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A STOCHASTIC PROJECTION MODEL WITH IMPLICATIONS FOR MULTISTATE DEMOGRAPHY AND MANPOWER ANALYSIS

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Abstract

A time- and age-continuous multistate projection model with exogeneously determined entries is presented. Whereas most other approaches are based on intuitive reasoning in terms of averages, our procedure is purely probabilistic in character. The survivorship proportions of the multistate life table come out as genuine probabilities and the demographic projection method is interpreted as a prediction by mean values. The stocastic unreliability of the projection is described in terms of exact distribution and variance. As a practical illustration, an example from the literature on manpower analysis is reconsidered. Finally, fertility aspects are introduced to provide a wider perspective.

Key words Demography; Multistate demographic population; Projection methods; Survivorship proportions; Continuity in time and age; Manpower systems.

Contents

1. Introduction

Population forecasting involves a great many problems and is subject to various levels of error. A comprehensive discussion of these issues including extensive bibliographical notes has been presented by Hoem (1973). In the present paper we consider the stochastic variation within the framework of a given projection model belonging to a certain class of models. Thus we deal with a specific aspect of the general forecasting question.

The traditional demographic projection problem may briefly be stated in this way: Assume that the state distribution of each cohort is known at time t_1. Given this information we want to make a qualified guess of the corresponding distribution at some future time t_2.

Entry into the population may be determined endogeneously (e.g. through birth) of exogeneously (e.g. through immigration). It may also have both endogeneous and exogeneous components. The time and age parameters of the population process may be either discrete or continuous. The state space is usually finite (with states corresponding to geographical regions, for instance). The projection period $[t_1,t_2]$ may be thought of as a fixed period which is shorter or not very much longer than the average life time of an individual.

Our description of the projection problem leaves out studies of population trends as time approaches infinity, even though others sometimes call them projections as well. Some central probabilistic references concerning the asymptotic properties of demographic populations are Pollard (1973), Keiding and Hoem (1976), Braun (1978) and Cohen (1981).

In the time- and age-discrete case, the projection problem has been subject to treatment in a probabilistic setting by Pollard (1968, 1970), Sykes (1969), Schweder (1971) and others. A stochastic approach to projections with continuous time and age aspects was given by Chiang (1972). The model of Chiang is not strictly continuous in time and age. For instance, all members of an age group observed in a given state at a given time are treated on an equal footing. This means that the central question of the age distribution inside each age interval is disregarded. Other authors such as Keyfitz (1968), Rogers (1975) and Leeson (1980) have used deterministic or semistochastic approaches to the continuous case. A common feature of the conventional continuous time approaches is the formation of what could be called projection coefficients. These coefficients determine the projection of the cohorts existing at the initial time t_1 and are usually taken to be survivorship proportions of the stationary population.

In this paper we present a stochastic population model with continuous time and age parameters and exogeneously determined entries. The population consists of independent individuals whose movements in the finite state space are determined by age-inhomogeneous Markov processes. The entrants are all of the same age and arrive according to one or several time-inhomogeneous Poisson processes. We note that the model may be seen as a time- and age-continuous counterpart to the discrete model of Pollard (1967). Some aspects of time continuous population models with Poisson recruitment have been discussed by Bartholomew (1973, pp. 157-160) and McClean (1976,1978,1980). Within the framework of the model the traditional demographic projection problem is shown to have the nicest possible solution. The above-mentioned projection coefficients (survivorship proportions), for example, turn out to be genuine probabilities. Likewise the conditional mean and variance of the cohort vector at time t_2 are easily computed since the associated conditional distribution is a convolution of multinomial distributions.

After the theoretical exposition an example from manpower analysis given by Leeson (1980) is reconsidered as a practical illustration. A procedure for stochastic model control is expounded. Furthermore, as the traditional matrix multiplication method is found to be inconsistent for this model, a modification of Leeson's projection technique is suggested. In the given situation the practical (numerical) effect of the modification turns out to be small, however. The variance of the projection is calculated and is found to be so small that it can only account for a minor part of the total unrealiability of the projection. This is a concomitant feature of this kind of projection models and the example accentuates the need for model improvements.

The paper will be closed by some remarks on a time continuous stochastic model for multistate populations with <u>endogeneously</u> determined entries. For such a model the traditional projection problem is considerably more difficult, both theoretically and computationally.

2. Theoretical Part

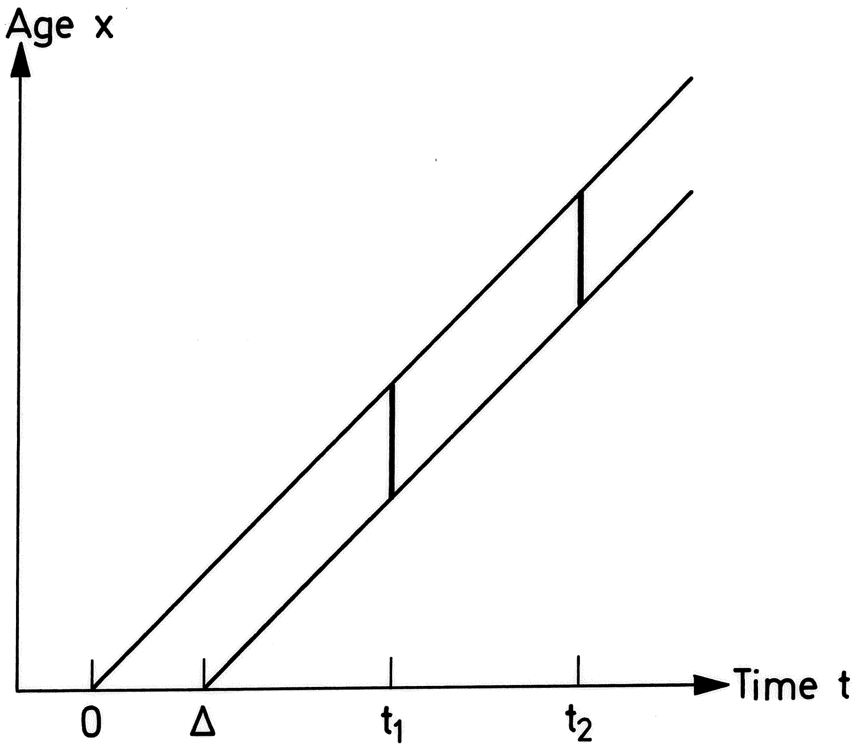
2.1. Model

Consider a population where entrants arrive at exact age (seniority) 0 according to a Poisson process. During some time interval $[0,\Delta]$ the Poisson process has a constant intensity λ. Everybody starts in a common state k. The individuals are mutually independent and each member of the population moves in the finite state space $S = \{1,2,...,n\}$ according to a Markov process with age dependent intensities $(\mu_{ij}(x))_{i \neq j, x \in [0,w]}$ and associated transition probabilities $(P_{ij}(x,y))_{i,j \in S,\ 0 \leq x \leq y \leq w}$. The intensitites are the same for all individuals who arrive during the time interval $[0,\Delta]$.The entry process is independent of the processes of movement in the state space.

2.2 Specification of the Projection Problem

Suppose that we know the cohort parameters λ and $(\mu_{ij}(x))$ and the distribution of the individuals over the state space at time $t_1 \geqq \Delta$ i.e. $(c_i(t_1))_{i \in S}$, where $c_i(t)$ denotes the number of individuals in state i at time t among those arrived during time $[0,\Delta]$. We wish to make a qualified guess of the corresponding distribution $(c_i(t_2))$ at time $t_2 > t_1$.

A. Lexis diagram for projecting an existing cohort.



Note that it is allowable to specify the projection problem separately for each cohort since the model implies that the cohorts are mutually independent.

2.3 Basic Results

Let $c_{ij}(t_1,t_2)$ denote the number of individuals who are in state i at time t_1 and in state j at time t_2. We have the following theorem.

Theorem 1 Conditional on $(c_i(t_1))_{i\in S}$, the variables $(c_{ij}(t_1,t_2))_{j\in S}$, $(c_{2j}(t_1,t_2))_{j\in S}$, ..., $(c_{nj}(t_1,t_2))_{j\in S}$ are mutually independent and $(c_{i1}(t_1,t_2), c_{i2}(t_1,t_2), \ldots, c_{in}(t_1,t_2))$ follows the multinomial distribution with parameters $c_i(t_1)$ and

$(s_{i1}^{(k)},\ldots,s_{in}^{(k)})$ where

$$s_{ij}^{(k)} = \int_{t_1-\Delta}^{t_1} P_{ki}(0,x)P_{ij}(x,x+t_2-t_1)dx / \int_{t_1-\Delta}^{t_1} P_{ki}(0,x)dx \ .$$

It follows that the conditional distribution $L((c_1(t_2),\ldots,c_n(t_2)) \mid (c_1(t_1),\ldots,c_n(t_1)))$ is the convolution of the n multinomial distributions with parameters

$$(c_i(t_1),(s_{i1}^{(k)},\ldots,s_{in}^{(k)})), \ i \in S \ .$$

Thus

$$E((c_1(t_2),\ldots,c_n(t_2)) \mid (c_1(t_1),\ldots,c_n(t_1)))$$
$$= \sum_{i=1}^{n} c_i(t_1)(s_{i1}^{(k)},\ldots,s_{in}^{(k)})$$

and

$$\mathrm{Var}((c_1(t_2),\ldots,c_n(t_2)) \mid (c_1(t_1),\ldots,c_n(t_1))) = \sum_{i=1}^{n} c_i(t_1)V_i^{(k)}$$

where $V_i^{(k)}$ is the matrix $(s_{ij}^{(k)}(\delta_{j\ell} - s_{i\ell}^{(k)}))_{j,\ell \in S}$.

The proof of Theorem 1 is based on a number of theorems concerning point processes (cf. Appendix). The point processes are either Poisson point processes or unnormalized empirical distributions of a given number of independent and identically distributed random variables. An exposition of the theoretical background may be found in Tjur (1980).

Proof of Theorem 1 The entry times in the time interval $[0,\Delta]$ constitute a Poisson point process M with intensity measure given by the constant density λ with respect to Lebesgue measure. From Theorem A3 (see Appendix) we then get that the age distribution at time t_1 is a Poisson point process on $[t_1-\Delta,t_1]$ with intensity measure given by the density λ (see Fig. A). The state i at time t_1 of an individual of exact age x at time t_1 is the outcome of an experiment with sample space $\{1,2,\ldots,n\}$ and point probabilities $P_{k1}(0,x),\ldots,P_{kn}(0,x)$. Applying Theorem A1 we see that the state/age-distribution at time t_1 can be represented by n independent Poisson point processes $M_1,\ldots,M_n$ on $[t_1-\Delta,t_1]$, where M_i has intensity measure given by the density $P_{ki}(0,x)\lambda$. From Theorem A4 we get that the conditional distribution of $(M_1,\ldots,M_n)$ given $(M_i[t_1-\Delta,t_1])_{i\in S} = (c_i(t_1))_{i\in S}$ is equal to the distribution of n independent point processes $\tilde{F}_1,\ldots,\tilde{F}_n$, where $\tilde{F}_i$ is the unnormalized empirical distribution of $c_i(t_1)$ independent, identically distributed random variables on $[t_1-\Delta,t_1]$ with distribution given by the density

$$f_i(x) = P_{ki}(0,x)\cdot\lambda/\int_{t_1-\Delta}^{t_1} P_{ki}(0,y)\lambda dy = P_{ki}(0,x)/\int_{t_1-\Delta}^{t_1} P_{ki}(0,y)\,dy \ .$$

For an individual of exact age x and in state i at time t_1, the state at time t_2 is chosen by an experiment with sample space $\{1,2,\ldots,n\}$ and point probabilities $P_{i1}(x,x+(t_2-t_1)),\ldots,$ $P_{in}(x,x+(t_2-t_1))$. From Theorem A5 we then get that $(c_{i1}(t_1,t_2),\ldots,c_{in}(t_1,t_2))$ follows the multinomial distribution with parameters $(c_i(t_1),(s_{i1}^{(k)},\ldots,s_{in}^{(k)}))$ where

$$s_{ij}^{(k)} = \int_{t_1-\Delta}^{t_1} f_i(x)P_{ij}(x,x+(t_2-t_1))dx$$

$$= \int_{t_1-\Delta}^{t_1} P_{ki}(0,x)P_{ij}(x,x+t_2-t_1)dx/\int_{t_1-\Delta}^{t_1} P_{ki}(0,x)dx \ .$$

The independence of

$$((c_{1j}(t_1,t_2))_{j\in S},\ (c_{2j}(t_1,t_2))_{j\in S},\ldots,(c_{nj}(t_1,t_2))_{j\in S}$$

follows from the independence of $\tilde{F}_1,\ldots,\tilde{F}_n$. □

In demographic terminology the $s_{ij}^{(k)}$'s are survivorship proportions of the multistate life table and they are sometimes called projection coefficients. The projection $(\tilde{c}_i(t_2))$ is given by

$$(\tilde{c}_1(t_2),\ldots,\tilde{c}_n(t_2)) = \sum_{i=1}^{n} c_i(t_1)(s_{i1}^{(k)},\ldots,s_{in}^{(k)})$$

or, if only the individuals in the nonabsorbing states (those "alive") are counted, by

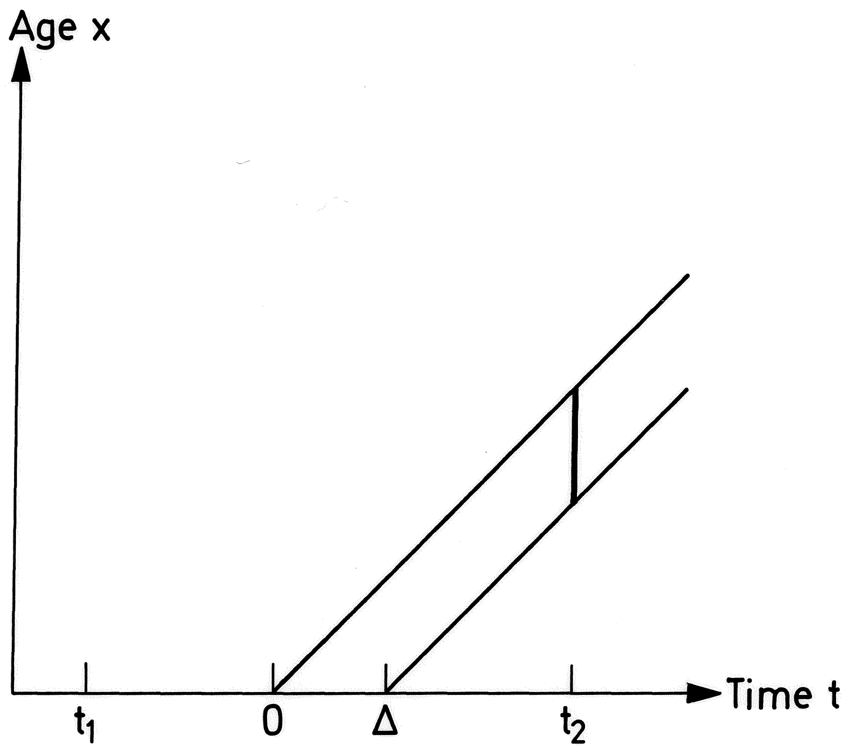
$$(\tilde{c}_1(t_2),\ldots,\tilde{c}_m(t_2)) = \sum_{i=1}^{m} c_i(t_1)(s_{i1}^{(k)},\ldots,s_{im}^{(k)})$$

with $L=\{1,2,\ldots,m\}$ being the set of nonabsorbing states. Within the framework of our model the demographic projection procedure may be characterized as a prediction by mean values.

The entry intensity λ does not appear in the formulas of the projection coefficients, in agreement with the fact that $(c_i(t_1))$ is sufficient for λ in the distribution of $((c_{i1}(t_1,t_2),\ldots,c_{in}(t_1,t_2)))$. Consequently λ need not be known for cohorts existing at time t_1.

By contrast, knowledge of λ will be required to project cohorts not yet born at t_1. To handle this situation imagine that the time interval $[0,\Delta]$ is located between t_1 and t_2 (see Figure B).

B. Lexis diagram for projecting a future cohort.



From the proof of Theorem 1 we immediately get that $c_1(t_2),\dots,c_n(t_2)$ are independent and that $c_i(t_2)$ follows the Poisson distribution with parameter $\lambda \int_{t_2-\Delta}^{t_2} P_{ki}(0,x)\,dx$. The projection then becomes

$$(\tilde{c}_1(t_2),\dots,\tilde{c}_n(t_2)) = (\lambda \int_{t_2-\Delta}^{t_2} P_{k1}(0,x)\,dx,\dots,\lambda \int_{t_2-\Delta}^{t_2} P_{kn}(0,x)\,dx)$$

The unreliability of the projection expressed by the variance of $(c_1(t_2),\dots,c_n(t_2))$ is

$$\mathrm{Var}(c_1(t_2),\dots,c_n(t_2)) = \mathrm{diag}\{\lambda \int_{t_2-\Delta}^{t_2} P_{ki}(0,x)\,dx \mid i = 1,2,\dots,n\}$$

2.4. A Generalization

By relaxing the model assumptions (cf. section 2.1) we get the following straightforward generalization:

1. The entry process is Poisson, now with intensity $\nu(t) = \lambda\,\nu_0(t)$.

2. The entrants may be born into different states. The allocation of the entrants arises from a random classification according to a known, time dependent probability $(\pi_1(t),\dots,\pi_n(t))$.

Comments: For $\nu_0|_{[0,\Delta]} \equiv$ constant and $(\pi_1,\pi_2,\dots,\pi_k,\dots,\pi_n)|_{[0,\Delta]} \equiv (0,0,\dots,1,\dots,0)$ we get the model presented in Section 2. In practical applications we presume that $\pi_i \equiv 0$ for $i \in S\setminus L$.

In the generalized model the projection coefficients are given by

$$s_{ij} = \frac{\sum_{k=1}^{n} \int_{t_1-\Delta}^{t_1} \nu_0(t_1 - x)\pi_k(t_1 - x)P_{ki}(0,x)P_{ij}(x,x + t_2 - t_1)\,dx}{\sum_{k=1}^{n} \int_{t_1-\Delta}^{t_1} \nu_0(t_1 - x)\pi_k(t_1 - x)P_{ki}(0,x)\,dx} \qquad i,j \in S \tag{2.1}$$

and the projection for a cohort not yet born is

$$\tilde{c}_i(t_2) = \sum_{k=1}^{n} \int_{t_2-\Delta}^{t_2} \lambda\,\nu_0(t_2 - x)\pi_k(t_2 - x)P_{ki}(0,x)\,dx \qquad i \in S \tag{2.2}$$

The properties of distributions etc. are strictly parallel to those earlier derived in section 2.3. The proof proceeds along the same lines as the proof of Theorem 1. In the passing it is worth noting that it is fairly straightforward to extend the results to a not necessarily Markovian case (with modified formulas for the projection coefficients, of course).

In the light of conceivable applications it may appear a little strange to construct the entry from a collective "birth" process and a subsequent allocation procedure. However, for the model where the entries follow n state-specific independent Poisson processes with intensities $\nu_1,\ldots,\nu_n$, the above mentioned conclusions hold if we replace $\lambda\cdot\nu_0\cdot\pi_k$ by ν_k for $k=1,\ldots,n$ in the formulas (2.1) and (2.2).

2.5. The Effects on the Coefficients of Projection of a Change in the Initial State Allocation

Let us consider the special case where ν_0 and $(\pi_1,\ldots,\pi_n)$ remain constant during time $[0,\Delta]$. With $(\pi_1,\ldots,\pi_n)$ representing the constant value the projection coefficients are

$$s_{ij} = \frac{\sum_{k=1}^{n} \pi_k \int_{t_1-\Delta}^{t_1} P_{ki}(0,x)\, P_{ij}(x,x+t_2-t_1)\,dx}{\sum_{k=1}^{n} \pi_k \int_{t_1-\Delta}^{t_1} P_{ki}(0,x)\,dx}$$

and this, again, is equal to

$$\frac{\sum_{k=1}^{n} \left(\pi_k \int_{t_1-\Delta}^{t_1} P_{ki}(0,x)\,dx\right) s_{ij}^{(k)}}{\sum_{k=1}^{n} \pi_k \int_{t_1-\Delta}^{t_1} P_{ki}(0,x)\,dx} \qquad i,j \in S \ .$$

Thus the projection coefficient s_{ij} turns out to be a weighted average of the projection coefficients $(s_{ij}^{(k)})_{k\in S}$ of the trivial initial allocations. It is easily seen that the projection coefficients will normally depend on the classification probabilities. This conclusion is fundamentally different from a finding by Rogers (1975, pp.78-81) based on a deterministically oriented analysis of the projection problem for multistate populations. Our result is equivalent to a conclusion made by Ledent (1980, pp.546) stating that the survivorship proportions of a multistate sta-stationary population are dependent on the chosen radix.

2.6. A Note on Generality

It is not our concern in this paper to present results in the widest possible generality. However, we should like to mention that the intensity measure(s) of the entry process(es) need not be absolutely continuous with respect to Lebesgue measure. In fact we may take any finite measure on $[0,\Delta]$ as the intensity measure(s) of the Poissonian entry process(es) and still derive the nice results of Theorem 1 (with obvious modifications of the expressions for the projection coefficients). This observation is important in relation to applications where some, or all, admission takes place at specific moments, provided of course that the Poisson assumption is reasonable. (Note that we use a wide definition of Poisson processes. Some authors only define the Poisson process for a non-atomic intensity measure.) It should be possible, moreover, to relax the assumption of homogenity of the transition intensities inside each cohort. The intensities could then, beside age, depend on exact time of birth. In this situation it would be necessary to make some regularity assumptions about the family $(M_t(\cdot))_{t\in[0,\Delta]}$ of time dependent intensity matrices.

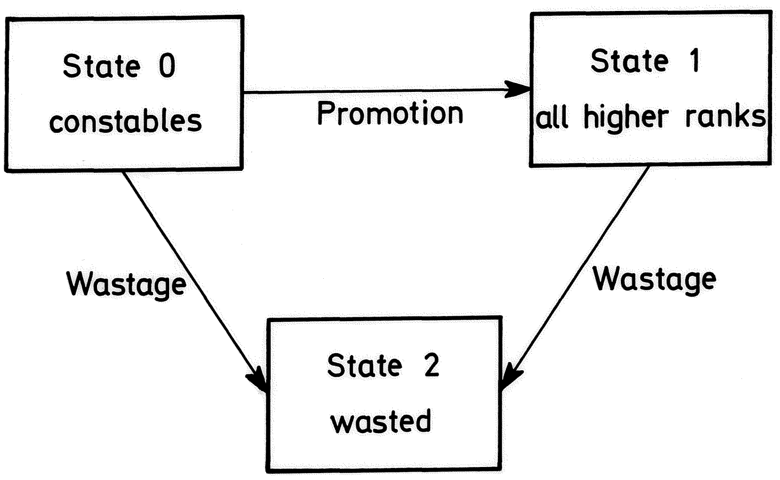
3. A Numerical Example

An example given by Leeson (1980) will now be reconsidered to illustrate the features of the simple projection model from our theoretical section.

3.1. Background

The example is concerned with a manpower system for a group of English County Police Constabularies. There are two occupational states. State 0 embraces constables, and state 1 is entered through promotion from state 0. Demotion is not permitted. State 2 (wastage) may be reached from any of these states. A flow diagram showing states and possible transitions is given in Figure C.

C. Flow diagram of possible transitions in the manpower system.



Entry into the police force leads into state 0. The approximate age of the recruits is 20 years. Starting with a known rank/length-og-service structure as of mid-year 1970 and assuming a simple police-to-population relationship Leeson's objective is to determine the trends in the internal rank/length-of-service structure of the police force. Estimates of future county population sizes are obtained from (unpublished) official projections. The rank/length-of-service-distribution as of mid-year 1970 is shown in Table 1. In reality these figures have been established through interpolation. For the sake of illustration, however, we consider the rounded figures as the result of an authentic enumeration.

1. Rank/Length-of-service distribution for a group of English County Police constabularies as of mid-year 1970.

k	Service interval $[x_k, x_{k+1}[$, years	Constables c_0^k (1970)	All higher ranks c_1^k (1970)	Total c^k (1970)
0	0-5	1662	0	1662
1	5-10	1053	121	1174
2	10-15	657	316	973
3	15-20	499	367	866
4	20-25	486	557	1043
5	25-30	158	150	308
6	30-35	7	61	68
	Total	4522	1572	6094

Source: Leeson (1980)

On the assumptions of a Markov chain on the individual level with constant intensities over five years service intervals, stochastic independence between individuals and homogenity Leeson computes the intensity estimates displayed in Table 2 (see also the figures D and E) based on experience from the period 1968-72. The estimates are (approximate) occurence/exposure rates. Since our purpose is to illustrate the stochastic projection model rather than to present a complete statistical account we will not discuss the estimation procedure or the properties of the estimates any further. The maximum possible service length is 35 years.

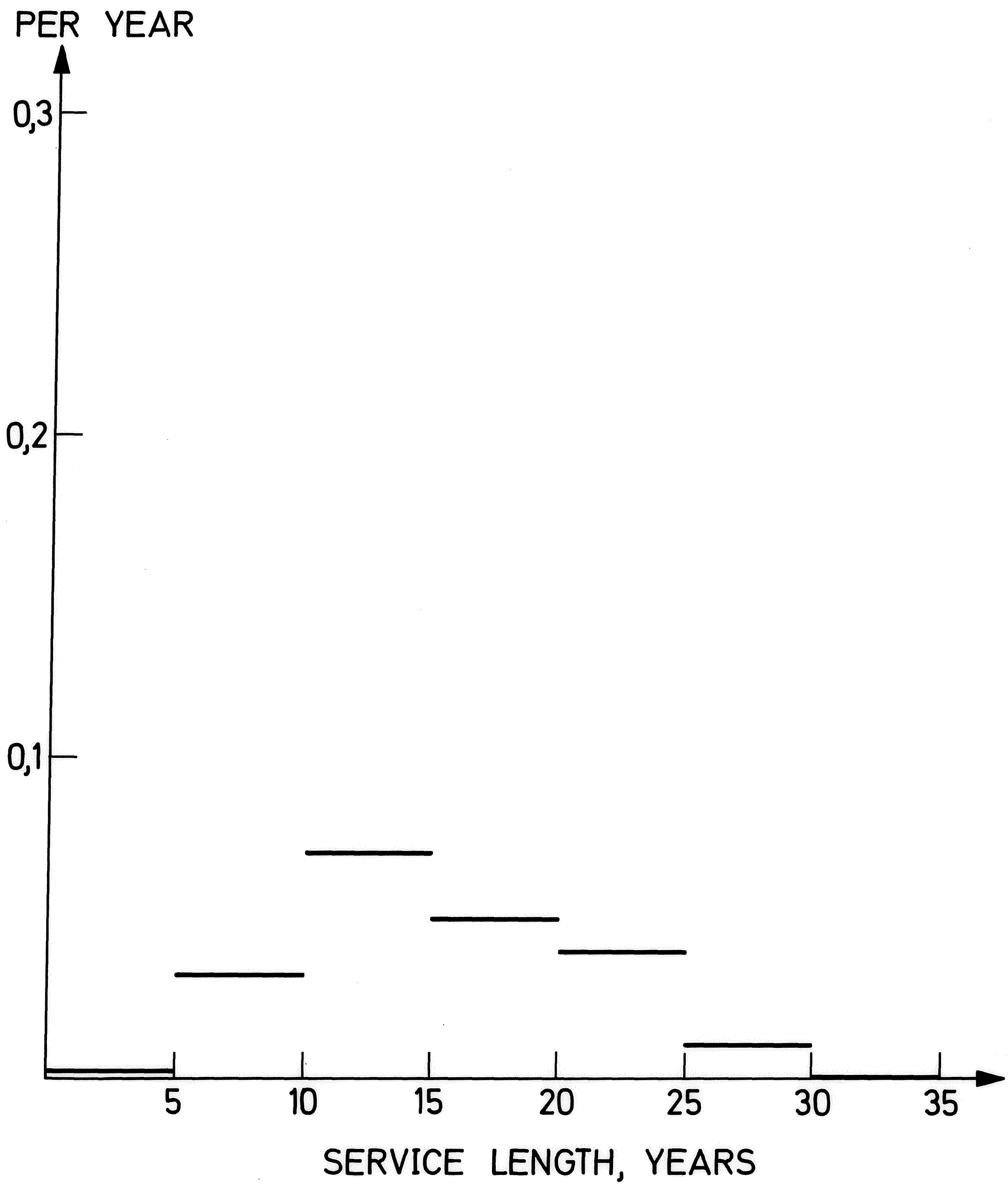
2. Transition intensities based on observed experience in 1968-1972.[a]

k	Service interval $[x_k, x_{k+1}[$	$\hat{f}_0^k$	$\hat{f}_{01}^k$	$\hat{f}_1^k$
0	0-5	0.1191	0.0026	0.0000
1	5-10	0.0665	0.0320	0.0115
2	10-15	0.0863	0.0707	0.0108
3	15-20	0.0534	0.0497	0.0050
4	20-25	0.0485	0.0392	0.0094
5	25-30	0.1218	0.0107	0.1095
6	30-35	0.1122	0.0000	0.3017

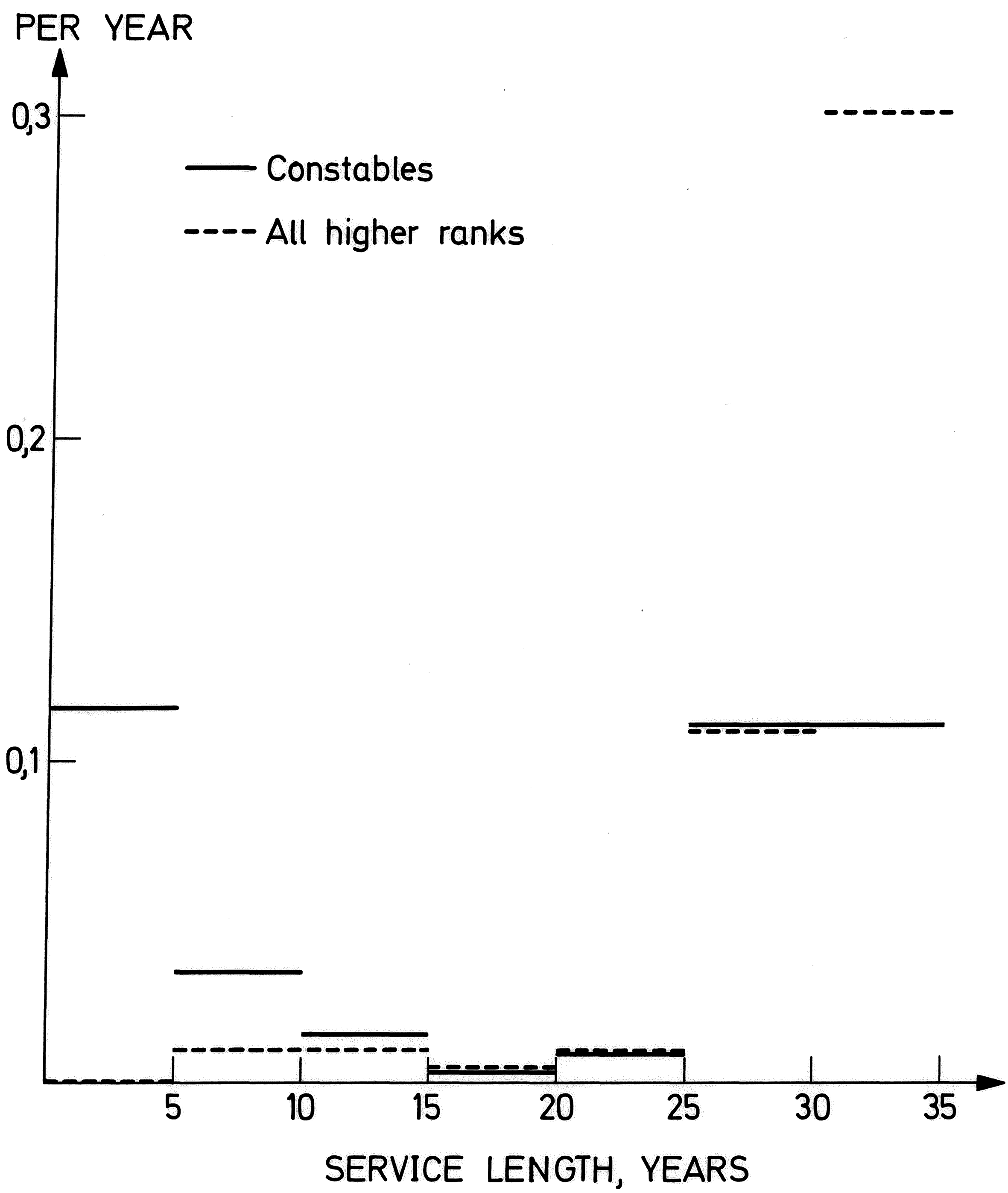
[a] f_0 is the total intensity for state 0, whereas f_{01}^k is the promotion intensity. f_1^k is the total transition intensity for state 1.

Source: Leeson (1980)

D. Estimated promotion intensities (see Table 2)



E. Estimated wastage intensities (see Table 2)



Leeson takes the five-year projection coefficients to be survivorship proportions of the stationary population. The projection is carried out by a stepwise procedure on a five-year basis in resemblance with the conventional demographic matrix method. Since the future size of the police force is predetermined as described above, the appropriate number of recruits may be determined for each five-year period (Table 3)

3. Required recruitment in five-year periods 1970-1990.

Period	Number of recruits
1970-75	3824[a]
1975-80	3509
1980-85	3852
1985-90	4043

[a] Reconstructed by the present author.

Source: Leeson (1980)

3.2. A supplement to the Manpower Model

To provide a stringent foundation for the projection method used for each cohort the model must be supplemented with some further assumptions. According to our theoretical part (Section 2) and disregarding that intensity estimates are used and that there are some special problems connected with the stepwise procedure (cf. Section 3.6), it will be sufficient to add:

The recruitment takes place according to a Poisson process which is independent of the processes of movement in the state space and have constant intensity over five-year periods.

3.3. On the Applicability of the Model

The model clearly is not a realistic description of the mechanisms of recruitment, promotion and wastage in the police force. From a pragmatic standpoint, however, the model might be applicable in certain situations. Considerations concerning this point should be based on thoroughgoing studies of institutional conditions and the given purpose, if possible supplemented with informative analyses of numerical data. For a discussion of institutional conditions and problem formulations cf. Leeson (1979[a],1979[b],1980). As far as informative analyses of numerical data are concerned a straightforward stochastic model control will be carried out in the following section. Test results concerning time independence of transition intensities within the estimation period 1968-72 are given by Leeson (1979[a],1979[b]). Final conclusions on the applicability of the model in the given situation lie outside the scope of our study. A wider perspective of manpower analysis may be found in Bartholomew and Forbes (1979).

3.4. Model Control

We now perform a simple control of that part of the model which refers to the time period prior to mid-year 1970. First imagine that the transition intensities of Table 2 are the model intensities in stead of estimates.

Let $c_i^k(1970)$ denote the number of constabularies in service interval k and in state i at mid-year 1970, $k=0,1,\ldots,6$; $i=0,1$. The assumptions of the projection model clearly imply that the pairs

$$(c_0^0(1970),c_1^0(1970)),(c_0^1(1970),c_1^1(1970)),\ldots,(c_0^6(1970),c_1^6(1970))$$

are mutually independent. Furthermore,

$$c_0^k(1970),\ c_1^k(1970)$$

are independent and follow Poisson distributions with parameters λ_0^k, λ_1^k. These parameters fulfill the condition

$$\lambda_i^k = \lambda^k \int_0^5 P_{0i}(0,5k+x)\,dx = \lambda^k r_{0i}^k, \qquad i=1,2,$$

where λ^k varies freely and r_{oi}^k is known for $i=1,2$.

We notice that the variable

$$(c^k(1970)) = (c_0^k(1970) + c_1^k(1970))$$

is a sufficient statistic, since

$$L((c_0^k(1970), c_1^k(1970)) \mid c^k(1970))$$

is the multinomial distribution with parameters

$$c^k(1970) \text{ and } (r_{00}^k(1970)/(r_{00}^k(1970)+r_{01}^k(1970)) \,,$$
$$r_{01}^k(1970)/(r_{00}^k(1970)+r_{01}^k(1970))$$

no matter what the value of λ^k is. In addition, since $L(c^k(1970))$ is the Poisson distribution with parameter

$$\lambda^k(r_{00}^k + r_{01}^k), \; \lambda^k \text{ varying freely,}$$

the model for $(c^k(1970))$ is universal (Barndorff-Nielsen, 1978). Then there is some basis (Barndorff-Nielsen, 1978, pp. 62-63) for examining the projection model by testing that $c_0^k(1970)$ has the binomial distribution with parameters

$$c^k(1970) \text{ and } r_{00}^k(1970)/(r_{00}^k(1970)+r_{01}^k(1970))$$

for $k=0,1,\ldots,6$ $(c_0^0(1970), c_0^1(1970), \ldots, c_0^6(1970)$ being independent).

The calculations for the model control are displayed in table 4.

4. Computations for model control.

k	Service interval $x_k - x_{k+1}$	$\frac{r_{00}^k}{r_{00}^k + r_{01}^k}$	$c^k(1970)$	$c_0^k(1970)$	$E(c_0^k(1970))$	Deviation	$\chi^2_{(1)}$
0	0-5	0.9929	1662	1662	1650.22	11.78	11.9
1	5-10	0.9065	1174	1053	1064.29	- 11.29	1.3
2	10-15	0.6970	973	657	678.19	- 21.19	2.2
3	15-20	0.5121	866	499	443.50	55.50	14.2
4	20-25	0.4106	1043	486	428.22	57.78	13.2
5	25-30	0.3617	308	158	111.41	46.59	30.5
6	30-35	0.4442	68	7	30.21	- 23.21	32.1
							105.

Since the 99,95 percent fractile of the χ^2-distribution with 7 degrees of freedom is equal to 26 we are forced to conclude that there is a notable deviation from the model. So far we have been reasoning as if the transition intensities in Table 2 were the model intensities. Recalling that they are only estimates as described in Section 3.1 the interpretation of the model control of Table 4 becomes more involved. However, it will be safe to conclude that the estimated model fails to explain the Rank/ Length-of-service distribution as of mid-year 1970. If we look at the single service intervals we observe that with the exception of the lower service interval the $\chi^2_{(1)}$-value tends to increase with length of service. Thus we get the largest deviations for cohorts with recruitment far back in time. This observation is quite understandable, of course. The signs of the deviations may be of interest. The over- or underrepresentation of promoted persons in the various length of service intervals could perhaps be explained from the historical development of the police force.

3.5. Calculation of Projection Coefficients

We carry out the projection as described in theory in Section 2. The projection coefficients are given by the following formulas:

$$s_{00}(x_k,5m) = P_{00}(x_k,x_{k+m})\,e^{00}_{x_{k+m}:5}/e^{00}_{x_k:5}$$

$$s_{01}(x_k,5m) = [P_{00}(x_k,x_{k+m})\,e^{01}_{x_{k+m}:5}+P_{00}(x_k,x_{k+1})P_{01}(x_{k+1},x_{k+m})\,e''_{x_{k+m}:5}$$

$$+ P_{11}(x_{k+1},x_{k+m})\frac{f^k_{01}}{f^k_0-f^k_{12}}$$

$$\left(\frac{P_{00}(x_k,x_{k+1})P_{11}(x_{k+m},x_{k+m+1}) - P_{11}(x_k,x_{k+1})}{f^k_{12}-f^k_0-f^{k+m}_{12}}\right.$$

$$\left.- P_{00}(x_k,x_{k+1})\,e''_{x_{k+m}:5}\right)]/e^{00}_{x_k:5}$$

$$s_{11}(x_k,5m) = [P_{00}(0,x_k)P_{11}(x_{k+1},x_{k+m})\frac{f^k_{01}}{f^k_0-f^k_{12}}(P_{11}(x_k,x_{k+1})\,e''_{x_{k+m}:5}$$

$$- \frac{P_{00}(x_k,x_{k+1})P_{11}(x_{k+m},x_{k+m+1})-P_{11}(x_k,x_{k+1})}{f^k_{12}-f^k_0-f^{k+m}_{12}})$$

$$+ P_{01}(0,x_k)P_{11}(x_k,x_{k+m})\,e''_{x_{k+m}:5}]/(P_{00}(0,x_k)\,e^{01}_{x_k:5}$$

$$+ P_{01}(0,x_k)\,e''_{x_k:5})$$

For the future recruitment cohorts we have the coefficients

$$r^k_{00} = P_{00}(0,x_k)\,e^{00}_{x_k:5}\ ,$$

$$r^k_{01} = P_{00}(0,x_k)\,e^{01}_{x_k:5} + P_{01}(0,x_k)\,e''_{x_k:5}\ .$$

$((s_{ij}(x_k,s))_{ij}$ denotes the coefficients used for a s year projections of the cohort with seniority $[x_k,x_{k+1}]$ at t_1, and $e^{ij}_{x:t}$ is the expected time spent in state j between seniority x and seniority $x+t$ by a person present in state i at age x).

The derivation of the formulas is based on repeated use of the Chapman-Kolmogorov equation as described by Leeson (1980). Observe that the entities included in the formulas are conventional computation results from standard demographic computer programs based on piecewise constant intensities (cf. e.g. Hansen 1981). Access to such a computer program considerably facilitates the computational work.

3.6. The Inconsistency of the Matrix Multiplication Method for the Calculation of Projection Coefficients

As far as the coefficients for five-year projections are concerned there is a complete agreement between Leeson's formulas and ours. For time horizons of 10,15 and 20 years Leeson's projection coefficients are obtained by matrix multiplication. For these projection periods there is a deviation from our results. This is connected with the fact that the movement process $(S_t)_{t \geq \Delta}$ (S_t = state at time t) of an individual whose time of entry is a random variable on $[0,\Delta]$, normally is non-Markovian (Pollard 1966,1969). It is easily seen that the matrix multiplication method is consistent for competing risk models (including the traditional single decrement life table), but it does not hold in the case of the police force model.

As a numerical illustration the projection coefficients of a time horizon of twenty years have been computed by both methods. The results are shown in Table 5a. In this example the deviation on the projection coefficients may amount to some 10 percent (consider $s_{11}(10,20)$ in Table 5a). A guess of the number remaining in state 1 by mid-year 1990 among the 316 promoted persons of seniority [10,15] by mid-year 1970 (cf. Table 1), would be 77 if based on the direct method and 85 if based on matrix multiplication.

5. Projection coefficients for a time horizon of twenty years.[a]

a. Coefficients for existing cohorts.

k	x_k	$s_{00}(x_k,20)$[b]	$s_{01}(x_k,20)$ Direct method	$s_{01}(x_k,20)$ Matrix mult.	$s_{11}(x_k,20)$ Direct method	$s_{11}(x_k,20)$ Matrix mult.
0	0	0,231707	0,326617	0,326615	0,846179	0,846460
1	5	0,246364	0,371043	0,367011	0,617984	0,657092
2	10	0,200004	0,143910	0,133455	0,244613	0,268664

b. Coefficients for future recruitment cohort.

k	$[x_k,x_{k+1}]$	r_{00}^k	r_{01}^k
0	0-5	3,767533	0,026905
1	5-10	2,345011	0,241733
2	10-15	1,605498	0,697928
3	15-20	1,126828	1,073475

[a] Computed on basis of the transition intensities of Table 2.

[b] For this coefficient no deviation should be found since it is computed in a pure decrement table.

If, like Leeson, we restrict the projection to embrace the total numbers of constables and promoted persons in each service interval, the bias on the guess will be considerably reduced since the errors on s_{01} and s_{11} are counteracting. Thus the guess of the number of promoted persons of service length [30,35] by mid-year 1990 will be 171.9 and 172.6 respectively (Table 6) with a relative deviation of 0.4 percent. In conclusion there are only minor deviations between the results obtained by either projection method for this particular manpower system by mid-year 1990. The same conclusion holds for similar computations for mid-years 1980 and 1985 (not given here). As is easily seen it will make no difference whether the future recruitment cohorts (entering after mid-year 1970) are subjected to direct or stepwise projection. The same will be valid in the situation where a recruitment cohort existing at mid-year 1970 has the expected relative distribution on states 0 and 1 (i.e. $\chi^2_{(1)} = 0$ in the model control). But this is not likely to happen, of course.

6. Projection of the police force to mid-year 1990 by each method.[a]

k	$[x_k, x_{k+1}]$	Constables	All higher ranks	
			Direct method	Matrix mult.
0	0-5	3046.43	21.76	21.76[b]
1	5-10	1806.60	186.23	186.23[b]
2	10-15	1126.74	489.81	489.81[b]
3	15-20	861.80	820.99	820.99[b]
4	20-25	385.10	542.84	542.83
5	25-30	259.42	465.48	465.97
6	30-35	131.40	171.85	172.58
	Sum	7617.	2699.	2700.

[a] The future recruitment intensities are based on the entry figures of Table 3.

[b] For these figures no deviation should be found since they refer to cohorts not yet recruited by mid-year 1970.

3.7. Projections and Variances of Projections for total Numbers

Projections and variances on projections for (total number of constables, total number of promoted persons) and for the entire police force as of mid-years 1980, 1985 and 1990 are displayed in Table 7.

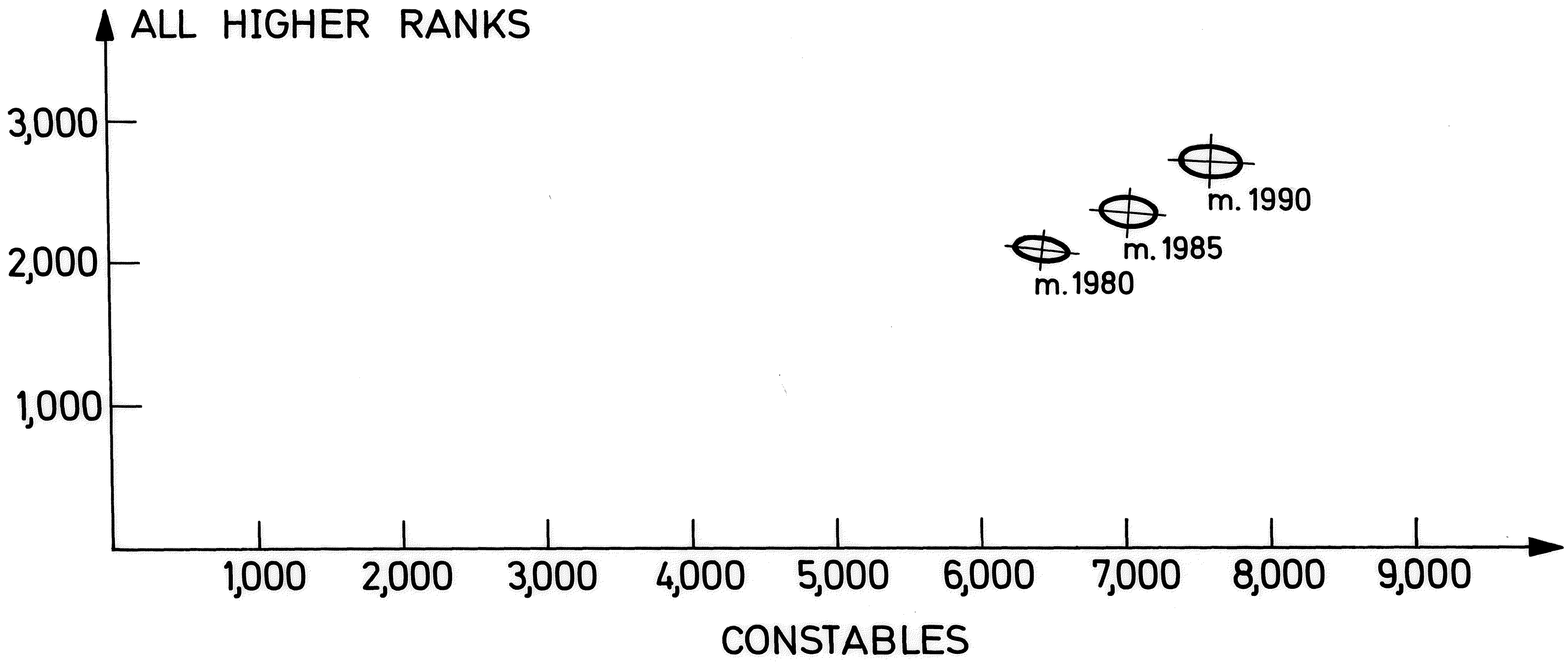
7. Projections, variances of projections etc. for total numbers.[a]

	time: t	mid-year 1980	mid-year 1985	mid-year 1990
Constables	$E(c_0(t))$	6444	7044	7617
$c_0(t)$	$Var(c_0(t))$	5507	6624	7438
	$D(c_0(t))/E(c_0(t))$	1.15 percent	1.16 percent	1.13 percent
	$\sqrt{E(c_0(t))}/E(c_0(t))$	1.25 "	1.19 "	1.15 "
All higher ranks	$E(c_1(t))$	2092	2347	2699
$c_1(t)$	$Var(c_1(t))$	1173	1665	2298
	$D(c_1(t))/E(c_1(t))$	1.64 percent	1.74 percent	1.78 percent
	$\sqrt{E(c_1(t))}/E(c_1(t))$	2.18 "	2.06 "	1.92 "
	$Cov(c_0(t),c_1(t))$	- 511	- 407	- 241
	$\rho(c_0(t),c_1(t))$	- 0.20	- 0.12	- 0.06
Total	$E(c(t))$	8536	9391	10316
$c(t) = c_0(t) + c_1(t)$	$Var(c(t))$	5659	7474	9254
	$D(c(t))/E(c(t))$	0.88 percent	0.92 percent	0.93 percent
	$\sqrt{E(c(t))}/E(c(t))$	1.08 "	1.03 "	0.98 "

[a] The future recruitment intensities are based on the entry figures of Table 3. The variances are calculated inside the projection model defined by these intensities and the transition intensities of Table 2.

It appears that the standard deviation is very small everywhere as compared with the mean value. The coefficient of variation amounts to less than two percent in all cases. By the way, observe that this entity approaches the ratio $\sqrt{\text{mean-value}}/\text{mean-value}$ from below as the Poisson element increases. Another concomitant feature of the projection model is the decreasing correlation between the numbers of constables and promoted persons. The impression of a rather small stochastic variation is sustained by the 95 percent contour ellipses based on normal approximations (Figure F).

F. 95 per cent contour ellipses for the total number of constables and persons of all other ranks, mid-years 1980, 1985 and 1990 (see Table 7).



Informed opinion will probably agree that not least for the longer time horizon the computed projection variances are so small that they can only account for a minor part of the total unrealiability of the projections. In the language of the model, there will be some uncertainty concerning the intensities, and this is not covered by the variances. Even if we are prepared to accept the model, some unrealiability of the projection is caused by lack of knowledge of the real intensities, for in reality the transition intensities used in the computations are estimates only and the future recruitment intensities are based on a (hardly interpretable) method of assessment. By dropping the very restrictive assumption of time independence of the transition intensities, another type of uncertainty is introduced. It seems natural to let the model catch some of the fundamental uncertainty by assuming that the intensities themselves are stochastic. Discrete time studies by Pollard (1968,1970), Sykes (1969), Schweder and Hoem (1972) and Bartholomew (1975) address such issues. For the present we will not go into these questions any further.

4. Some Remarks on the Projection Problem in Relation to a Model with Entry Through Birth

For populations where entry at age 0 is through own reproduction the model assumption of exogeneously determined entry at age 0 is unreasonable. To sketch some problems involved with endogeneously determined entries we specify a new model, as follows.

4.1. Model

The individual level: The individual moves in the state space $S=\{1,2,\dots,n\}$ according to a Markov process $(S_x)_{x\in[0,w]}$ with age-dependent intensities $(\mu_{ij}(x))_{i\neq j,x\in[0,w]}$. Furthermore n^2 Poisson point processes B_{ij} $i,j\in S$ on $[0,w]$ are associated with the individual. The processes $(S_x), B_{11},\dots,B_{nn}$ are mutually independent and the intensity measure of B_{ij} is given by the density $\lambda_{ij}(x)$ with respect to Lebesgue measure for $i,j\in S$. The realized birth process $(\tilde{B}_1,\dots,\tilde{B}_n)$ is constructed by

$$\tilde{B}_j = \sum_{i=1}^{n} 1_{\{S=i\}} B_{ij} \qquad \text{for } j\in S$$

and we interpret the atoms (epochs) of $\tilde{B}_j$ as single births into state j. It is assumed that with probability 1 no explosion takes place. (This assumption was not stated explicitly for the individual movement process in Section 2 but it was meant to hold there as well).

The population level: The individuals are mutually independent and each individual has a behaviour as described above with the same intensities applying to all individuals (or at least to all individuals born in the period $[t_1-w,t_2]$). New individuals are brought into the population through birth in accordance with the interpretation of the realized birth process of an individual. The population is left by the attainment of age w.

4.2. A Demographic Interpretation of the Model

The model may be pictured as relating to the female part of a closed human population. Then only girl births are considered. At least one of the states in S is absorbing (a state of death). In an absorbing state j normally no births can take place so $\lambda_{j\cdot} \equiv 0$ ($\lambda_{j\cdot} = \sum_{i \in S} \lambda_{ji}$). Births into the absorbing states represent stillbirths. The non-absorbing states,which constitute a set denoted L, refer to demographic status such as marital status and/or region of residence. In the gross maternity function $(\lambda_{ij})_{i,j \in L}$ normally some $\lambda_{ij} \equiv 0$. If for example the states of L refer to marital status, everybody is born into the state j_0 = 'never married' and therefore $\lambda_{ij} \equiv 0$ for $i \in L$ and $j \in L \setminus \{j_0\}$. If, alternatively, the states of L refer to regions of residence, a child is born into the state of her mother and we have $\lambda_{ij} \equiv 0$ unless $i = j$. The age w could be thought of as the maximum age attainable if such a concept is accepted, or it may be the highest age considered by the investigator. Of course, w should be higher than the age at menopause if the emphasis is on the population process.

4.3. Relations to Other Models

The model of Section 4.1, like some of the models considered by Braun (1978), is a natural generalization of Kendall's age-dependent birth-and-death process (Kendall 1949) and it embraces the major part of the time and age continuous population models of modern multistate demography (cf. Keyfitz 1980). On the other hand, the model may be seen as a simple multitype Crump-Mode-

Jagers process and the relevant tools are those of branching processes (cf. Jagers 1975).

4.4. Consideration on the Projection Problem

A way of attacking the projection problem could be as follows:

1. Derivation of renewal equations for first and perhaps second moments of the branching process started with a single ancestor in a given age $u \in [0,w]$ and a given state $i \in S$ at time t_1.
2. Aggregation of the means, and perhaps the variances, connected with single ancestors and their descendants to mean and variance of the total population vector at time t_2 (here the word "population vector" denotes the distribution of the population on age groups and states).

Comment: Normally we only consider the population in the states of L (cf. Section 4.2).

The renewal equations can be established by direct derivation. Concerning the aggregation described under 2 it is not clear to the present author how this should be carried out in the case of the traditional demographic projection problem, where the starting point is the population vector at time t_1 rather than full information on the state-age distribution at this time.

Assume, however, that this theoretical problem is solved (or at least bypassed by introducing some further assumptions concerning the age distribution at time t_1) so that one could think of applying the model in practical projection work. Then we would surely be faced with rather cumbersome computations even for a simple model like Kendall's age-dependent birth-and-death process (cf.

e.g. the renewal equations given by Doney (1972)).

The introduction of endogeneously determined entries in the time and age continuous model makes the traditional demographic projection problem considerably more difficult, both theoretically and computationally. This circumstance probably explains the widespread use of time and age discrete counterparts to the model of section 4.1 for projection purposes (compare the discussion of Pollard 1973, pp. 112-113, 130-133).

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Appendix: Some Theorems on Point Processes

Theorem A1 Let M denote a Poisson point process on [a,b] with intensity measure given by the density $\nu(x)$ with respect to Lebesgue measure, and let $p = (p_1, \ldots, p_n) : [a,b] \to [0,1]^n$ denote a continuous function with $\sum_{i=1}^{n} p_i(x) = 1$ for all x. We make a partition of M into n point processes $M_1, \ldots, M_n$ using the following prescription: Every atom (epoch) x of M is to become atom of exactly one of the point processes $M_1, \ldots, M_n$. The point process M_i, where the atom is placed, is chosen by letting i be the outcome of an experiment with sample space $\{1,2,\ldots,n\}$ and point probabilities $p_1(x), \ldots, p_n(x)$. The experiments are performed so that the outcomes for the different atoms are stochastically independent.

The point processes $M_1, \ldots, M_n$ are then stochastically independent and M_1 is a Poisson point process on [a,b] with intensity measure given by the density $\nu_i(x) = p_i(x) \cdot \nu(x)$.

Theorem A2 Let $M_1, \ldots, M_n$ denote stochastically independent Poisson point processes on [a,b] with intensity measures given by the densities $\nu_1(x), \ldots, \nu_n(x)$ with respect to Lebesgue measure. Define $M = M_1 + \ldots + M_n$. Then M is a Poisson point process with intensity measure given by the density $\nu(x) = \nu_1(x) + \ldots + \nu_n(x)$.

Theorem A3 Let M denote a Poisson point process on [a,b] with intensity measure given by the density $\nu(x)$ with respect to Lebesgue measure. Let Θ denote the affine mapping given by $\Theta(x) = \theta - x$, $x \in [a,b]$. Define M_1 by $M_1 = \Theta(M)$.

M_1 is then a Poisson point process on $[\theta - b, \theta - a]$ with intensity measure given by the density $\nu_1(x) = \nu(\theta - x)$ with respect to Lebesgue measure.

Theorem A4 Let $M_1,\dots,M_n$ denote stochastically independent Poisson point processes on [a,b] with intensity measures given by the densities $\nu_1(x),\dots,\nu_n(x)$ with respect to Lebesgue measure. Consider the conditional distribution of $(M_1,\dots,M_n)$ given $(M_1[a,b],\dots,M_n[a,b]) = (c_1,\dots,c_n)$. In the conditonal distribution $M_1,\dots,M_n$ are still stochastically independent and

$$\begin{aligned} &L(M_i \mid (M_1[a,b],\dots,M_n[a,b]) = (c_1,\dots,c_n)) \\ &= L(M_i \mid M_i[a,b] = c_i), \end{aligned}$$

which is the distribution of the unnormalized empirical distribution of c_i independent, identically distributed random variables with distribution given by the density

$$f_i(x) = \nu_i(x)/\int_a^b \nu_i(y)\,dy$$

with respect to Lebesgue measure.

Theorem A5 Let $\tilde{F}$ denote the unnormalized empirical distribution of c independent, identically distributed variables on [a,b] with distribution given by the density f(x) with respect to Lebesgue measure. Let $(p_1,\dots,p_n) : [a,b] \to [0,1]^n$ denote a continuous function with $\sum_{i=1}^{n} p_i(x) = 1$ for all x. We make a partition of $\tilde{F}$ into n point processes $\tilde{F}_1,\dots,\tilde{F}_n$ using the following prescription: Every atom x of $\tilde{F}$ is to become atom of exactly one of the point processes $\tilde{F}_1,\dots,\tilde{F}_n$. The point process $\tilde{F}_i$, where the atom is placed, is chosen by letting i be the outcome of an experiment with sample space $\{1,2,\dots,n\}$ and point probabilities $p_1(x),\dots,p_n(x)$. The experiments are performed so that the outcomes for the different atoms are stochastically independent. Define $(c_1,\dots,c_n) = (\tilde{F}_1[a,b],\dots,\tilde{F}_n[a,b])$. Then $(c_1,\dots,c_n)$ is multinomially distributed with parameters $(c,(q_1,\ldots,q_n))$, where

$$q_i = \int_a^b f(x)\,p_i(x)\,dx \quad .$$

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