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Preprint 1980 No. 3

INSTITUTE OF MATHEMATICAL STATISTICS
UNIVERSITY OF COPENHAGEN

April 1980

# EQUILIBRIUM PROPERTIES OF THE M/G/1 QUEUE

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Summary Various aspects of the equilibrium M/G/1 queue at large values are studied subject to a condition on the service time distribution closely related to the tail to decrease exponentially fast. A simple case considered is the supplementary variables (age and residual life of the current service period), the distribution of which conditioned upon queue length n is shown to have a limit as $n\to\infty$. Similar results hold when conditioning upon large virtual waiting times. More generally, a number of results are given which describe the input and output streams prior to large values e.g. in the sense of weak convergence of the associated point processes and incremental processes. Typically, the behaviour is shown to be that of a different transient M/G/1 queueing model with a certain stochastically larger service time distribution and a larger arrival intensity. The basis of the asymptotic results is a geometrical approximation for the tail of the equilibrium queue length distribution.

## 1. INTRODUCTION

We consider the M/G/1 queue and let $\alpha$ denote the arrival intensity, G the service time distribution and

$$\rho = \alpha\nu \quad (\nu = \nu^1 = \int_0^\infty x dG(x) = \int_0^\infty (1 - G(x))dx)$$

the traffic intensity. We assume throughout $\rho < 1$. It is then well-known that a number of quantities associated with the queueing process at time t converge in distribution as $t \to \infty$. E.g. this holds for the queue size $Q_t$, the virtual waiting time $v_t$ (residual amount of work in the system), the age $A_t$ of the present service time and the residual service time $B_t$ (the $A_t$, $B_t$ are defective, being defined on $\{Q_t > 0\}$ only). The limiting distributions are the <u>equilibrium distributions</u> (e.d.) or <u>steady states</u> and a standard point of view in queueing theory is to measure the characteristics of the system by means of the e.d., cf. e.g. Cox and Smith (1961). This motivates a detailed study of the process at equilibrium.

To facilitate notation, we let $P_e$, $E_e$ refer to the equilibrium case so that e.g.

$$\pi_n = P_e(Q_t = n) = \lim_{s\to\infty} P(Q_s = n)$$

It is well-known and easily proved, cf. Miller (1972), that $P_e$ can also be interpreted as the probability law governing a strictly stationary process.

The investigations of the present paper start in Section 2 by pointing out an estimate

(1.1) $$\pi_n \simeq c\delta^{-n} \quad \text{as } n \to \infty$$

for the equilibrium queue size distribution which, though simple in form and easy to prove, we could not find in the literature. The formula (1.1) should be compared to the exact geometric form $\pi_n = (1-\rho)\rho^n$ in the M/M/1 case and explicit expressions for special cases as $M/E_k/1$ and M/D/1 as given e.g. in Saaty (1961) Ch. 6 (there is also some relation to diffusion approximations, see Section 2). The conditions for (1.1) are the same as the main ones for the rest of the paper, viz. the existence of a solution $\gamma > 0$ to the equation

(1.2) $$\alpha \int_0^\infty e^{\gamma x}(1-G(x))dx = 1$$

with the additional property

(1.3) $$\kappa = \int_0^\infty xe^{\gamma x}(1-G(x))dx < \infty \quad .$$

The connection between $\delta,\gamma,c,\kappa$ is given by

(1.4) $$\alpha(\delta-1) = \gamma, \qquad c = \frac{1-\rho}{\alpha^2\kappa}$$

One should note that conditions similar to (1.2), (1.3) come up in other aspects of queueing theory too, see Kingman (1964), Cohen (1968), (1969), (1973) and Iglehart (1972).

In Section 3, we study the joint e.d. of $Q_t$ and $A_t$, $B_t$, i.e.

$$U_n(\xi) = P_e(A_t \leqq \xi \mid Q_t = n), \quad V_n(\xi) = P_e(B_t \leqq \xi \mid Q_t = n) \quad .$$

The interest in these quantities arise largely from the role of either of $A_t$, $B_t$ as <u>supplementary variables</u>, who in conjunction with $Q_t$ form the minimal information needed to make the process Markovian. In fact, the e.d. of any the above quantities are derivable from the $\pi_n$, $U_n$. As examples, note the formulae

(1.5) $$1 - V_n(\xi) = \int_0^\infty \frac{1 - G(x+\xi)}{1 - G(x)} \, dU_n(x)$$

(1.6) $$P_e(v_t \leqq \xi) = \pi_0 + \sum_{n=1}^\infty \pi_n V_n * G^{*(n-1)}(\xi) \ .$$

However, discussions like those of Gnedenko and Kovalenko (1968) pg. 157-160, Cohen (1969) II.6.2 or Hokstad (1975) place little emphasis on the supplementary variables *per se*. The main result in that direction seems to be that (up to the defect) the marginal e.d. of $A_t, B_t$ coincide with the common e.d. of the backwards and forwards recurrence times in a renewal process with interarrival distribution G. That is (cf. Cohen (1976) Ch. I and Feller (1971) Ch. XI),

(1.7) $$P_e(A_t \leqq \xi) = \sum_{n=1}^\infty \pi_n U_n(\xi)$$

$$= P_e(B_t \leqq \xi) = \sum_{n=1}^\infty \pi_n V_n(\xi) = \frac{1-\rho}{\nu} \int_0^\xi (1 - G(x))\,dx$$

Section 3 starts off by computing the $U_n, V_n$ by means of the embedded Markov chain and the basic formula

(1.8) $$E_e W_t = \frac{1}{E\underline{c}} E \int_0^{\underline{c}} W_s ds$$

(with $\underline{c}$ the busy cycle) for functionals $W_t$ of the process which are regenerative w.r.t. the renewal process formed by the successive ends of busy cycles and satisfies some path conditions automatic in all cases considered in the present paper. Cf. Smith

(1955), Feller (1971) Ch. XI, Miller (1972) and Cohen (1976). The expressions obtained are explicit, though maybe not as simple as one could have hoped from (1.7). However, as one of our main results we show that $U_n, V_n$ have weak limits as $n \to \infty$. A corollary is a similar behaviour of the length $C_t = A_t + B_t$ of the current service period.

These results raise the more general question of the behaviour of the process prior to the large value $Q_t = n$. This problem is the topic of sections 4 and 5, where we obtain a number of limit results describing the entire past, e.g. in terms of the input and output point processes or the incremental processes. In addition to the queue length, we also consider the virtual waiting time, motivated, of course, from the fact that the virtual waiting time in many applications is a more relevant measure of the amount of congestion than the queue length. The results obtained seem to be of a genuinely new type (except that in the M/M/1 case there is a close relation to the well-known time reversibility) and a more detailed statement is deferred to the body of the paper. A typical result is, however, that the input and output point processes prior to a large virtual waiting time behave like two independent stationary point processes, which are, respectively, a Poisson process with intensity $\tilde{\alpha} = \alpha\delta$ (rather than $\alpha$) and a renewal process with interarrival distribution $d\tilde{G}(x) = \delta^{-1} e^{\gamma x} dG(x)$ (rather than $dG(x)$).

## 2. THE IMBEDDED MARKOV CHAIN AND THE QUEUE LENGTH

Unless when considering the equilibrium situation, we suppose that $Q_0 = 0$. Define $\tau(0) = 0$, $\tau(n)$ as the instant where the $n^{th}$ service period is completed. It is then well-known, that $\{X_n\} = \{Q_{\tau(n)}\}$ is a aperiodic positive recurrent Markov chain, the e.d. of which coincides with the e.d. $\{\pi_n\}$ of the queue length $Q_t$. Let $p_n(t)$ be the probability of n arrivals in an interval of length t, $q_n = Ep_n(Z)$ probability of n arrivals during a service period Z, i.e.

$$p_n(t) = e^{-\alpha t} \frac{(\alpha t)^n}{n!} \quad , \quad q_n = \int_0^\infty p_n(t)\,dG(t) \quad .$$

Also let $s_n = 1 - q_0 - \ldots - q_n$. For future reference, we state

2.1 LEMMA <u>The expected amount of time during a service period where n customers have arrived since the start of the period is</u>

$$(2.1) \qquad E\int_0^Z I(n \text{ arrivals in } [0,t])dt = \int_0^\infty p_n(t)(1 - G(t))dt = \frac{s_n}{\alpha} \quad .$$

Indeed, the first equality in (2.1) follows immediately and the second upon integration by parts, noting that $\frac{d}{dt}(1 - p_0(t) - \ldots - p_n(t)) = \alpha p_n(t)$.

Now the transition matrix of $\{X_n\}$ is

$$\begin{pmatrix} q_0 & q_1 & q_2 & \cdots \\ q_0 & q_1 & q_2 & \cdots \\ 0 & q_0 & q_1 & \cdots \\ 0 & 0 & q_0 & \cdots \end{pmatrix}$$

so that the equations determining the $\pi_n$ are $\Sigma_0^\infty \pi_n = 1$ and

$$\pi_0 = q_0\pi_0 + q_0\pi_1$$

(2.2)
$$\pi_1 = q_1\pi_0 + q_1\pi_1 + q_0\pi_2$$
$$\vdots$$
$$\pi_n = q_n\pi_0 + q_n\pi_1 + q_{n-1}\pi_2 + \dots + q_0\pi_{n+1}$$
$$\vdots \qquad .$$

Usually these equations are used to compute the generating function of $\{\pi_n\}$ (see, however, Cinlar (1975) Ch. 6), but here we shall follow a different line of attack. Letting $\overset{*}{\pi}_1 = \pi_0 + \pi_1 (= (1-\rho)/q_0)$, $\overset{*}{\pi}_n = \pi_n$ $n > 1$, $f_n = s_n/q_0$ and summing the equations $0,\dots,n$ yields after some elementary algebra

(2.3)
$$\overset{*}{\pi}_{n+1} = \overset{*}{\pi}_1 f_n + \dots + \overset{*}{\pi}_n f_1 \; .$$

That is, up to a constant $\{\overset{*}{\pi}_{n+1}\}_{n=0,1,2,\dots}$ is <u>renewal sequence</u>, cf. Orey (1971) Ch. 3 and also Kingman (1972), Feller (1968) Ch. XIII. The underlying distribution $\{f_n\}$ is defective since (using $\Sigma_0^\infty nq_n = \rho$)

(2.4)
$$\sum_{n=1}^{\infty} f_n = \frac{1}{q_0}\{\sum_{n=0}^{\infty} s_n - s_0\} = 1 - \frac{1-\rho}{q_0} < 1 \; .$$

This observation permits a standard proof of (1.1). In fact, defining $\delta$ by (1.2), (1.4) and letting $u_n = q_0\delta^n \overset{*}{\pi}_{n+1}/(1-\rho)$ (so that $u_0 = 1$), $g_n = \delta^n f_n$ in (2.3) yields $u_n = u_0 g_n + \dots + u_{n-1}\, g_1$ and hence by (2.1)

$$(2.5)\quad \sum_{n=1}^{\infty} g_n = \frac{1}{q_0}\sum_{n=1}^{\infty}\delta^n s_n = \frac{\alpha}{q_0}\int_0^{\infty}\sum_{n=1}^{\infty}\delta^n p_n(t)(1-G(t))dt =$$

$$\frac{\alpha}{q_0}\int_0^{\infty}\{e^{\alpha t(\delta-1)} - p_0(t)\}(1-G(t))dt = \frac{1}{q_0} - \frac{s_0}{q_0} = 1 ,$$

$$\sum_{n=1}^{\infty} ng_n = \frac{\alpha}{q_0}\int_0^{\infty}\sum_{n=1}^{\infty} n\delta^n p_n(t)(1-G(t))dt =$$

$$\frac{\alpha}{q_0}\int_0^{\infty} e^{\alpha t(\delta-1)}\alpha t\delta(1-G(t))dt = \frac{\alpha^2\delta\kappa}{q_0} < \infty .$$

Hence by discrete time renewal theory (Feller (1968) Ch. XIII), $u_n \to q_0/\alpha^2\delta\kappa$ which is equivalent to (1.1), (1.4).

2.2 REMARK A slightly different proof of (1.1) can be given using the representation of $\{\pi_n\}$ as the distribution of the maximum of a random walk, cf. Prabhu (1965) pg. 127, and appealing to results like those of Spitzer (1964) Ch. IV or Feller (1971) Ch. XII. Some reformulation in the spirit of Iglehart (1972) is needed to compute the constants.

2.3 REMARK In heavy traffic with $\rho$ close to one, it seems reasonable to write $e^{\gamma x} \simeq 1 + \gamma x$ in (1.2), which (with $\nu^2$ the second moment of G) yields $\delta \simeq 1 + 2(1-\rho)/\alpha^2\nu^2$. This agrees with heuristic diffusion approximations (see e.g. Kleinrock (1976) pg. 74-75), where it has been suggested that the $\pi_n$ decrease geometrically at rate

$$\exp\left(-\frac{2(1-\rho)}{\rho+\alpha^2\nu^2/\rho^2-1}\right) \cong 1 - \frac{2(1-\rho)}{\alpha^2\nu^2} \cong \delta^{-1} .$$

When Conditions (1.2), (1.3) fail, the extensive literature on renewal sequences does not seem to provide much information on the tail behaviour of $\{\pi_n\}$. However, the existence of $\delta_1 = \lim_{n\to\infty} \pi_n^{-1/n}$ ($= \lim_{n\to\infty} \pi_n^{*-1/n}$) follows quite easily (cf. Orey (1971) pg. 68) as well as a chain of inequalities:

2.4 PROPOSITION <u>Define</u>

$$\delta_2 = \sup\{\delta \geqq 1 : \alpha \int_0^\infty e^{\alpha(\delta-1)x}(1 - G(x))dx \leqq 1\} ,$$

$$\underline{\delta} = \varliminf_{n\to\infty} \frac{\pi_{n-1}}{\pi_n} \quad \left(= \varliminf_{n\to\infty} \frac{\pi^*_{n-1}}{\pi^*_n}\right) , \quad \overline{\delta} = \varlimsup_{n\to\infty} \frac{\pi_{n-1}}{\pi_n} .$$

<u>Then</u>

(2.6) $$1 \leqq \underline{\delta} \leqq \delta_2 \leqq \delta_1 \leqq \overline{\delta} < \infty ,$$

(2.7) $$\underline{\delta} \geqq \overline{\delta}\left(1 - \frac{1-\rho}{q_0}\right) .$$

PROOF Noting that $f_{n+1}/f_n = (1 - q_0 - \dots - q_{n+1})/(1 - q_0 - \dots - q_n) \leqq 1$, one can check that the arguing of Orey (1971) pg. 89-90 (given for the proper case $\Sigma f_n = 1$) goes through without change to yield $0 < \underline{\delta} \leqq \overline{\delta} < \infty$, while the inequality $\delta_1 \leqq \overline{\delta}$ follows by elementary calculus (writing $\pi_n$ as a telescope product). Since $\alpha \int_0^\infty e^{\alpha(\delta_2-1)x}(1 - G(x))dx \leqq 1$ (by monotone convergence or Fatou's lemma), it follows as in (2.5) that $\Sigma_1^\infty \delta_2^n f_n \leqq 1$ so that $\{\delta_2^{n+1} \pi^*_{n+1}\}_{n=0,1,2,\dots}$ (up to a constant) is a renewal sequence. Hence $\lim(\delta_2^{n+1} \pi^*_{n+1})^{-1/n}$ exists and is at least one (Orey (1971) pg. 68) and $\delta_1 \geqq \delta_2$ follows. Dividing (2.3) by $\pi^*_n$ and using Fatou's lemma yields

$$(2.8)\qquad \frac{1}{\delta} = \underline{\lim}_{n\to\infty} \sum_{k=1}^{n} \frac{\pi^*_{n+1-k}}{\pi^*_n} f_k \geqq \sum_{k=1}^{\infty} \underline{\lim}_{n\to\infty} \frac{\pi^*_{n+1-k}}{\pi^*_n} f_k \geqq \frac{1}{\underline{\delta}} \sum_{k=1}^{\infty} \underline{\delta}^k f_k$$

so that $\Sigma_1^\infty \underline{\delta}^k f_k \leqq 1$ which as in (2.5) implies $\underline{\delta} \leqq \delta_2$. Thus for (2.5), it only remains to prove $\underline{\delta} \geqq 1$. But if $\underline{\delta} < 1$, we may conclude as in Orey (1971) pg. 94-96 that $\Sigma_1^\infty \underline{\delta}^k f_k = 1$, contradicting (2.4). Finally (2.7) follows from (2.8), $\underline{\delta} \geqq 1$ and (2.4). □

## 3. THE SUPPLEMENTARY VARIABLES

We start off by computing $U_n$, $V_n$.

3.1 PROPOSITION *The distributions* $U_1, U_2, \ldots, V_1, V_2, \ldots$ *have densities* $u_n, v_n$ *given by*

(3.1) $$\pi_n u_n(\xi) = \alpha(1 - G(\xi)) \sum_{m=1}^{n} \pi_m^* \, p_{n-m}(\xi)$$

(3.2) $$\pi_n v_n(\xi) = \alpha \int_{\xi}^{\infty} \sum_{m=1}^{n} \pi_m^* \, p_{n-m}(x - \xi)\, dG(x)$$

PROOF We let $W_t = I(Q_t = n, A_t \leqq \xi)$ in (1.8), recall the imbedded Markov chain defined in Section 2 and write

(3.3) $$\int_0^c W_s ds = \sum_{k=0}^{\underline{k}-1} J_k \ , \text{ with } J_k = \int_{\tau(k)}^{\tau(k+1)} W_s ds$$

and $\underline{k}$ the number of customers served during the busy cycle, i.e. the time of the first return of $\{X_n\}$ to 0. Now suppose first $k \geqq 1$. Then a new service period starts at time $\tau(k)$ and $X_k = m$ (say) customers are present. Thus in order for the event $\{A_t \leqq \xi,\ Q_t = n,\ t < \tau(k+1)\}$ to occur, $n - m$ new customers must have arrived within $u = t - \tau(k)$ time units, the service period must not have terminated and we must have $u \leqq \xi$. Conditioning upon $\mathcal{H}_k$ (the $\sigma$-algebra containing all relevant information up to time $\tau(k)$) shows that

$$EJ_k I(1 \leqq k < \underline{k}, X_k = m) = P(1 \leqq k < \underline{k}, X_k = m) \int_0^{\xi} p_{n-m}(u)(1 - G(u))du$$

For $k = 0$ an exponentially distributed period elapses before service starts and a slight modification of the argument yields

$$EJ_0 = \int_0^{\xi} p_{n-1}(u)(1 - G(u))du$$

Thus, combining these expressions by (1.8), (3.3), $E\underline{c} = 1/\alpha(1-\rho)$ and the fact that the expected number of visits of $X_n$ to m before $\underline{k}$ is $\pi_m/\pi_0$, it follows that $P_e(Q_t = n, A_t \leq \xi)$ equals

$$\alpha(1-\rho)\int_0^{\xi}\{p_{n-1}(u) + \sum_{m=1}^{n} E\sum_{k=1}^{\underline{k}-1} I(X_k = m)\, p_{n-m}(u)\}(1-G(u))\,du =$$

$$\alpha(1-\rho)\int_0^{\xi}\{(1+\frac{\pi_1}{\pi_0})p_{n-1}(u) + \sum_{m=2}^{n}\frac{\pi_m}{\pi_0}p_{n-m}(u)\}(1-G(u))\,du =$$

$$\alpha\int_0^{\xi}(1-G(u))\sum_{m=1}^{n}\pi_m^* p_{n-m}(u)\,du$$

and (3.1) follows by differentation. (3.2) could be derived in a similar manner, but follows more directly from (3.1), (1.5). We get

$$\pi_n(1-V_n(\eta)) = \pi_n\int_0^{\infty}\frac{1-G(u+\eta)}{1-G(u)}\,dU_n(u) = \alpha\int_0^{\infty}\sum_{m=1}^{n}\pi_m^* p_{n-m}(u)(1-G(u+\eta))\,du =$$

$$\alpha\int_0^{\infty}\int_0^{\infty}\sum_{m=1}^{n}\pi_m^* p_{n-m}(x-\xi)\, I(\eta \leq \xi \leq x < \infty)\,dG(x)\,d\xi ,$$

which is the same as $\int_\eta^\infty w_n(\xi)\,d\xi$, with $w_n$ the r.h.s. of (3.2). Hence $v_n = w_n$. □

3.2 REMARK In equilibrium, the rate of upcrossings $n \to n+1$ is the same as the rate of downcrossings $n+1 \to n$. Hence $\alpha\pi_n = \pi_{n+1}v_{n+1}(0)$ and it follows that the equilibrium equations for $\{\pi_n\}$, $\{V_n\}$ as given by Gnedenko and Kovalenko (1968) pg. 158 can be written as

(3.4) $$\pi_1 v_1(x) = \alpha\pi_1^*(1 - G(x)) - \alpha\pi_1(1 - V_1(x))$$

(3.5) $$\pi_n v_n(x) = \alpha\pi_{n-1}(1 - V_{n-1}(x)) - \alpha\pi_n(1 - V_n(x)) + \alpha\pi_n(1 - G(x)) .$$

An alternative verification of (3.2) is possible using (3.4), (3.5) and induction. For some purposes, (3.4), (3.5) are quite convenient. Consider e.g. $\mu_n^r$, the $r^{th}$ moment of $V_n$. Then multiplying (3.4), (3.5) by $x^r$ and integrating yields the set

(3.6) $$\pi_1\mu_1^r = \alpha\pi_1^* \frac{\nu^{r+1}}{r+1} - \alpha\pi_1\frac{\mu_1^{r+1}}{r+1} ,$$

(3.7) $$\pi_n\mu_n^r = \alpha\pi_{n-1}\frac{\mu_{n-1}^{r+1}}{r+1} - \alpha\pi_n\frac{\mu_n^{r+1}}{r+1} + \alpha\pi_n\frac{\nu^{r+1}}{r+1} ,$$

of equations (with $\nu^r$ the $r^{th}$ moment of G), which combined with $\mu_n^0 = 1$ determines the $\mu_n$. E.g. in this manner one can check (after some tedious algebra) that

(3.8) $$\mu_n^1 = \frac{1-\rho}{\alpha\pi_n} \sum_{k=n+1}^{\infty} \pi_k , \quad \mu_n^2 = \frac{2(1-\rho)}{\alpha^2\pi_n} \sum_{k=n+2}^{\infty} (k-n-1)\pi_k - \frac{\nu^2}{\pi_n} \sum_{k=n+1}^{\infty} \pi_k .$$

A set of equations similar to (3.4), (3.5) involving the $U_n$ rather than $V_n$ seems only to hold if G is absolutely continuous, cf. Cohen (1969) II.6.2 (adapted to the equilibrium situation). In any case, moments are available directly from (3.1). We omit the details.

We can now easily prove

3.3 THEOREM <u>If Conditions</u> (1.2), (1.3) <u>hold, then the distributions</u> $U_\infty$, $V_\infty$ with densities

$$u_\infty(\xi) = \alpha e^{\gamma\xi}(1 - G(\xi)), \quad v_\infty(\xi) = \alpha \int_\xi^\infty e^{\gamma(x-\xi)} dG(x)$$

<u>are proper</u>, $U_\infty(\infty) = V_\infty(\infty) = 1$, <u>and</u> $u_n(\xi) \to u_\infty(\xi)$, $v_n(\xi) \to v_\infty(\xi)$ $\forall \xi \geq 0$. <u>In particular</u> (cf. Billingsley (1968) pg. 224), $U_n$ <u>and</u> $V_n$ <u>converge weakly and in total variation to</u> $U_\infty$, <u>resp</u>. $V_\infty$. <u>Conversely, if</u> $U_n$ <u>has a proper limit as</u> $n \to \infty$, <u>then Condition</u> (1.2) <u>holds</u>.

PROOF That $U_\infty$ is proper is inherent in (1.2), and that $V_\infty$ is so follows by the obvious integration by parts. Furthermore, from (1.1) it follows that there is a constant $c_1$ such that for all n and k, $\pi^*_{n-k}/\pi_n \leq c_1\delta^k$, and also that $\pi^*_{n-k}/\pi_n \to \delta^k$ as $n \to \infty$. Hence by dominated convergence

$$u_n(\xi) = \alpha(1 - G(\xi)) \sum_{k=0}^{n-1} \frac{\pi^*_{n-k}}{\pi_n} p_k(\xi) \to \alpha(1 - G(\xi)) \sum_{k=0}^{\infty} \delta^k p_k(\xi) = u_\infty(\xi) .$$

In a similar manner it follows that $v_n(\xi) \to v_\infty(\xi)$.

Suppose conversely that $U_n$ has a proper limit $U_\infty$. Then, appealing to (1.5), $V_n$ has a proper limit $V_\infty$. It can be assumed that the support of G is unbounded ( since otherwise (1.2) is automatic) and then passing to the limit in (1.5) shows that $V_\infty$ is not degenerate at zero. Let $\xi$ be some continuity point of $V_\infty$ with $V_\infty(\xi) < 1$. Then integrating (3.5) from 0 to $\xi$ shows that

$$\int_0^\xi v_n(x)\,dx = \alpha\{\frac{\pi_{n-1}}{\pi_n} \int_0^\xi (1 - V_{n-1}) - \int_0^\xi (1 - V_n) + \int_0^\xi (1 - G)\}$$

has a limit (viz. $V_\infty(\xi)$). Since $\int_0^\xi (1 - V_{n-1}) \to \int_0^\xi (1 - V_\infty) \neq 0$, $\pi_{n-1}/\pi_n$ must have a limit, say $\delta$. Let $\tilde{\delta} > \delta$, $\gamma = \alpha(\delta - 1)$, $\tilde{\gamma} = \alpha(\tilde{\delta} - 1)$ and choose K such that $\pi^*_{n-k}/\pi_n \leq K\tilde{\delta}^k$ for all n,k. Then by (3.1), $u_n(\xi)$ is dominated by $K\alpha(1 - G(\xi))e^{\tilde{\gamma}\xi}$ and tends to $\alpha(1 - G(\xi))e^{\gamma\xi}$. Thus for any continuity point x of $U_\infty$,

$$U_\infty(x) = \lim_{n\to\infty} \int_0^x u_n(\xi) = \alpha \int_0^x (1-G(\xi))e^{\gamma\xi}\, d\xi \quad .$$

Letting $x \to \infty$ shows that Condition (1.2) is satisfied. □

One might note as a contrast to (1.7), that in general $U_\infty \neq V_\infty$. A marked difference is that the tail $1 - V_\infty(x)$ tend to decrease more rapidly (always as $o(e^{-\gamma x})$) than $1 - U_\infty(x)$. E.g. (1.2),(1.3) suffice for the existence of all moments of $V_\infty$ but only the mean of $U_\infty$.

As an obvious application of 3.3, consider the length $C_t = A_t + B_t$ of the current service period:

3.4 COROLLARY <u>Conditions</u> (1.2),(1.3) <u>imply the existence of</u> $W_\infty(\xi) = \lim_{n\to\infty} P_e(C_t \leq \xi | Q_t = n)$. $W_\infty$ <u>is larger than</u> G <u>in the stochastical ordering and is absolutely continuous w.r.t.</u> G <u>with density</u>

$$\frac{dW_\infty(\xi)}{dG(\xi)} = \frac{\alpha}{\gamma}\{e^{\gamma\xi} - 1\} \tag{3.9}$$

PROOF Since $\int f dU_n \to \int f dU_\infty$ if f is bounded and a.e. continuous,

$$P_e(C_t > \xi | Q_t = n) = 1 - U_n(\xi) + \int_0^\xi \frac{1-G(\xi)}{1-G(u)} dU_n(u) \to$$

$$1 - U_\infty(\xi) + \int_0^\xi \frac{1-G(\xi)}{1-G(u)} dU_\infty(u) = \alpha\int_\xi^\infty e^{\gamma u}(1-G(u))\,du + \alpha(1-G(\xi))\int_0^\xi e^{\gamma u}\,du =$$

$$\frac{\alpha}{\gamma}\int_\xi^\infty (e^{\gamma u} - 1)\, dG(u) \quad .$$

Since the r.h.s. of (3.9) has G-integral one according to (1.2), it follows that indeed $W_\infty$ exists and has the form (3.9). The stochastical domination follows from the fact that (3.9) is non-decreasing in $\xi$. Indeed, if $W_\infty(\xi) > G(\xi)$ for some $\xi$, then necessarily $\frac{\alpha}{\gamma}\{e^{\gamma\xi} - 1\} > 1$ so that a contradiction results from

$$1 = W_\infty(\xi) + \int_\xi^\infty \frac{\alpha}{\gamma}\{e^{\gamma u} - 1\}\, dG(u) > G(\xi) + \int_\xi^\infty dG(u) = 1 \quad . \qquad \square$$

## 4. THE GROWTH TO LARGE VALUES

The main result of the present section (and one of the main ones of the whole paper) could informally be described by the statement that (in equilibrium and subject to the limit $n \to \infty$) prior to the large value $Q_t = n$, the process has behaved as if the arrival intensity were $\tilde{\alpha} = \alpha\delta$ and the service time distribution were the distribution $\tilde{G}$ with density $\frac{1}{\delta}e^{\gamma t}$ w.r.t. G. Note that $\tilde{G}$ is stochastically larger than G, cf. the proof of 3.4, and that the M/G/1 model specified by $\tilde{\alpha}, \tilde{G}$ is transient since

$$\tilde{\rho} = \tilde{\alpha} \int_0^\infty x d\tilde{G}(x) = \alpha \int_0^\infty x e^{\gamma x} dG(x)$$

$$= \alpha \int_0^\infty \{e^{\gamma x} + \gamma x e^{\gamma x}\}(1 - G(x))dx = 1 + \alpha\gamma\kappa > 1 \ .$$

Various formal statements of this result is possible. We start off in 4.1 with the version readily provided by means of regenerative processes and reformulate two corollaries 4.2, 4.3 in more abstract terms.

In order to be able to describe the whole past prior to t, it will be convenient to take $t = 0$ and assume the equilibrium queue length process represented as a stationary process $\{Q_t\}_{-\infty<t<\infty}$ with double infinite time scale (cf. Breiman (1968) Prop. 6.5) and left-continuous path with right-hand limits. Then the growth prior to 0 is described by means of the random element $(Q_0 - Q_{-t})_{t\geq 0}$ of $D[0,\infty)$. Let $0 > -Y_0 > -Y_0 - Y_1 - > \ldots > -Y_0 - \ldots - Y_j > \ldots$ be the instants in $(-\infty, 0]$ where service is completed, $T_0, T_j$ the number of arrivals in $(-Y_0, 0]$, resp. $(-Y_0 - \ldots - Y_j, Y_0 - \ldots - Y_{j-1}]$, let the arrival instants be of the form $-Y_0 - \ldots - Y_{j-1} - Z_r^j$ with

$0 < Z_1^j < \ldots < Z_{T_j}^j < Y_j$ and let finally $\Phi_j = (Y_j, Z_1^j, \ldots, Z_{T_j}^j)$. Then $\Phi_j$ is a random element of $\Omega = \cup_0^\infty (0,\infty)^{k+1}$, $\Phi_j$ taking its value in the $k^{th}$ component on $\{T_j = k\}$, and equipping $\Omega$ with the obvious topology, we have

4.1 THEOREM Suppose that Conditions (1.2), (1.3) hold. Then for any r, the e.d. of $\Phi_0, \ldots, \Phi_r$ given $Q_0 = n$ has a limit as $n \to \infty$, which can be described as follows: (i) $\Phi_0, \ldots, \Phi_r$ are independent; (ii) the distribution of $Y_j$ is $U_\infty$ for $j = 0$ and $\tilde{G}$ for $j > 0$; (iii) given $Y_j = y$, $T_j$ is Poisson distributed with mean $\tilde{\alpha} y$; and (iv) given $Y_j = y$, $T_j = k$, the distribution function of $Z_1^j, \ldots, Z_k^j$ is $F_{y,k}$, the k-variate d.f. of the order statistics corresponding to k drawings from a uniform distribution on $(0,y)$.

PROOF Let $\Phi_j(t) = (Y_j(t), Z_1^j(t), \ldots, Z_{T_j}^j(t))$ be defined relative to time t rather than time 0, let F(t) be the event that at time t the server is busy and the r preceding service periods fall within the present busy period and define

$$E'(t) = I(Y_j(t) \leqq y_j, T_j(t) = k_j,\ Z_i^j \leqq z_i^j;\ j = 0, \ldots, r, i = 1, \ldots, k_j),$$

$E''(t) = E'(t)F(t)$. Then the assertion amounts to

$$(4.1) \qquad \lim_{n\to\infty} P_e(E'(0) \mid Q_0 = n) =$$

$$\int_0^{y_0} p_{k_0}(\delta u_0) F_{k_0,u_0}(z_1^0, \ldots, z_{k_0}^0)\, dU_\infty(u_0) \ .$$

$$\prod_{j=1}^{r} \int_0^{y_j} p_{k_j}(\delta u_j) F_{k_j,u_j}(z_1^j, \ldots, z_{k_j}^j)\, d\tilde{G}(u_j) \ .$$

Now $I(E''(t), Q_t = n)$ is regenerative and hence $P_e(E''(0), Q_0 = n)$

computable by means of (1.8). We use the imbedded Markov chain in a similar manner as in the proof of 3.1. In order for $E''(t)\{Q_t = n\}$ to occur, $X_k = n - k_0 - \ldots - k_r + r$ customers must have been present at the start $\tau(k)$ of the $r^{th}$ among the preceding service periods and we must have all $n - k_0 - \ldots - k_j + j \geqq 1$ (since otherwise the queue is empty between $\tau(k)$ and t). The latter requirement is satisfied if n is sufficiently large, say $n \geqq k_0 + \ldots + k_j$ and similar arguments as in the proof of 3.1 then yield the expression

$$\alpha(1-\rho)\frac{\pi^*_{n-k_0-\ldots-k_r+r}}{\pi_0}\int_0^{y_0} p_{k_0}(u_0)F_{k_0,u_0}(z_1^0,\ldots,z_{k_0}^0)(1-G(u_0))du_0 \cdot$$

$$\prod_{j=1}^{r}\int_0^{y_j} p_{k_j}(u_j)F_{k_j,u_j}(z_1^j,\ldots,z_{k_j}^j,\ldots,z_{k_j}^j)dG(u_j) ,$$

for $P_e(E''(0), Q_0 = n)$. Dividing by $\pi_n$ and using (1.1) shows that

(4.2)

$$\lim_{n\to\infty} P_e(E''(0) \mid Q_0 = n) =$$

$$\int_0^{y_0} p_{k_0}(u_0)\delta^{k_0}F_{k_0,u_0}(z_1^0,\ldots,z_{k_0}^0)\alpha(1-G(u_0))du_0 \cdot$$

$$\prod_{j=1}^{r}\int_0^{y_j} p_{k_j}(u_j)\delta^{k_j}F_{k_j,u_j}(z_1^j,\ldots,z_{k_j}^j)\frac{1}{\delta}dG(u_j) = \text{r.h.s. of } (4.1) ,$$

using

$$p_k(u)\delta^k = e^{-\alpha u}\frac{(\alpha u\delta)^k}{k!} = e^{\gamma u}p_k(u\delta) .$$

Thus (4.1) will follow if $\lim_{n\to\infty} P_e(F(0) \mid Q_0 = n) = 1$. But summing (4.2) shows that

$$\varliminf_{n\to\infty} P_e(F(0)\,|\,Q_0=n) \geqq \sum_{k_0=0}^{K_0} \cdots \sum_{k_r=0}^{K_r} \text{r.h.s. of (4.1)}$$

which can be taken arbitrarity close to 1 upon choosing $y_0,\dots,y_r,K_0,\dots,K_r$ large enough. □

Let $N_t^{'}, N_t^{''}$ be the number of departures, resp. arrivals in $[-t,0]$ and $N^{'}, N^{''}$ the corresponding point processes, i.e. random elements of the space $N$ of counting measures on $[0,\infty)$. The vague topology on $N$ defines the concept of weak convergence of point processes in the usual manner, cf. e.g. Neveu (1977).

4.2 COROLLARY. *As $n\to\infty$, the e.d. of $(N^{'},N^{''})$ given $Q_0=n$ converges weakly to the distribution of $(K^{'},K^{''})$ where : $K^{'},K^{''}$ are independent; $K^{'}$ is a renewal process with delay distribution $U_\infty$ and interarrival distribution $\tilde{G}$; $K^{''}$ is a stationary Poisson process with intensity $\tilde{\alpha}$.*

Note that (except for special cases like G exponential) $u_\infty(\xi)$ is not proportional to $1-\tilde{G}(\xi)$ and hence $K^{'}$ not stationary. This irregularity is shown to vanish in the set-up of Section 5.

PROOF. The statement of 4.2 is almost obvious from 4.1, but a formal proof may proceed along the following lines. The statement of 4.1 may be reformulated that the e.d. of the sequence $\{\Phi_j\}_{j\in\mathbb{N}}$ given $\{Q_0=n\}$ converges weakly in $\Omega^{\mathbb{N}}$ to the product probability measure $\mu$ described in 4.1, weak convergence in $\Omega^{\mathbb{N}}$ meaning just weak convergence of coordinates $0,\dots,r$ for any r. For $\phi_0,\phi_1,\dots\in\Omega$, write $S=S(\phi_0,\phi_1,\dots)=y_0+y_1+\dots$ , and consider the mapping $\Delta^{'}:\Omega^{\mathbb{N}}\to N$ which takes $\{\phi_j\}$ into the counting measure placing unit weights at the points $y_0+\dots+y_r$ with (say)

$y_0 + \ldots + y_r \leqq S/2$ (i.e. all $y_0 + \ldots + y_r$ if $S = \infty$). It is then a matter of routine to check that $\Delta$ is continuous at every $\{\phi_j\}$ with $S = \infty$ and $\mu$ being concentrated on $\{S = \infty\}$, it follows that the departure process $N' = \Delta'(\Phi_0, \Phi_1, \ldots)$ indeed converges weakly to $K'$. A mapping $\Delta'' : \Omega^{\mathbb{N}} \to \mathcal{N}$ constructed in a similar spirit produces the arrival process and since clearly $(\Delta', \Delta'') : \Omega^{\mathbb{N}} \to \mathcal{N}^2$ maps $\mu$ into the distribution of $(K', K'')$, the proof is complete. □

4.3 COROLLARY <u>As</u> $n \to \infty$, <u>the</u> <u>e.d.</u> <u>of</u> $\{Q_0 - Q_{-t}\}_{t \geqq 0} = \{N_t' - N_t''\}_{t \geqq 0}$ <u>given</u> $\{Q_0 = n\}$ <u>converges</u> <u>weakly</u> <u>in</u> $D[0,\infty)$ (cf. Lindvall (1973)) <u>to</u> <u>the</u> <u>distribution</u> of $\{K_t' - K_t''\}_{t \geqq 0}$.

The proof is an similar obvious application of the continuous mapping theorem.

It is instructive to review the above results in the M/M/1 case, where $1 - G(x) = e^{-\sigma x}$ with $\rho = \alpha/\sigma$. Straightforward calculations then show that $\tilde{\alpha} = \sigma$ and that $1 - U_\infty(x) = 1 - \tilde{G}(x) = e^{-\alpha x}$. Hence by 4.2, 4.3 in the limit, $\{Q_0 - Q_{-t}\}_{t \geqq 0}$ is the difference between two independent stationary Poisson processes with intensities $\alpha$, repsectively $\sigma$. However, it is well-known (Reich (1957)) that $\{Q_t\}_{-\infty < t < \infty}$ is time-reversible at equilibrium. Thus $N', N''$ are the arrival, resp. departure, processes of the time-reversed process. In particular $N'$ is stationary Poisson with intensity $\alpha$. Now conditioning $\{Q_t\}$ on the final value $Q_0 = n$ amounts to starting the time reversed process at n. But the departure process of a M/M/1 queue started at n is readily verified to approach a Poisson process with intensity $\sigma$ as $n \to \infty$. Hence the results of 4.2, 4.3 are exactly the ones implied by the time reversibility.

## 5. THE VIRTUAL WAITING TIME

The notation and main results of Sections 3-4 are used without further reference. Our first objective is to reformulate the results of Section 4 in terms of large virtual waiting times. That is, rather than $\lim_{n\to\infty} P_e(\cdot\,|Q_0=n)$ we consider $\lim_{x\to\infty} P_e(\cdot\,|v_0>x)$. Let $\tilde{\mu}$ denote the mean of $\tilde{G}$.

5.1 THEOREM <u>Suppose that Conditions</u> (1.2), (1.3) <u>hold</u>. <u>Then</u> :

(i) <u>For all</u> $\xi$,

$$\lim_{x\to\infty} P_e(A_t \leq \xi\,|v_t > x) = \tag{5.1}$$

$$\lim_{x\to\infty} P_e(B_t \leq \xi\,|v_t > x) = \frac{1}{\tilde{\mu}}\int_0^{\xi} (1-\tilde{G}(y))\,dy \;.$$

(ii) <u>The e.d. given</u> $v_t > x$ <u>of</u> $(N', N'')$ <u>converges weakly as</u> $x \to \infty$ <u>to the distribution of</u> $(L', L'')$ <u>where</u>: $L', L''$ <u>are independent</u>; $L'$ <u>is a stationary renewal process with interarrival distribution</u> $\tilde{G}$; $L''$ <u>is a stationary Poisson process with intensity</u> $\tilde{\alpha}$.

5.2 REMARK Of course, the r.h.s. of (5.1) represents the stationary wait and delay in a renewal process with interarrival distribution $\tilde{G}$.

PROOF An analogue of (1.1) for $Z(x) = P_e(v_t > x)$ rather than $\pi_n$ is well-known (though not stated in all textbooks on queueing theory), viz.

$$Z(x) \cong de^{-\gamma x} \text{ as } x\to\infty \quad \left(d = \frac{1-\rho}{\alpha\gamma\kappa}\right) \;. \tag{5.2}$$

For a simple proof, note that the renewal equation satisfied by $1-Z$ (cf. e.g. Cohen (1976) pg. 35) coincides with the one treated by Feller (1971) pg. 377-378 (the reason for this is discussed in Seal (1972)) .

We shall also need the estimate

(5.3) $$P_e(v_t > x \mid Q_t = n) = o(e^{-\gamma x}) = o(P_e(v_t > x)) \quad \text{as } x \to \infty$$

valid for any fixed n. In view of (1.6) it suffices to show $1 - G^{*(n-1)}(x) = o(e^{-\gamma x})$. But, using induction and dominated convergence,

$$e^{\gamma x}(1 - G^{*n}(x)) = \int_0^x e^{\gamma(x-y)}(1 - G^{*(n-1)}(x-y))e^{\gamma y}dG(y) \to 0 .$$

Now let the arrivals prior to 0 take place at times $0 > -D_1 > -D_1 - D_2 > \dots$ and define

$$F = \{Y_i \leq y_i \quad i = 1,\dots,r, D_j > \eta_j \quad j = 1,\dots,s\} ,$$

$$f = \lim_{n\to\infty} P_e(F \mid Q_0 = n) = \tilde{G}(y_1)\dots\tilde{G}(y_r)e^{-\alpha(\eta_1 + \dots + \eta_s)} .$$

Then, in view of (1.1), (5.2) and (5.3),

(5.4) $$P_e(F, Y_0 \leq y_0, B_0 > b \mid v_0 > x) =$$

$$\frac{\sum_{n=1}^{\infty} \pi_n P_e(F, Y_0 \leq y_0, B_0 > b, v_0 > x \mid Q_t = n)}{P_e(v_0 > x)} \cong$$

$$d^{-1}e^{\gamma x}\sum_{n=1}^{\infty} c\delta^{-n} f \int_0^{y_0} u_\infty(z)dz \frac{1}{1-G(z)} \int_{z+b}^{\infty} (1-G^{*(n-1)}(x-v+z))dG(v) =$$

$$\gamma f \int_0^{y_0} dz \int_{z+b}^{\infty} e^{\gamma(x-v+z)} \sum_{n=1}^{\infty} \delta^{-n}(1-G^{*(n-1)}(x-v+z)) \; e^{\gamma v}dG(v)$$

Now consider the transient renewal function $U = \Sigma_0^\infty F^{*n}$ where $F = \delta^{-1}G$. Since $\int_0^\infty e^{\gamma x}\, dF(x) = 1$, it follows from Feller (1971) pg. 374-377 that

$$U(\infty) - U(x) \cong \frac{e^{-\gamma x}}{\gamma\int_0^\infty x e^{\gamma x} dF(x)} = \frac{e^{-\gamma x}}{\gamma\tilde{\mu}} .$$

Thus

$$\text{(5.5) r.h.s. of (5.4)} \cong \frac{f}{\delta\tilde{\mu}} \int_0^{y_0} dz \int_{z+b}^\infty e^{\gamma v} dG(v) = \frac{f}{\tilde{\mu}} \int_0^{y_0} (1 - \tilde{G}(z+b))dz$$

Taking first $b = 0$, it follows that the limiting distribution of $Y_0,\ldots,Y_r,D_1,\ldots,D_s$ is as asserted in Part (ii) and Part (ii) follows easily . For Part (i), take first $y_0 = y_1 = \ldots = y_r = \infty$, $\eta_1 = \ldots = \eta_s = 0$ so that $f = 1$ and (5.5) reads

$$P_e(B_0 > b | v_0 > x) \cong \frac{1}{\tilde{\mu}} \int_b^\infty (1 - \tilde{G}(z))dz$$

which is equivalent to the assertion on $B_t$.

For the one on $A_t$, let again $t = 0$, let $F(0)$ be as in the proof of 4.1 with $r = 0$ and recall that $P_e(F(0) | Q_0 = n) \to 1$ as $n \to \infty$. Hence $P_e(F(0) | v_t > x) \to 1$ as $x \to \infty$ in view of (5.3). But $A_0 = Y_0$ on $F(0)$ so that Part (ii) applies. □

The results of Section 4 and 5.1 describe the behaviour of the increments of the queue length process. We next turn to the increments of the virtual waiting time process, which are shown to behave as the difference between a linear function and a compound Poisson process. Let as before the paths of $\{v_t\}_{-\infty<t<\infty}$ be normalized to be left-continuous at the jump points (i.e. times of arrivals) .

5.3 THEOREM <u>Suppose that Conditions</u> (1.2), (1.3) <u>hold. Then as</u> $n \to \infty$, <u>the e.d. given</u> $Q_0 = n$ <u>of</u> $\{v_0 - v_{-t}\}_{t \geq 0}$ <u>converges weakly in</u> $D[0,\infty)$ <u>to the distribution of</u> $\{t - \Sigma_1^{M_t} Z_j\}_{t \geq 0}$, <u>where</u> $M$ <u>is a stationary Poisson process with intensity</u> $\tilde{\alpha}$ <u>and</u> $Z_1, Z_2, \ldots$ <u>are independent of</u> $M$ <u>and i.i.d. with distribution</u> $G$. <u>If rather than</u> $\lim_{n\to\infty} P_e(\cdot | Q_0 = n)$ <u>one considers</u> $\lim_{x\to\infty} P_e(\cdot | v_0 > x)$, <u>the same conclusion holds except that the common distribution of the</u> $Z_j$ <u>is now</u> $\tilde{G}$.

PROOF Let $Z_j$ be the service time of the $j^{th}$ customer arriving before 0 and $M_t$ the number of arrivals in $[-t,0]$. Then the paths of $\{v_0 - v_{-t}\}_{t \geq 0}$ and $\{t - \Sigma_1^{M_t} Z_j\}_{t \geq 0}$ coincide on $[0,\tau]$, with $-\tau$ the last time before 0 where the queue has been empty. It follows from the above results and proofs that, subject to the limits considered, $\tau \to \infty$ in distribution and that the distribution of $M$ is as claimed. Hence the theorem follows in a routine manner once the $Z_j$ in the limit are shown to have the distributional properties asserted.

Now let $F$ be any measurable subset of $N$ and $f$ the probability assigned to $F$ by the Poisson process with intensity $\tilde{\alpha}$. Fix $r, z_1, \ldots, z_r$ and let $H$ be the event that the $r^{th}$ customer arriving before 0 starts his service after 0. It then follows easily from Section 4 that $P_e(H | Q_0 = n) \to 1$ as $n \to \infty$. Hence

$$P_e(M \in F, Z_j \leq z_j \; j = 1,\ldots,r | Q_0 = n) \cong$$

$$P_e(M \in F, Z_j \leq z_j \; j = 1,\ldots,r, H | Q_0 = n) =$$

$$G(z_1)\ldots G(z_r) P_e(M \in F, H | Q_0 = n) \cong$$

$$G(z_1)\ldots G(z_r) P_e(M \in F | Q_0 = n) \cong G(z_1)\ldots G(z_r) f$$

(conditioning upon the past prior to 0 to obtain the equality sign) and the claim follows subject to the conditioning upon $Q_0$. For the one upon $v_0$, we get as in the proof of 5.1

$$P_e(M \in F, Z_j \leqq z_j\ j=1,\dots,r \mid v_0 > x) \cong$$

$$\frac{\sum_{n=1}^{\infty} \pi_n P_e(M \in F, Z_j \leqq z_j\ j=1,\dots,r, v_0 > x, H \mid Q_0 = n)}{P_e(v_0 > x)} \cong$$

$$d^{-1} e^{\gamma x} \sum_{n=r+1}^{\infty} c\delta^{-n} f \int_0^{\infty} dV_\infty(b) \int_0^{z_1} dG(y_1) \dots$$

$$\int_0^{z_r} dG(y_r)\,(1 - G^{*(n-1-r)}(x-b-y_1-\dots-y_r)) \cong$$

$$\frac{\gamma f}{\alpha} \int_0^{\infty} dV_\infty(b) \int_0^{z_1} dG(y_1) \dots \int_0^{z_r} dG(y_r)\, \delta^{-1-r} e^{\gamma x} \sum_{k=0}^{\infty} \delta^{-k}(1-G^{*k}(x-b-y_1-\dots-y_r)) \cong$$

$$\frac{\gamma f}{\alpha} \int_0^{\infty} dV_\infty(b) \int_0^{z_1} dG(y_1) \dots \int_0^{z_r} dG(y_r)\, \delta^{-1-r} \frac{e^{\gamma(b+y_1+\dots+y_r)}}{\gamma\tilde{\mu}} =$$

$$\frac{f}{\delta\alpha\tilde{\mu}} \tilde{G}(z_1) \dots \tilde{G}(z_r) \int_0^{\infty} e^{\gamma b} dV_\infty(b) =$$

$$\frac{f}{\delta\tilde{\mu}} \tilde{G}(z_1) \dots \tilde{G}(z_r) \int_0^{\infty} e^{\gamma b} db \int_b^{\infty} e^{\gamma(\xi-b)} dG(\xi) =$$

$$\frac{f}{\tilde{\mu}} \tilde{G}(z_1) \dots \tilde{G}(z_r) \int_0^{\infty} \xi d\tilde{G}(\xi) = f\, \tilde{G}(z_1) \dots \tilde{G}(z_r)\ . \quad \square$$

ACKNOWLEDGEMENT I would like to thank my colleagues at the Universities of Gothenburg and Copenhagen for stimulating the present research through their comments on the preliminary versions presented to them.

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