^{\frac{T-\bar{\lambda}u}{\bar{\kappa}u^{\frac{1}{2}}}-u^{-\frac{1}{4}}} U*k_{u,b}(T-\bar{\lambda}u-\bar{\kappa}u^{\frac{1}{2}}s)\,dK_u(b,s) \ ,$$

$$q_3(T,u) = e^{\gamma u}\int_0^{\infty}\int_{-\infty}^{\frac{T-\bar{\lambda}u}{\bar{\kappa}u^{\frac{1}{2}}}-u^{-\frac{1}{4}}} \tilde{k}_b\,dK_u(b,s) \ ,$$

$$q_4(T,u) = e^{\gamma u}\int_0^{\infty}\int_{-\infty}^{\frac{T-\bar{\lambda}u}{\bar{\kappa}u^{\frac{1}{2}}}} \tilde{k}_b\,dK_u(b,s) \ ,$$

$$q_5(T,u) = \int_0^{\infty}\int_{-\infty}^{\frac{T-\bar{\lambda}u}{\bar{\kappa}u^{\frac{1}{2}}}} \tilde{k}_b\,dK(b,s) = C^+E\tilde{k}_B+\Phi\left(\frac{T-\bar{\lambda}u}{\bar{\kappa}u^{\frac{1}{2}}}\right) \ ,$$

$\delta_i = \overline{\lim_{u\to\infty}} \sup_T|q_i(T,u)-q_{i+1}(T,u)|$. It follows from (6.3), (6.4) and Billingsley (1968) pg.31-33 that the measures $e^{\gamma u}b\,dK_u(b,t)$ are bounded and converge weakly to the bounded measure $b\,dK(b,t)$. Invoking the continuity of the limit and Lemma 6.1 we may conclude that

$$\delta_1 \leq \overline{\lim_{u\to\infty}} \sup_{-\infty<x<\infty} \int_0^{\infty}\int_{x-u^{-\frac{1}{4}}}^{x}(c_1+c_2b)e^{\gamma u}dK_u(b,s) = 0$$

and similarly $\delta_3 = 0$ since $\tilde{k}_b = \lim_{t\to\infty} U*k_b(t) \le c_1 + c_2 b$.
Also $\delta_4 = 0$ is an immediate consequence.
Finally

$$\delta_2 \le \overline{\lim_{u\to\infty}}\, e^{\gamma u} \int_0^\infty \int_{-\infty}^\infty \sup_{t \ge \bar{K} u^{1/4}} |U*k_{u,b}(t) - \tilde{k}_b| \, dK_u(b,s) = 0 ,$$

using tightness and Lemma 6.2. Thus the theorem is proved except that we need to show that $C^+ E\tilde{k}_{B^+} = C_1$. Presumably this could be verified directly, but it seems easier just to invoke the validity of the argument for $T = \infty$ and (6.1). □

In view of models like D/G/1, the case of a lattice $F^{(2)}$ is not entirely without interest. An argument just along the above lines works, only one has to apply the lattice version of the key renewal theorem.

7. Conditioned limit theorems for the GI/G/1 queue in the steady state.

We shall only consider the discrete time case and the actual waiting time. As is well-known (e.g. Miller (1972) and Breiman (1968) Prop.6.5), the steady state can be represented as a strictly stationary process $\{W_n\}_{-\infty<n<\infty}$ with doubly infinite time scale and governing probability measure say P^e. Our object of study is $P_u^e = P^e(\cdot|W_0 > u)$.

Assume that the busy cycle comprising $N = 0$ started at time $-\underline{m} = \sup\{n \le 0 : W_n = 0\}$ and define

$$\tilde{H}(t_1,t_2) = \frac{1}{\underline{m}} \sum_{n=-\underline{m}}^{-1} I(X_n^{(1)} \le t_1 , X_n^{(2)} \le t_2) ,$$

$$\tilde{\zeta}^{(1)}(t_1) = \tilde{H}(t_1,\infty) , \tilde{\zeta}^{(2)}(t_2) = \tilde{H}(\infty,t_2) .$$ Then:

Corollary 7.1 (i) $\|\tilde{H} - \bar{F}^{(1)} \otimes \bar{F}^{(2)}\| \overset{P_u^e}{\to} 0$;

(ii) $(\zeta^{(1)} , \zeta^{(2)}) \overset{d_u^e}{\Rightarrow} (\zeta^{(1)} , \zeta^{(2)})$ *in* $D(-\infty,\infty) \times D(-\infty,\infty)$.

Corollary 7.2 $$\left(\left(\frac{W_{[(t-1)\underline{m}]}-tu}{(\bar{\sigma}^2u/\bar{\mu})^{\frac{1}{2}}}\right)_{0\le t\le 1} , \frac{\underline{m}-u/\bar{\mu}}{(\bar{\sigma}^2u/\bar{\mu}^3)^{\frac{1}{2}}}\right) \overset{d_u^e}{\Rightarrow} (\overset{\circ}{\xi},-\xi(1)) .$$

Again, the two proofs follows just the same lines so we only give the (more difficult)

Proof of Corollary 7.2 Denote the random quantity under investigation by (η_u,Y_u) and let for $0 \le n < \underline{n}$

$$\eta_{u,n}(t) = \frac{W_{[nt]}-tu}{(\bar{\sigma}^2u/\bar{\mu})^{\frac{1}{2}}} , Y_{u,n} = \frac{n-u/\bar{\mu}}{(\bar{\sigma}^2u/\bar{\mu}^3)^{\frac{1}{2}}} .$$

Then for $f \in C_b(D[0,1])$, $g \in C_b(-\infty,\infty)$

$$(7.1) \quad E^e f(\eta_u)g(Y_u)I(W_0 > u) = \frac{1}{E\underline{n}} E \sum_{n=0}^{\underline{n}-1} f(\eta_{u,n})g(Y_{u,n})I(W_n > u)$$

by means of a standard formula for regenerative processes. Appealing to Theorem 5.2 and Remark 5.1 we have

$$(7.2) \qquad (\eta_{u,\nu(u)} , Y_{u,\nu(u)} , B(u)) \overset{d_u^+}{\Rightarrow} (\overset{\circ}{\xi} , -\xi(1) , B^+)$$

with B^+ independent of ξ and distributed as in (5.3).

The idea in the investigation of (7.1) is loosely that only terms with n close to $\nu(u)$ matter. Here $(\eta_{u,n} , Y_{u,n})$ is close to $(\eta_{u,\nu(u)} , Y_{u,\nu(u)})$ so that (7.1) should be approximately

$$(7.3) \qquad C_2 P(\nu(u) < \underline{n}) E_u^+ f(\eta_{u,\nu(u)}) g(Y_{u,\nu(u)}) \cong$$

$$C_3 e^{-\gamma u} E f(\overset{\circ}{\xi}) g(-\xi(1)) \ .$$

Taking $f \equiv 1$, $g \equiv 1$ and combining with (1.1) yields the identification $C = C_3$ and Corollary 7.2. We now proceed to fill in the details, which involve steps similar as when studying (6.2). Define now

$$k_{u,b}(n) = P(S_n + b > 0 , \inf_{0 \le k \le n} S_k > -u-b) ,$$

$$k_b(n) = \lim_{u\to\infty} k_{u,b}(n) = P(S_n + b > 0) ,$$

$$K_u^*(b,n) = P(B(u) \le b , \nu(u) \le n < \underline{n}) ,$$

$$K_u(b,n) = K_u^*(b, u/\bar{\mu} + (\bar{\sigma}^2 u/\bar{\mu}^3)^{\frac{1}{2}} n) \ .$$

Then

$$(7.4) \qquad e^{\gamma u} E \sum_{n=\nu(u)+N}^{\underline{n}-1} I(W_n > u) =$$

$$\int_0^\infty \int_0^\infty \sum_{n=\nu(u)+N}^{\infty} k_{u,b}(n-t) e^{\gamma u} d\tilde{K}_u(b,t) \le$$

$$\int_0^\infty \int_0^\infty \sum_{n=N}^{\infty} k_b(n) e^{\gamma u} d\tilde{K}_u(b,t) \to \int_0^\infty \int_0^\infty \sum_{n=N}^{\infty} k_b(n) dK(b,t) < \infty ,$$

using arguments similar to those of Section 6 to show the existence and finiteness of the limit.

It will now be convenient to restrict attention to functions $f \in C_b(D[0,1])$, $g \in C_b(-\infty,\infty)$ satisfying

(7.5) $\|f\| \le \frac{1}{2}$, $\|g\| \le \frac{1}{2}$; $|f(\eta_1) - f(\eta_2)| \le \|\eta_1 - \eta_2\|$, $|g(y_1) - g(y_2)| \le |y_1 - y_2|$.

In fact, $(\eta_u, Y_u) \overset{d_u^e}{\Rightarrow} (\overset{o}{\xi}, -\xi(1))$ will follow if $E_u^e f(\eta_u) g(Y_u) \to Ef(\xi)g(-\xi(1))$

for all f,g satisfying (7.5), cf. the discussion in Aldous (1978) and the fact that $d(\eta_1,\eta_2) \le \|\eta_1 - \eta_2\|$ for the metric d on D considered by Billingsley (1968). Subject to (7.5), we have for any fixed n

(7.6) $$|E\{f(\eta_{u,\nu(u)+n}) g(Y_{u,\nu(u)+n}) - f(\eta_{u,\nu(u)}) g(Y_{u,\nu(u)})\} I(W_{\nu(u)+n} > u\, ,\, \nu(u)+n < \underline{n})| \le$$

$$EI(\nu(u) < \underline{n})\{\|\eta_{u,\nu(u)+n} - \eta_{u,\nu(u)}\| \wedge 1 + |Y_{u,\nu(u)+n} - Y_{u,\nu(u)}| \wedge 1\} ,$$

(7.7) $$EI(\nu(u) < \underline{n}) |Y_{u,\nu(u)+n} - Y_{u,\nu(u)}| \wedge 1 \le P(\nu(u) < \underline{n}) \frac{n}{(\bar{\sigma}^2 u/\bar{\mu}^3)^{\frac{1}{2}}} = o(e^{-\gamma u}) .$$

The inspection of the first term in (7.6) requires more care. Let first $0 \le t \le \nu(u)/(\nu(u) + n)$. Then $(\nu(u) + n)t = \nu(u)s$ with $0 \le s \le 1$, $|s-t| \le n/\nu(u)$, and defining the modulus of continuity w as in Section 3, we have

(7.8) $$|\eta_{u,\nu(u)+n}(t) - \eta_{u,\nu(u)}(t)| \le \frac{|s-t|u}{(\bar{\sigma}^2 u/\bar{\mu})^{\frac{1}{2}}} + w(\eta_{u,\nu(u)}\, ,\, |s-t|)$$

$$\le \frac{nu/\nu(u)}{(\bar{\sigma}^2 u/\bar{\mu})^{\frac{1}{2}}} + w(\eta_{u,\nu(u)}\, ,\, \frac{n}{\nu(u)}) .$$

Let next $\nu(u)/(\nu(u) + n) < t \le 1$ and define

$m_n = \max_{k=0,\dots,n} |X_{\nu(u)+1} + \dots + X_{\nu(u)+k}| / (\bar{\sigma}^2 u/\bar{\mu})^{\frac{1}{2}}$. Then

(7.9) $$|\eta_{u,\nu(u)+n}(t) - \eta_{u,\nu(u)}(t)| \le \left|\frac{W_{\nu(u)} - tu}{(\bar{\sigma}^2 u/\bar{\mu})^{\frac{1}{2}}} - \eta_{u,\nu(u)}(t)\right| + m_n \le$$

$$\frac{(1-t)u}{(\bar{\sigma}^2 u/\bar{\mu})^{\frac{1}{2}}} + w(\eta_{u,\nu(u)}\, ,\, 1-t) + m_n \le$$

$$\frac{nu/\nu(u)}{(\bar{\sigma}^2 u/\bar{\mu})^{\frac{1}{2}}} + w(\eta_{u,\nu(u)} , \frac{n}{\nu(u)}) + m_n .$$

Combining (7.2), (7.8), (7.9) with the $X_{\nu(u)+k}$ being i.i.d. with d.f. F w.r.t. P_u^+ , we may conclude that $\|\eta_{u,\nu(u)+n} - \eta_{u,\nu(u)}\| \overset{P_u^+}{\to} 0$,

$$(7.10) \quad EI(\nu(u) < \underline{n})\|\eta_{u,\nu(u)+n} - \eta_{u,\nu(u)}\| \wedge 1 =$$

$$P(\nu(u) < \underline{n})E_u^+\|\eta_{u,\nu(u)+n} - \eta_{u,\nu(u)}\| \wedge 1 = O(e^{-\gamma u})o(1) = o(e^{-\gamma u}) .$$

Hence in view of (7.6), (7.7), (7.10)

$$(7.11) \quad Ef(\eta_{u,\nu(u)+n})g(Y_{u,\nu(u)+n})I(W_n > u, \nu(u) + n < \underline{n}) =$$

$$Ef(\eta_{u,\nu(u)})g(Y_{u,\nu(u)})I(W_n > u, \nu(u)+n < \underline{n}) + o(e^{-\gamma u})$$

and combining with (7.1) and (7.4) with N large, it follows that

$$E^e f(\eta_u)g(Y_u)I(W_0 > u) =$$

$$\frac{1}{E\underline{n}}EI(\nu(u) < \underline{n})f(\eta_{u,\nu(u)})g(Y_{u,\nu(u)}) \sum_{n=\nu(u)}^{\underline{n}-1} I(W_n > u) + o(e^{-\gamma u}) =$$

$$C^+/E\underline{n}\ Ef(\mathring{\xi})g(-\xi(1)) \int_0^\infty \int_0^\infty \sum_{n=0}^\infty k_{u,b}(n)dK_u^*(b,m) + o(e^{-\gamma u}) =$$

$$C^+/E\underline{n}\ Ef(\mathring{\xi})g(-\xi(1))\ E \sum_{n=0}^\infty k_{B^+}(n)e^{-\gamma u} + o(e^{-\gamma u}) ,$$

using arguments similar to the proof of Theorem 6.2 for the last step. Hence, since $P^e(W_0 > u) \cong Ce^{-\gamma u}$,

$$E_u^e f(\eta_u)g(Y_u) \to C_4 Ef(\mathring{\xi})g(-\xi(1)) .$$

Taking $f \equiv 1$, $g \equiv 1$ shows that $C_4 = 1$ and completes the proof. □

Individual limit theorems for the steady state are given in Asmussen (1980). It should be noted that at least in some cases the limits are simpler than for the first passage time situation considered in the next section.

8. Some individual limit theorems.

For simplicity, we treat only the random walk setting. Our aim is to investigate the distribution of $(\ldots, X_{\nu(u)-1}, X_{\nu(u)})$ w.r.t. $\bar{P}$ and P_u for u large. Note that strictly speaking we are dealing with sequences of finite length $\nu(u)$: $(\ldots, X_{\nu(u)-1}, X_{\nu(u)}) \overset{d_u}{\Rightarrow} (\ldots, Y_{-1}, Y_0)$ in $\mathbb{R}^{\{\ldots,-1,0\}}$ means that for any $n < \infty$ and any $g \in C_b(\mathbb{R}^{n+1})$

$$(8.1) \qquad E_u g(X_{\nu(u)-n}, \ldots, X_{\nu(u)}) I(\nu(u) > n) \to Eg(Y_{-n}, \ldots, Y_0)$$

An alternative formalism is the standard device of introducing an extra point Δ and working with the random element $(\ldots, \Delta, \Delta, X_1, \ldots, X_{\nu(u)})$ of $(\mathbb{R} \cup \{\Delta\})^{\{\ldots,-1,0\}}$ so that (8.1) asserts the existence of a limit concentrated on $\mathbb{R}^{\{\ldots,-1,0\}}$. Similar remarks apply to the interpretation of many results and arguments in the following.

From the point of view of the rest of the paper, the main result is the following:

Theorem 8.1 *There exists a random element* $(\ldots, Y_{-1}, Y_0)$ *of* $\mathbb{R}^{\{\ldots,-1,0\}}$ *such that* $(\ldots, X_{\nu(u)-1}, X_{\nu(u)}) \overset{d_u}{\Rightarrow} (\ldots, Y_{-1}, Y_0)$ *as* $u \to \infty$ *and the* Y_{-n} *have the property that* $(\ldots, Y_{-n-1}, Y_{-n}) \Rightarrow (\ldots, Z_{-1}, Z_0)$ *as* $n \to \infty$, *with the* Z_{-n} *i.i.d. with common d.f.* $\bar{F}$.

(that the d.f. of Y_k, with k fixed and finite, cannot be $\bar{F}$ is a fact related to the waiting time paradox; e.g. for $k = 0$ we have $P(Y_0 < 0) \le \overline{\lim}_{u\to\infty} P_u(X_{\nu(u)} < 0) = 0$ whereas $\bar{F}(0-0) > 0$) . However, we shall also be able to give a complete explicit description of the distribution of $(\ldots, Y_{-1}, Y_0)$ as well as related $\bar{P}$-properties.

In view of Lemma 2.2, the problem is equivalent to the study of the $\bar{P}$-distribution of $(B(u), \ldots, X_{\nu(u)-1}, X_{\nu(u)})$, to which we first address ourselves.

The idea is to decompose the sequence $\{X_{\nu(u)-n}\}$ into blocks separated by the ladder epochs $0 = \tau(0) < \tau(1) < \tau(2) < \ldots$ Note that $\tau(1) = \nu(0)$ and, more generally, that any $\nu(u)$ can be written as $\nu(u) = \tau(m(u))$. We let $\mathcal{T} = \bigcup_{n=1}^{\infty} \mathbb{R}^n$ be the space of finite sequences $\underset{\sim}{x} = (x_1,\ldots,x_n)$, equipped with the topology as a disjoint union (so that in particular sequence length $|\underset{\sim}{x}| = |(x_1,\ldots,x_n)| = n$ is a continuous function), and define $\underset{\sim}{X}(n) = (X_{\tau(n)+1},\ldots,X_{\tau(n+1)})$

Proposition 8.1 *For any fixed* k ,

$$(B(u),\underset{\sim}{X}(m(u)-k),\ldots,\underset{\sim}{X}(m(u)-1),\underset{\sim}{X}(m(u))) \overset{d}{\Rightarrow} (B(\infty),\underset{\sim}{U}(k),\ldots,\underset{\sim}{U}(1),\tilde{\underset{\sim}{U}}(0))$$

where $\underset{\sim}{U}(1),\ldots,\underset{\sim}{U}(k)$ *are i.i.d. with the common* $\bar{P}$*-distribution of the* $\underset{\sim}{X}(n)$ *and independent of* $(B(\infty),\tilde{\underset{\sim}{U}}(0))$, *the distribution of which is in turn described by*

$$(8.2) \qquad \bar{E}f(B(\infty)\,,\tilde{\underset{\sim}{U}}(0)) = \frac{1}{\bar{E}S_{\tau(1)}}\,\bar{E}\int_0^{S_{\tau(1)}} f(S_{\tau(1)}-u\,,\,\underset{\sim}{X}(0))du \quad .$$

Note that $S_{\tau(1)}$ has the d.f. of the ascending ladder height. Thus with f depending on $B(\infty)$ alone, (8.2) reduces to Lemma 2.3.

Proof Let $f_0 \in C_b([0,\infty) \times \mathcal{T})$, $f_1,\ldots,f_k \in C_b(\mathcal{T})$ be given and define for $i = 0,1,\ldots,k$

$$Z_i(u) = \bar{E}I(m(u)>k, m(u) = i \bmod(k+1))f_0(B(u),\underset{\sim}{X}(m(u)) \prod_{\ell=1}^{k} f_\ell(\underset{\sim}{X}(m(u)-\ell) \ .$$

Then, with the convention above for weak convergence of defect sequences, the assertion of the lemma is equivalent to

$$(8.3) \qquad Z(u) = Z_0(u)+\ldots+Z_k(u) \to \bar{E}f_0(B(\infty),\tilde{\underset{\sim}{U}}(0)) \prod_{\ell=1}^{k} \bar{E}f_\ell(\underset{\sim}{U}(\ell)) \ .$$

Now Z_0 satisfies the renewal equation $Z_0 = z_0+\bar{H}^{*(k+1)}*Z_0$ with

$$z_0(u) = \bar{E}I(S_{\tau(k)} \le u < S_{\tau(k+1)})f_0(S_{\tau(k+1)}-u,\underset{\sim}{X}(k)) \prod_{\ell=1}^{k} f_\ell(\underset{\sim}{X}(k-\ell) \ .$$

Hence applying the key renewal theorem and conditioning upon $\underset{\sim}{X}(1),\ldots,\underset{\sim}{X}(k)$ to compute the integral yields

$$Z_0(\infty) = \lim_{u\to\infty} Z_0(u) = \frac{1}{\bar{h}(k+1)} \int_0^\infty z_0(u)du =$$

$$\frac{1}{\bar{h}(k+1)} \bar{E} \int_{S_{\tau(k)}}^{S_{\tau(k+1)}} f_0(S_{\tau(k+1)}-u,\underset{\sim}{X}(k)) \prod_{\ell=1}^{k} f_\ell(\underset{\sim}{X}(k-\ell))du$$

$$\frac{1}{\bar{h}(k+1)} \bar{E} \int_0^{S_{\tau(1)}} f_0(S_{\tau(1)}-u,\underset{\sim}{X}(0))du\ \bar{E} \prod_{\ell=1}^{k} f_\ell(\underset{\sim}{X}(k-\ell)) =$$

$$\frac{1}{\bar{h}(k+1)} \bar{E}\ f_0(B(\infty),\underset{\sim}{\tilde{U}}(0)) \prod_{\ell=1}^{k} E\ f_\ell(\underset{\sim}{U}(\ell)) .$$

Furthermore, for $i = 1,\ldots,k$

$$Z_i(u) = \int_0^u Z_0(u-y)d\bar{H}^{*i}(y) \to \int_0^\infty Z_0(\infty)d\bar{H}^{*i}(y) = Z_0(\infty) ,$$

using dominated convergence, so that $Z(u) \to (k+1)Z_0(\infty)$ which is equivalent to (8.3). □

Now define the continuous mapping

$$T : \mathcal{T}^{\mathbb{N}} \to (\mathbb{R} \times \{0,1\})^{\{\ldots,-1,0\}}$$

$$\{\underset{\sim}{x}(n)\}_{n\in\mathbb{N}} = \{(x_1^{(n)}\ldots x_{k(n)}^{(n)})\}_{n\in\mathbb{N}} \overset{T}{\to}$$

$$(\ldots,(x_{k(2)}^{(2)},1),(x_1^{(1)},0),\ldots,(x_{k(1)}^{(1)},1),(x_1^{(0)},0),\ldots,(x_{k(0)-1}^{(0)},0),(x_{k(0)}^{(0)},1))$$

(with the obvious extension to defective sequences, say by adding an extra point Δ to $\mathcal{T}$ and $\mathbb{R}\times\{0,1\}$) . Time points with mark 1 may be thought of as ladder epochs. Writing $T(\underset{\sim}{\tilde{U}}(0),\underset{\sim}{U}(1),\underset{\sim}{U}(2),\ldots) = \{(\tilde{Y}_n,i(n))\}_{n\le 0}$, the continuous mapping theorem yields

Corollary 8.1 $(B(u),\ldots,X_{\nu(u)-1},X_{\nu(u)}) \overset{\bar{d}}{\Rightarrow} (B(\infty),\ldots,\tilde{Y}_{-1},\tilde{Y}_0)$

and we shall show

Proposition 8.2 *As* $n \to \infty$, $(B(\infty),\ldots,\tilde{Y}_{-n-1},\tilde{Y}_{-n}) \overset{d}{\Rightarrow} (B(\infty),\ldots,Z_{-1},Z_0)$ *where the* Z_{-k} *are i.i.d. with common d.f.* $\bar{F}$ *and independent of* $B(\infty)$.

Proof. When time is read from $n = 0$ to $n = -\infty$, the process $\{(\tilde{Y}_n,i(n))\}_{n \le 0}$ is made up of independent cycles $0,1,2\ldots$ separated by the k with $i(k) = 1$, and with the property that cycles $1,2,\ldots$ are i.i.d. In fact, the r^{th} cycle is simply $\underset{\sim}{U}(r)$ marked at its end. Thus the distribution of cycle length for $r \ge 1$ is that of $\tau(1)$, which is non-periodic with finite mean, and hence we may conclude by time-reversion and standard results on regenerative processes that the backwards shift $\{(\tilde{Y}_{n-k},i(n-k))\}_{n \le 0}$ for $k \to \infty$ converges weakly to a strictly stationary process $\{(Z_n,j(n))\}_{n \le 0}$. In view of the independence of the $\underset{\sim}{U}(n)$, $n \ge 1$, of $(B(\infty)$, $\tilde{\underset{\sim}{U}}(0))$, it is not difficult to see that the limit is independent of $B(\infty)$. Now let $\ell(n) = I(\tau(m) = n$ for some m). The process $(\{X_n,\ell(n)\})_{n \ge 1}$ is regenerative and hence has a strictly stationary version $\{(Z'_n,j'(n))\}_{-\infty<n<\infty}$ which can be realized as the weak limit of the forwards shift $\{(X_{n+k},\ell(n+k))\}_{n \ge -k}$ as $k \to \infty$ and therefore is easily seen to have the property of the Z'_n being i.i.d. with common d.f. $\bar{F}$. But the process $\{(Z'_n,j'(n))\}_{n \le 0}$ has the same regenerative properties as $\{(\tilde{Y}_n,i(n))\}_{n \le 0}$ (except for the distribution of the first cycle) and in view of its stationarity therefore the same distribution as $\{(Z_n,j(n))\}_{n \le 0}$. Thus the Z_{-k} are i.i.d. with common d.f. $\bar{F}$. □

Proof of Theorem 8.1 It follows from (2.3) and Proposition 8.1 that

$$(\underset{\sim}{X}(m(u)-k),\ldots,\underset{\sim}{X}(m(u)) \overset{d_u}{\Rightarrow} (\underset{\sim}{U}(k),\ldots,\underset{\sim}{U}(1),\underset{\sim}{U}(0)) ,$$

where the $\underset{\sim}{U}(k)$ $(k \geq 1)$ are i.i.d. with the $\bar{P}$-distribution of $\underset{\sim}{X}(n)$ and

$$Eg(\underset{\sim}{U}(0)) = \frac{1}{\bar{E}S_{\tau(1)}} \bar{E} \int_0^{S_{\tau(1)}} e^{-\gamma(S_{\tau(1)}-u)} g(\underset{\sim}{X}(0))du \,/\, C .$$

The rest of the proof is just the same as for Corollary 8.1 and Proposition 8.2. □

9. Bibliographical comments.

As the present paper and its references once more illustrates, the problems and methods in queueing and risk theory bear a considerable resemblance. The simplest example is of course the coincidence of the ruin probability $\psi(u)$ and the tail $P(W_\infty > u)$ of the waiting time d.f. in the steady state with the random walk quantity $P(M > u)$. In the present paper, we have focused on approximations, but also bounds and inequalities have been extensively discussed, frequently without recognizing the interrelation between the areas. Thus the classical Lundberg's inequality $\psi(u) \le e^{-\gamma u}$, extended by Sparre Andersen (1957) to the model considered here, is essentially the same as a queueing inequality by Kingman (1962), (1970), and extensions like those of Ross (1974) and Taylor (1976) apply equally well in both areas. Note that Lundberg's inequality follows immediately from Lemma 2.2 and $\bar{E}e^{-\gamma B(u)} \le 1$ (the same method works also under the condition $Ee^{\gamma X_1} \le 1$ which is slightly weaker than Condition 1.2).

One of the purposes of the calculation of ruin probabilities is to assess the risk premium loading, viz. the excess of c over the critical value $\mu^{(1)}/\mu^{(2)}$, cf. Seal (1969) Ch.5. With a large initial risk reserve u, this will be a small quantity so that from the point of view of queues we are in the heavy traffic situation. If $F^{(1)}$, $F^{(2)}$ are fixed, it follows from Kingman (1965) that as $c \downarrow \mu^{(1)}/\mu^{(2)}$, it holds for any fixed t that

$$(9.1) \qquad \psi\left(\frac{\sigma^{(1)^2} + (\mu^{(1)}\sigma^{(2)}/\mu^{(2)})^2}{c\mu^{(2)} - \mu^{(1)}}t\right) \to e^{-2t}$$

This if also u is fixed and one wishes to determine c so as to make $\psi(u)$ equal to a given small value α, (9.1) suggest to compute c by means of

$$e^{-2t} = \alpha \ , \quad \frac{\sigma^{(1)^2} + (\mu^{(1)}\sigma^{(2)}/\mu^{(2)})^2}{c\mu^{(2)} - \mu^{(1)}} t = u \ , \qquad \text{i.e.}$$

$$(9.2) \qquad c = \frac{\mu^{(1)}}{\mu^{(2)}} + \frac{|\log\alpha|(\mu^{(2)^2}\sigma^{(1)^2} + \mu^{(1)^2}\sigma^{(2)^2})}{2u\mu^{(2)^3}} \ .$$

Note that only the first two moments are involved, whereas γ , c depend heavily on the whole tail of the $F^{(i)}$. Of course, further investigations (into which the author has not looked) are needed concerning the relevance of (9.2) as well as to clarify the relation to classical methods in the actuarial literature.

We finally remark, that the assumption $c > 0$ is not crucial for the above discussion and the results of Section 4. In fact, one may apply the method for reducing to random walks used by Thorin (1971) pg.30 and von Bahr (1974) in the case $c < 0$. The constants become different.

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