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Abstract

Given a Markov chain in discrete time with stationary transition probabilities, consider random times τ determined by the evolution of the Markov chain such that conditionally on the transition performed by the chain from time $\tau - 1$ to time τ either the pre-τ or post-τ process is Markov with stationary transition probabilities that may depend on the value of the transition at time τ. Such random times are called death times, respectively birth times with conditional independence. Various characterisations of all such random times and of some particularly interesting subclasses are given.

1. Introduction and notation

This paper contains generalisations and ramifications of the results presented in Jacobsen and Pitman [3] and is thus concerned with characterisations of certain classes of birth times and death times for a given Markov chain in discrete time with stationary transition probabilities.

The setup and notation to be used are very much like those of [3]: given a countable state space J, let Ω denote the space of all sequences $\omega = (\omega_0, \omega_1, \ldots)$ in J indexed by the non-negative integers N, let $(X_n, n \in N)$ be the coordinate process on Ω, i.e. $X_n(\omega) = \omega_n$, and denote by $(Y_n, n \in N_+)$ the sequence of transitions $Y_n = (X_{n-1}, X_n)$ defined for $n \in N_+ = \{1,2, \ldots\}$. Writing $\mathcal{F}$ for the usual σ-algebra on Ω, a probability P on $(\Omega, \mathcal{F})$ is said to be Markov or Markov (p) if P makes (X_n) a Markov chain with stationary transitions p. If μ is the P-law of X_0, P^μ may be written instead of P and, as is the custom, P^x if μ is degenerate at x. The following convention is adapted throughout: the same letter is used to denote a Markov probability (capital letter) and its transition function (small letter).

Adjoining a state Δ to J, write $J_\Delta = J \cup \{\Delta\}$ and let Ω_Δ be the space of all sequences in J_Δ that remain in Δ once they get there. The life time of a sequence $\omega \in \Omega_\Delta$ is $\zeta(\omega) = \inf\{n \in N : X_n(\omega) = \Delta\}$.

The space Ω_Δ will be used mainly in Section 4 on death times. For objects pertaining to Ω_Δ, the same notation will be used as for the corresponding objects on Ω.

For $n \in N$, the killing operator $K_n : \Omega \to \Omega_\Delta$ and shift operator $\theta_n : \Omega \to \Omega$ are defined by

$$K_n(\omega_0,\omega_1, \ldots) = (\omega_0, \ldots ,\omega_{n-1},\Delta,\Delta, \ldots),$$

$$\theta_n(\omega_0,\omega_1, \ldots) = (\omega_n,\omega_{n+1}, \ldots).$$

A <u>random time</u> is a measurable mapping from Ω to the extended time set $\bar{N} = N \cup \{\infty\}$. Given a random time τ, X_τ, Y_τ, K_τ, θ_τ are defined by local identification, e.g. $X_\tau = X_n$ on $(\tau = n)$. Also, X_τ, K_τ, θ_τ are defined only on the set $(\tau < \infty)$ and Y_τ on $(0 < \tau < \infty)$. As a consequence, for instance $(Y_\tau = (a,b))$ will be the notation for the subset $\{\omega : 0 < \tau(\omega) < \infty, Y_\tau(\omega) = (a,b)\}$ of Ω.

For a fixed $n \in N$ the pre - n σ - algebra $\mathcal{F}_n$ is the σ - algebra spanned by $(X_0, \ldots ,X_n)$. The atoms $\mathcal{A}_n$ are the sets of the form $A = (X_0 = x_0, \ldots ,X_n = x_n)$. For τ a random time, the pre - τ σ - algebra $\mathcal{F}_\tau$ consists of the sets which are countable unions of sets of the form (i) $A\ (\tau = n)$ where $n \in N$, $A \in \mathcal{A}_n$ or (ii) one - point sets $\{\omega\}$ where $\tau(\omega) = \infty$.

A random time τ splits the process (X_n) into two parts, the <u>pre - τ process</u>, conveniently identified with and therefore labelled K_τ, given as

$$(X_n \circ K_\tau,\ n \in N) = (X_0, \ldots ,X_{\tau-1},\Delta,\Delta, \ldots),$$

and the <u>post - τ process</u> θ_τ given as

$$(X_n \circ \theta_\tau,\ n \in N) = (X_\tau,X_{\tau+1}, \ldots).$$

To avoid misunderstandings it is finally stressed that the inclusion symbols $\subset$, $\supset$ always are used to denote non-strict inclusions, allowing for equality.

The purpose of this paper is to study random times such that with respect to a given Markov probability P and subject to a conditioning on Y_τ either K_τ or θ_τ is Markov with stationary transitions. The main difference between the results to be presented here and those contained in [3] consists in allowing the transition function of the Markovian K_τ or θ_τ-fragment to depend on the value of the conditioning variable Y_τ.

In the paper three different types of definitions of random times will be used: (i) operational definitions, (ii) implicitly algebraic definitions and (iii) explicitly algebraic definitions. The first of these three types defines the properties of a random time relative to a Markov probability, while the other two are concerned exclusively with the properties of a random time as a function on Ω. The difference between the implicit case (ii) and the explicit case (iii) may amount for example to a description of the random time involving a collection of parameters which may be chosen independently of each other in the explicit case, but are interrelated in the implicit case, or to a functional equation in the implicit case and the solution to that equation in the explicit case.

For an example, consider stopping times. Firstly, given a Markov probability P, τ is an operationally defined stopping time for P if conditionally on $\mathcal{F}_\tau$ within $(\tau < \infty)$, the post-τ process is Markov with the same transition function as P.

Secondly, a random time τ is an implicitly algebraically defined stopping time if $(\tau = n) \in \mathcal{F}_n$ (or if $(\tau \leqq n) \in \mathcal{F}_n$) for every $n \in N$. (Writing $F_n = (\tau = n)$, $(G_n = (\tau \leqq n))$, the side conditions on the F_n, (G_n) are of course that they be mutually disjoint (increase with n)).

Thirdly, a random time τ is an explicitly algebraically defined stopping time if there is a sequence $(F_n, n \in N)$ of sets $F_n \in \mathcal{F}_n$ such that $\tau(\omega) = \inf\{n \in N : \omega \in F_n\}$.

The characterisation theorems to be given here, as those presented in [3], provide probabilistic equivalences between operationally defined classes of random times on the one side and implicitly algebraically or explicitly algebraically defined classes on the other.

For instance, for stopping times the following result is valid: a random time τ is an operationally defined stopping time for the Markov probability P, if and only if it is P - equivalent to an implicitly or an explicitly algebraically defined stopping time. (This may be readily verified from the results in Section 3 of [3]).

2. CI - birth times and CI - death times

The paper [3] contained characterisations of regular birth times and regular death times. A random time τ is called a regular birth time for the Markov probability P if, relative to P, the post - τ process is Markov (q) for some transition function q with the pre - τ and post - τ processes being conditionally independent given $\tau < \infty$ and X_τ. Similarly τ is called a regular death time for P if the pre - τ process is Markov (r) for some transition function r with the pre - τ and post - τ processes being conditionally independent given $0 < \tau < \infty$ and $X_{\tau-1}$.

An alternative characterisation of regular birth times is of course the following: τ is a regular birth time for P iff there is a transition function q such that conditionally on $\mathcal{F}_\tau$ within $(\tau < \infty)$, the post - τ process is Markov (q) (with initial law trivially degenerate at X_τ).

The class of birth times to be considered in this paper is now obtained by still conditioning on $\mathcal{F}_\tau$ but allowing the transition function of the conditional post - τ process to depend on $Y_\tau = (X_{\tau-1}, X_\tau)$.

Before presenting the precise definition and the analogous definition of the relevant class of death times, I need the following generalisation of Definition 3.11 of [3].

(2.1). Definition. A random time τ is called a conditional independence time for the Markov probability P, if under P the pre - τ and post - τ processes are conditionally independent given Y_τ, i.e.

if there is a conditional distribution of θ_τ given $(X_0, \ldots, X_\tau)$ within $(0 < \tau < \infty)$ or equivalently of K_τ given $(X_{\tau-1}, X_\tau, \ldots)$ within $(0 < \tau < \infty)$, which is a function of Y_τ alone.

Remark. Conditioning on $(X_0, \ldots, X_\tau)$ is equivalent to conditioning on $\mathcal{F}_\tau$ and involves in particular the conditioning on the value of τ. By contrast, conditioning on $(X_{\tau-1}, X_\tau, \ldots)$ does not imply knowledge of the exact value of τ, which implies in particular, as is essential, that the conditional pre - τ process has a random life time.

Whereas two different forms of conditional independence were needed to define regular birth times and regular death times respectively, this new definition applies to both situations.

(2.2). Definition. A random time τ is a birth time with conditional independence (in short a CI - birth time) for the Markov probability P if it is a conditional independence time for P and if conditionally on Y_τ within $(0 < \tau < \infty)$ the post - τ process is Markov with a stationary transition function (depending possibly on Y_τ).

(2.3). Definition. A random time τ is a death time with conditional independence (in short a CI - death time) for the Markov probability P if it is a conditional independence time for P and if conditionally on Y_τ within $(0 < \tau < \infty)$ the pre - τ process is Markov with a stationary transition function (depending possibly on Y_τ).

From time reversal arguments (see below) it is clear that in Definition 2.3 one may instead of the pre - τ process consider the pre - τ process run backwards from time $\tau - 1$.

The main motivation for introducing CI - birth times and CI - death times comes from the basic path decompositions established by Williams [7] for the one - dimensional Brownian motion.

Considering for instance a Brownian motion starting at 0 and killed at the time it hits 1, Williams showed that if τ is the time where the killed path attains its ultimate minimum, then conditionally on the value X_τ of that minimum the pre - τ and post - τ processes are independent, both being Markov with stationary transitions depending on X_τ. Thus, ignoring the discrepancy between the continuous time Brownian motion and the discrete time setup used here, τ is both a CI - birth time and a CI - death time.

The discrete time analogue of Williams' result is not covered by the theory developed in [3], the reason being simply that in the Brownian motion case, although both the conditional pre - τ and post - τ processes are Markov, neither of the unconditional processes are. Hence the need for objects like CI - birth times and CI - death times.

While in the Brownian (continuous path) case, the transitions of the conditional post - τ or pre - τ process depend on X_τ alone, they will in general for a real valued right continuous process with left limits (still considering the time τ of the ultimate minimum) depend on the transition $(X_{\tau-}, X_\tau)$. Translating this into the discrete time situation makes it natural to study K_τ and θ_τ given the transition Y_τ.

For more general results in continuous time that the time of the minimum is a CI-birth time, see Jacobsen [2] and Millar [5].

While it is quite clear that any regular birth time is a CI-birth time, the analogous statement about death times is not so transparent and requires an argument.

Recall that if R is a Markov probability on Ω_Δ with initial measure ν and transition function r, such that (X_n) has positive probability of having finite life time, then the process reversed from the life time defined by the reversal transformation $\rho : \Omega_\Delta \to \Omega_\Delta$ given by

$$X_n \circ \rho = \begin{cases} X_{\zeta-1-n} & \text{if } n < \zeta < \infty \\ \Delta & \text{otherwise} \end{cases}$$

is again Markov with a substochastic transition function $\hat{r}$ on J given by

$$\hat{r}(x,y) = \begin{cases} r(y,x)\,\dfrac{e^\nu(y)}{e^\nu(x)} & \text{if } e^\nu(x) > 0 \\ \text{anything} & \text{if } e^\nu(x) = 0 \end{cases} \tag{2.4}$$

where e^ν is the occupation measure

$$e^\nu(z) = \sum_{n\geq 0} R(X_n = z) = \sum_{u\in J} \nu(u)\, G_r(u,z) \qquad (z \in J), \tag{2.5}$$

writing G_r for the potential kernel

$$G_r(u,z) = \sum_{n\geq 0} r^{(n)}(u,z)\,.$$

Therefore, if τ is a regular death time for the Markov probability P on Ω, the reversed pre - τ process $\rho \circ K_\tau$ is Markov with stationary transitions and for this process conditioning on $X_{\tau-1}$ simply amounts to freezing the initial state, so that $\rho \circ K_\tau$, and therefore also $\rho \circ \rho \circ K_\tau = K_\tau$, conditionally on $X_{\tau-1}$ is Markov with stationary transitions. By the conditional independence property shared by all regular death times it now follows that any regular death time is a CI - death time.

Consider the Markov probability P^μ on Ω and let τ be a regular death time for P^μ. The unconditional pre - τ process is then Markov with initial law ν and transition function r, where for $x, y \in J$, $n \in N$

$$\nu(x) = P^\mu(X_0 = x, \ \tau > 0),$$

(2.6)

$$r(x,y) = P^\mu(X_{n+1} = y, \ \tau > n + 1 \mid X_n = x, \ \tau > n).$$

Now condition on $X_{\tau-1} = a$, $0 < \tau < \infty$. By computation it is readily found that the conditional pre - τ process has transition function

$$r_a(x,y) = \begin{cases} r(x,y) \dfrac{G_r(y,a)}{G_r(x,a)} & \text{if } G_r(x,a) > 0 \\ \text{anything} & \text{if } G_r(x,a) = 0 \end{cases}$$

for $x, y \in J$, while the transition function $\hat{r}$ of the conditional pre - τ process reversed is just that of $\rho \circ K_\tau$ itself and therefore given by (2.4) with ν as in (2.6) and e^ν as in (2.5).

Thus the transition function of K_τ given $X_{\tau-1} = a$ is determined by r and a alone while that of $\rho \circ K_\tau$ depends on ν and r but not on a.

These considerations are relevant to Definition 4.19 below.

The following result provides a useful characterisation of conditional independence times.

(2.7). Lemma. A random time τ is a conditional independence time for the Markov probability P iff for every $n \in N_+$ and every $(a,b) \in J^2$ there exists events $F_n \in \mathcal{F}_n$, $G_{ab} \in \mathcal{F}$ respectively such that

$$(2.8) \qquad (\tau = n, Y_\tau = (a,b)) = (F_n, Y_n = (a,b), \theta_n \in G_{ab}) \qquad P\text{-a.s.}$$

or equivalently iff for every $n \in N_+$ and every $(a,b) \in J^2$ there exists $F_{n-1,ab} \in \mathcal{F}_{n-1}$, $G \in \mathcal{F}$ such that

$$(2.9) \qquad (\tau = n, Y_\tau = (a,b)) = (F_{n-1,ab}, Y_n = (a,b), \theta_{n-1} \in G) \quad P\text{-a.s.}$$

Proof. Proceeding exactly as in the proof of Lemma 3.12 of [3] one finds that τ is a conditional independence time for P iff for every choice of $n \in N_+$, $(a,b) \in J^2$ there exists $F_{n-1,ab} \in \mathcal{F}_{n-1}$, $G_{ab} \in \mathcal{F}$ such that

$$(\tau = n, Y_\tau = (a,b)) = (F_{n-1,ab}, Y_n = (a,b), \theta_n \in G_{ab}) \qquad P\text{-a.s.}$$

This is equivalent to (2.8) and (2.9) being valid with

$$F_n = \bigcup_{(a,b)} (F_{n-1,ab}, Y_n = (a,b)),$$

$$G = \bigcup_{(a,b)} (Y_1 = (a,b), \theta \in G_{ab})$$

and the lemma is proved. □

Remark. Conditional independence times satisfying (2.8) or (2.9) exactly are not splitting times as originally defined by Williams, see [2], equation (3.3). It appears most natural to generalise the definition there and call τ a splitting time if

$$(\tau = n) = (F_n, \theta_{n-1} \in G_n) \qquad (n \in N_+)$$

for some $F_n \in \mathcal{F}_n$, $G_n \in \mathcal{F}$.

This definition is implicitly algebraic. It may be shown that for a Markov probability P, a random time is a conditional independence time iff it is P - a.s. equal to a splitting time defined in this manner. Thus the definition of conditional independence times may be viewed as the operational definition of splitting times.

3. CI - birth times and the class $\mathcal{BO}$

The definitions of regular birth times and CI - birth times are both operational. The main result of Section 3 of [3], Theorem 3.9, provides a probabilistic equivalence between regular birth times and an explicitly algebraically defined class $\mathcal{B}$ of random times (with an implicitly algebraic description of the times in $\mathcal{B}$ provided by (3.16) of [3]).

The main purpose of this section is to give similar characterisations of suitable classes of CI - birth times.

The first results, Propositions 3.7 and 3.8 below, provide implicitly algebraic characterisations of the class of all CI - birth times for a given Markov probability. To obtain explicitly algebraic characterisations it is necessary to restrict attention to CI - birth times satisfying a condition relating the properties of the random times to the structure of the state space of the process. The basic definitions and the main result appear as Definition (3.22) and Theorem (3.24) below.

The entire section relies heavily on the theory presented in [3], and it is therefore necessary to recall some of the results from there.

Given a Markov probability P on Ω, consider an event $D \in \mathcal{F}$ with $P(D) > 0$ and let P_D denote the conditional probability $P(\cdot \mid D)$.

It was shown in [3], Theorem 2.3, that P_D is Markov iff D is P-equivalent to an event of the form $(X_0 \in H, C)$ where $H \subset J$ and C is coterminal, i.e.

$$C = C_V \, C_\infty \, , \qquad C_V = (Y_n \in V, n \in N_+)$$

for some $V \subset J^2$ and some invariant (under θ) event $C_\infty \in \mathcal{F}$. In particular for any $x \in J$, P_D^x is Markov iff D is P^x-equivalent to a coterminal event.

The reader is reminded that if all states are recurrent for P, then all coterminal events are P-equivalent to Ω or $\emptyset$, see Corollary 2.4 of [3].

If D is an event such that P_D is Markov, denote by J_D the (probabilistic) range of the conditional chain given by

$$J_D = \{x \in J : \sum_{n \geq 0} P_D(X_n = x) > 0\} .$$

Then the transition function q of P_D is stochastic on J_D and uniquely determined there. The following fact is implicit in Section 2 of [3], but is stressed here for later reference.

(3.1). Lemma. Suppose that $D \in \mathcal{F}$ is such that P_D is Markov (q) and let J_D denote the range of P_D. If C is any coterminal event such that

$$(3.2) \qquad D = (X_0 \in H, C) \qquad P\text{-a.s.}$$

for some $H \subset J$, then for all $x \in J_D$

$$P^x(C) > 0, \qquad Q^x = P^x(\cdot \mid C) .$$

Proof. Suppose C satisfies (3.2) and let $x \in J_D$. For $n \in N$ chosen so that $P_D(X_n = x) > 0$,

$$Q^x = P_D(\theta_n \in \cdot \mid X_n = x) = P(\theta_n \in \cdot, X_n = x, D) / P(X_n = x, D) .$$

But $C = (F_n, \theta_n \in C)$ for a suitable $F_n \in \mathcal{F}_n$ and hence

$$0 < P(X_n = x, D) = P(X_0 \in H, X_n = x, F_n) P^x(C) ,$$

in particular $P^x(C) > 0$, and

$$Q^x = P(X_0 \in H, X_n = x, F_n, \theta_n \in \cdot \cap C) / P(X_0 \in H, X_n = x, F_n, \theta_n \in C)$$

$$= P^x(\cdot \cap C) / P^x(C) = P^x(\cdot \mid C) . \qquad \square$$

Given a transition function p, a state $b \in J$ and a coterminal event $C \in \mathcal{F}$, write

$$(3.3) \qquad \beta(b,p,C) = (J_C, q)$$

for the pair consisting of the range J_C of the Markov probability $P^b(\cdot \mid C)$ and its transition function q. This makes sense if $P^b(C) > 0$, but it is convenient to define β for all b, p and C, simply letting $J_C = \emptyset$ and q be the 'empty transition function' on $\emptyset$ if $P^b(C) = 0$.

Note that $b \in J_C$ if $P^b(C) > 0$. Also note that $\beta(b,p,C) = \beta(b,p,C')$ if and only if $C = C'$ P^b - a.s.

Assume that $P^b(C) > 0$. By Lemma 3.1, $P^x(C) > 0$, $Q^x = P^x(\cdot \mid C)$ for all $x \in J_C$ and from this it follows easily that if $C = C_V C_\infty$ is a representation of C with $V \subset J^2$ and $C_\infty \in \mathcal{F}$ invariant, then

(3.4) $$q(x,y) = p(x,y)\, 1_V(x,y)\, \frac{P^y(C)}{P^x(C)} \qquad (x, y \in J_C)$$

no matter which of the possible V and C_∞'s representing C are used.

The result on conditioning events quoted above was used in [3] to establish Theorem 3.9 on the characterisation of regular birth times. Defining $\mathcal{B}$ to be the class of random times of the form

(3.5) $$\tau_C + \rho$$

where C is an arbitrary coterminal event, τ_C is the associated coterminal time

$$\tau_C = \inf\{n \in N : \theta_n \in C\},$$

and ρ is a stopping time for the family $(\mathcal{F}_{\tau_C + n}, n \in N)$ of σ-algebras, it was shown that τ is a regular birth time for P iff τ is P-equivalent to a random time in $\mathcal{B}$. (For the analogous result in continuous time, see the recent paper by Pittenger [6]).

It is not difficult to see that (3.5) alternatively may be expressed as follows: $\tau \in \mathcal{B}$ iff there is a coterminal C and events $F_n \in \mathcal{F}_n$, $n \in N$ such that

(3.6) $$\tau(\omega) = \inf\{n \in N : \omega \in F_n, \theta_n\omega \in C\}.$$

The first characterisation of CI - birth times to be given is an observation due to J.W. Pitman (private communication).

(3.7). Proposition. A random time τ is a CI - birth time for the Markov probability P if and only if, for every $(a,b) \in J^2$ the random time τ_{ab} defined by

$$\tau_{ab} = \begin{cases} \tau & \text{on } (Y_\tau = (a,b)) \\ \infty & \text{otherwise} \end{cases}$$

is a regular birth time for P.

Proof. If τ is given with the τ_{ab} defined as above one finds that

$$P(\theta_\tau \in \cdot \mid \mathcal{F}_\tau) = P(\theta_{\tau(ab)} \in \cdot \mid \mathcal{F}_{\tau(ab)})$$

on the set $(Y_\tau = (a,b)) = (\tau_{ab} < \infty)$. Appealing to the definitions of CI - birth times and regular birth times, the result now follows immediately. □

This characterisation of CI - birth times is implicitly algebraic because although each τ_{ab} may be described in an explicitly algebraic fashion by (3.5) or (3.6), it is not at all clear how the τ_{ab} may be chosen simultaneously so as to satisfy the necessary requirement that the sets $(\tau_{ab} < \infty)$ for (a,b) varying be mutually disjoint.

The next result is a straightforward consequence of Lemma 2.7.

(3.8). Proposition. A random time τ is a CI - birth time for P if and only if for every $n \in N_+$ and every $(a,b) \in J^2$ there exists $F_n \in \mathcal{F}_n$ and coterminal C_{ab} respectively such that

$$(3.9) \qquad (\tau = n, Y_\tau = (a,b)) = (F_n, Y_n = (a,b), \theta_n \in C_{ab}) \qquad P\text{-a.s.}$$

Proof. If (3.9) is satisfied, τ is a conditional independence time for P by (2.8), and since the law of the conditional post-τ process given F_τ on the set $(Y_\tau = (a,b))$, is the Markov probability $P^b(\cdot \mid C_{ab})$, τ is a CI-birth time. If conversely τ is a CI-birth time, (2.8) holds in particular which makes $P^b(\cdot \mid G_{ab})$ the law of the post-τ process given $(Y_\tau = (a,b))$. But Theorem 2.3 of [3] forces G_{ab} to be P^b-equivalent to a coterminal event C_{ab} whence, as is easily seen using the Markov property for P, the identity

$$(F_n, Y_n = (a,b), \theta_n \in G_{ab}) = (F_n, Y_n = (a,b), \theta_n \in C_{ab}) \qquad P\text{-a.s.}$$

follows, and the proof is complete. □

The proposition may also be proved from the preceding proposition, using (3.16) of [3] to describe each τ_{ab}.

Propositions 3.7 and 3.8 both give implicitly algebraic characterisations of the operationally defined CI-birth times. It does not appear possible (cf. the examples at the end of this section) to obtain an explicitly algebraic characterisation of the class of all CI-birth times.

The chief object of the remainder of the section is now to introduce a subclass of CI-birth times and to characterise that explicitly.

Suppose again that τ is a CI-birth time for P. Thus, as a consequence of Proposition 3.8, there is a coterminal event C_{ab} associated with each possible value (a,b) of Y_τ such that $P^b(\cdot \mid C_{ab})$

is the law of the post - τ process given F_τ inside $(Y_\tau = (a,b))$.

Instead of $J_{C(ab)}$ I shall write J_{ab} for the range of the $P^b(\cdot \mid C_{ab})$ process whenever $P(Y_\tau = (a,b)) > 0$ and q_{ab} for its transition function. If $P(Y_\tau = (a,b)) = 0$, J_{ab} is defined to be empty. Thus using the mapping β from (3.3),

(3.10) $$\beta(b,p,C_{ab}) = (J_{ab}, q_{ab})$$

whenever $P(Y_\tau = (a,b)) > 0$. Also in this case, although C_{ab} is determined from τ only up to a P^b - equivalence, the right hand side does not depend on the choice of C_{ab}.

Now define a relation $\underset{\tau,P}{\succ}$ on J^2, the <u>birth time relation for τ with respect to P</u>, by

$$(a,b) \underset{\tau,P}{\succ} (c,d) \Leftrightarrow \sum_{n\geq 1} P(Y_\tau = (a,b), Y_{\tau+n} = (c,d)) > 0 ,$$

i.e. $(a,b) \underset{\tau,P}{\succ} (c,d)$ if it is possible to have an (a,b) transition at time τ and a (c,d) transition at a time (strictly) after that. Notice that $(a,b) \underset{\tau,P}{\succ} (c,d)$ iff

(3.11) $$P(Y_\tau = (a,b)) > 0, \qquad \sum_{n\geq 1} Q^b_{ab}(Y_n = (c,d)) > 0 .$$

To ease the notation I shall frequently write $\succ$ instead of $\underset{\tau,P}{\succ}$. For the next definition, the notation used so far for the various objects connected with a CI - birth time is retained.

(3.12). <u>Definition</u>. A CI - birth time for P is called <u>transition reproducing</u> if

$$(a,b) \underset{\tau,P}{\succ} (c,d) \Rightarrow \beta(d, q_{ab}, C_{cd}) = (J_{cd}, q_{cd})$$

whenever $P(Y_\tau = (c,d)) > 0$.

Remark. If $(a,b) \succ (c,d)$, then in particular $d \in J_{ab}$. Assume that $P(Y_\tau = (c,d)) > 0$. Then although C_{cd} is not uniquely determined from τ, I claim that the definition is consistent in the sense that $\beta(d,q_{ab},C_{cd})$ does not depend on the choice of C_{cd}. To see this amounts to showing that all possible C_{cd} are Q^d_{ab} - equivalent. But as noted above they are certainly P^d - equivalent and since $p(x,y) = 0 \Rightarrow q_{ab}(x,y) = 0$ (cf. (3.4)), it is clear that Q^d_{ab} is absolutely continuous with respect to P^d, whence the desired conclusion.

As defined by (3.3), $\beta(b,p,C)$ merely identifies the probability $P^b(\cdot \mid C)$. Thus the requirement in Definition 3.12 is that

(3.13) $$(a,b) \underset{\tau,P}{\succ} (c,d) \Rightarrow Q^d_{ab}(C_{cd}) > 0 \text{ and } Q^d_{ab}(\cdot \mid C_{cd}) = P^d(\cdot \mid C_{cd})$$

whenever $P(Y_\tau = (c,d)) > 0$.

As Definition 3.12 stands it has an analogue in the death time case (see Definition 4.19 below), while a formulation using (3.13) instead would have had no obvious parallel.

For an example, consider Williams' decomposition of a killed Brownian motion at the time of its ultimate minimum discussed in Section 2. That τ is transition reproducing follows because while loosely speaking the Brownian motion conditioned to stay above a given level is a three - dimensional Bessel process measuring its origin at that level, it is also true that a Bessel process conditioned to stay above a level is again a Bessel process with that level as origin.

One consequence of Definition 3.12 is the property exhibited in the next result.

(3.14). Proposition. If τ is a transition reproducing CI-birth time for P, then the relation $\succ_{\tau,P}$ is transitive.

Proof. Suppose that τ is transition reproducing and let $(a,b) \succ (c,d)$, $(c,d) \succ (e,f)$. Then in particular for some $m, n \in N_+$ (see (3.11))

$$Q^b_{ab}(Y_m = (c,d)) > 0, \qquad Q^d_{cd}(Y_n = (e,f)) > 0$$

and $Q^d_{ab}(C_{cd}) > 0$ with (see (3.13))

$$Q^d_{ab}(\cdot \mid C_{cd}) = Q^d_{cd} .$$

But then also

$$\begin{aligned} Q^b_{ab}(Y_{m+n} = (e,f)) &\geq Q^b_{ab}(Y_m = (c,d), \theta_m \in C_{cd}, Y_{m+n} = (e,f)) \\ &= Q^b_{ab}(Y_m = (c,d))\; Q^d_{ab}(Y_n = (e,f), C_{cd}) \\ &= Q^b_{ab}(Y_m = (c,d))\; Q^d_{ab}(C_{cd})\; Q^d_{cd}(Y_m = (e,f)) \\ &> 0 \end{aligned}$$

so from (3.11) it follows that $(a,b) \succ (e,f)$. □

When the relation $\succ$ is transitive an order relation may be induced in the following manner: the relation $\sim$ on J^2 defined by $(a,b) \sim (c,d)$ iff either $(a,b) = (c,d)$ or $(a,b) \succ (c,d)$, $(c,d) \succ (a,b)$ is an equivalence relation and the relation $\succsim$ on the equivalence classes (which are subsets of J^2) given by $V \succsim W$ iff $(a,b) \succ (c,d)$ for any and hence for all $(a,b) \in V$, $(c,d) \in W$ is consequently a reflexive partial ordering. For the time of the ultimate minimum of a real - valued process considered earlier this ordering which, it should be remembered, depends on the underlying probability, can of course almost be identified with the natural ordering of the real numbers. Thus transition reproducing CI - birth times appear as reasonable generalisations of the time of the minimum. See also Proposition 3.31 below.

(3.15). Proposition. A random time τ is a transition reproducing CI - birth time for the Markov probability P if and only if for every $n \in N_+$ and $(a,b) \in J^2$ there exists $F_n \in \mathcal{F}_n$ and coterminal $C_{ab} \in \mathcal{F}$ respectively such that

(3.16) $$(\tau = n, Y_\tau = (a,b)) = (F_n, Y_n = (a,b), \theta_n \in C_{ab}) \qquad P\text{-a.s.}$$

and

(3.17) $$(a,b) \underset{\tau,P}{\succ} (c,d) \Rightarrow C_{ab} \supset C_{cd}\,.$$

Remark. It will be clear from the proof that (3.17) may be replaced by the weaker

(3.18) $$(a,b) \underset{\tau,P}{\succ} (c,d) \Rightarrow C_{ab} \supset C_{cd} \qquad P^d\text{-a.s.}$$

Proof. (3.16) is merely the necessary and sufficient condition (3.9) from Proposition 3.8 for τ to be a CI - birth time for P. Hence it suffices to consider CI - birth times satisfying (3.16).

Suppose first that τ is also transition reproducing. It must be shown that the C_{ab} may be redefined without affecting (3.16) and in such a way that (3.17) holds. For each (a,b) define a new coterminal event C^*_{ab} as follows: if $P(Y_\tau = (a,b)) = 0$ define $C^*_{ab} = \emptyset$. If $P(Y_\tau = (a,b)) > 0$ so that in particular $P^b(C_{ab}) > 0$, write J_{ab} for the range of $Q^b_{ab} = P^b(\cdot \mid C_{ab})$ and define

(3.19) $$C^*_{ab} = C_{ab} \cap \bigcap_{(x,y)} C_{xy} ,$$

where the intersection extends over all (x,y) such that $(x,y) \succ (a,b)$. The aim then is to show that $(3.16)^*$ and $(3.17)^*$ hold where the * indicates that in (3.16) and (3.17), C_{ab} and C_{cd} have been replaced by C^*_{ab} and C^*_{cd}.

If $P(Y_\tau = (a,b)) = 0$, $(3.16)^*$ holds trivially. To show $(3.16)^*$ when $P(Y_\tau = (a,b)) > 0$ is, since $C^*_{ab} \subset C_{ab}$, equivalent to showing

$$P(F_n, Y_n = (a,b), \theta_n \in C^*_{ab}) = P(F_n, Y_n = (a,b), \theta_n \in C_{ab})$$

which by the Markov property will follow from

(3.20) $$P^b(C^*_{ab}) = P^b(C_{ab}) .$$

It is for the proof of this and $(3.17)^*$ that the assumption that τ be transition reproducing is needed. The fact required to prove (3.20) is that

(3.21) $$(a,b) \succ (c,d) \Rightarrow C_{ab} \supset C_{cd} \quad P^d\text{-a.s.}$$

since this implies $C_{xy} \supset C_{ab}$ P^b-a.s. for all (x,y) appearing in the intersection in (3.19).

Because τ is transition reproducing, if $(a,b) \succ (c,d)$ and $P(Y_\tau = (c,d)) > 0$ the Markov probability $Q^d_{ab}(\cdot \mid C_{cd})$ is well defined and equal to $P^d(\cdot \mid C_{cd})$, see (3.13). But since $d \in J_{ab}$, Lemma 3.1 implies that $Q^d_{ab} = P^d(\cdot \mid C_{ab})$. Thus

$$P^d(\cdot \mid C_{cd}) = P^d(\cdot \mid C_{ab}\, C_{cd})$$

and (3.21) follows.

Finally (3.17)* is a trivial consequence of the transitivity of $\succ$ established in Proposition 3.14: each C_{xy} appearing in the definition (3.19) of C^*_{ab} (including C_{ab} itself) is one of the C_{xy} appearing in the corresponding expression for C^*_{cd}.

Now suppose conversely that τ is a CI - birth time satisfying (3.16) and (3.17). To see that τ is transition reproducing, assume that $(a,b) \succ (c,d)$ with $P(Y_\tau = (c,d)) > 0$. Then $P^b(C_{ab}) > 0$, $P^d(C_{cd}) > 0$, $d \in J_{ab}$ and $Q^d_{ab} = P^d(\cdot \mid C_{ab})$, $Q^d_{cd} = P^d(\cdot \mid C_{cd})$ while by (3.17), $C_{ab} \supset C_{cd}$. Therefore trivially $Q^d_{ab}(C_{cd}) > 0$ and $Q^d_{ab}(\cdot \mid C_{cd}) = Q^d_{cd}$ so that τ is transition reproducing. □

The main result of this section, to be stated shortly, provides an explicitly algebraic characterisation of transition reproducing CI - birth times.

(3.22). Definition. Let $\mathcal{BO}$ be the class of random times τ such that for $n_0 \in N_+$

(3.23) $\tau(\omega)=n_0 \Leftrightarrow n_0=\inf\{n\in N_+ : \omega\in F_n, \theta_n\omega\in C_{\omega_{n-1}\omega_n}, C_{\omega_{n-1}\omega_n}\supset C_{\omega_{k-1}\omega_k}, k>n\}$,

where for each $n \in N_+$, F_n is an arbitrary event in $\mathcal{F}_n$ and for each transition $(a,b) \in J^2$, C_{ab} is an arbitrary coterminal event.

With the string of inclusions appearing in (3.23), this definition embodies an algebraic equivalent of (3.17). For an alternative description of $\mathcal{BO}$ involving a transitive relation on J^2, see Proposition 3.31 below.

Notice that in the definition nothing is said about the structure of the sets $(\tau = 0)$ and $(\tau = \infty)$. Given only the sets $(\tau = n)$ for

$n \in N_+$ they may be any two disjoint measurable sets, also disjoint from all $(\tau = n)$ for $n \in N_+$, such that $\Omega = \bigcup_{0 \leq n \leq \infty} (\tau = n)$.

Also notice that the class $\mathcal{B}$ (Definition 3.8 of [3], see also (3.6) above), is obtained by taking all the C_{ab} to equal the same coterminal event C, requiring in addition that the set $(\tau = 0)$ have a suitable form.

(3.24). Theorem. A random time τ is a transition reproducing CI-birth time for the Markov probability P if and only if τ is P-equivalent to a random time in $\mathcal{BO}$.

Proof. By Proposition 3.15 it suffices to show that a random time τ satisfies (3.16) and (3.17) iff it is P-equivalent to a time in $\mathcal{BO}$.

Suppose first that τ satisfies (3.16) and (3.17). Then for N a suitable P-null set, the identities

$$(3.25) \qquad (\tau = n, Y_\tau = (a,b), N^c) = (F_n, Y_n = (a,b), \theta_n \in C_{ab}, N^c)$$

hold exactly for all $n \in N_+$, $(a,b) \in J^2$. But since the sets appearing on the right of (3.25) are mutually disjoint for n or (a,b) varying, it is clear that for $\omega \in N^c$ there is at most one value of $n \in N_+$ such that

$$(3.26) \qquad \omega \in F_n, \quad \theta_n\omega \in C_{\omega_{n-1}\omega_n}.$$

Introducing τ' by

$$(3.27) \quad \tau'(\omega) = n_0 \Leftrightarrow n_0 = \inf\{n \in N_+ : \omega \in F_n, \theta_n\omega \in C_{\omega_{n-1}\omega_n}, C_{\omega_{n-1}\omega_n} \supset C_{\omega_{k-1}\omega_k}, k > n\}$$

for $n_0 \in N_+$, the proof that τ is P-a.s. equal to a random time in

$\mathcal{BO}$ will be completed by showing that

(3.28) $$(\tau = n) = (\tau' = n) \qquad P\text{-a.s.}$$

for every $n \in N_+$.

Obviously, (3.26) with the additional requirement $C_{\omega_{n-1}\omega_n} \supset C_{\omega_{k-1}\omega_k}$ for $k > n$ has at most one solution in $n \in N_+$. Therefore for $n \in N_+$

$$\bigcap_{k>n} (F_n, Y_n = (a,b), \theta_n \in C_{ab}, C_{ab} \supset C_{Y_k}, N^c) \subset (\tau' = n, Y_{\tau'} = (a,b), N^c)$$

$$\subset (\tau = n, Y_\tau = (a,b), N^c) = (F_n, Y_n = (a,b), \theta_n \in C_{ab}, N^c),$$

and so, to prove (3.28) it suffices to show that

$$(F_n, Y_n = (a,b), \theta_n \in C_{ab}) \subset (C_{ab} \supset C_{Y_k}) \qquad P\text{-a.s.}$$

for all $n \in N_+$, $k > n$, $(a,b) \in J^2$, or that

$$P(F_n, Y_n = (a,b), \theta_n \in C_{ab}, Y_k = (c,d)) > 0 \Rightarrow C_{ab} \supset C_{cd}$$

for all $n \in N_+$, $k > n$, (a,b), $(c,d) \in J^2$. But that is an immediate consequence of (3.16) and (3.17).

To show the converse assertion of the theorem, it is of course sufficient to show that every $\tau \in \mathcal{BO}$ is a transition reproducing CI-birth time for every Markov probability P. Hence assume that τ satisfies (3.23) and consider the set $M = (\tau = n, Y_\tau = (a,b))$ where $n \in N_+$ and $(a,b) \in J^2$ are given. Clearly

$$M = (F_n, Y_n = (a,b), \theta_n \in C_{ab}, C_{ab} \supset C_{Y_k}, k > n) \cap \bigcap_{k=1}^{n-1} A_k$$

where

$$A_k(Y_k = (x,y)) = (Y_k = (x,y)) \cap [F_k^c \cup (\theta_k \in C_{xy}^c) \cup \bigcup_{\ell=k+1}^{\infty} (C_{xy} \not\supset C_{Y_\ell})].$$

Suppose that $\omega \in M$, fix k with $1 \leqq k < n$ and write $(x,y) = (\omega_{k-1},\omega_k)$. Because $C_{ab} \supset C_{\omega_{k-1}\omega_k}$ for $k > n$ either

(i) $\qquad \omega \in F_k^c$ $\qquad$ or

(ii) $\qquad \omega \in \bigcup_{\ell=k+1}^{n} (C_{xy} \not\supset C_{Y_\ell})$ $\qquad$ or

(iii) $\qquad \theta_k\omega \in C_{xy}^c$.

Here the first two possibilities correspond to $\mathcal{F}_n$-measurable events and since, writing $C_{xy} = (Y_k \in V_{xy}, 1 \leqq k \leqq m, \theta_m \in C_{xy})$ for some $V_{xy} \subset J^2$ and all $m \in N_+$,

$$\omega \in F_k \cap \bigcap_{\ell=k+1}^{n} (C_{xy} \supset C_{Y_\ell}) \cap (\theta_k \in C_{xy}^c)$$

happens for $\omega \in M$ iff

$$\omega \in F_k \cap \bigcap_{\ell=k+1}^{n} (C_{xy} \supset C_{Y_\ell}) \cap \bigcup_{\ell=k+1}^{n} (Y_\ell \in V_{xy}^c)$$

because $\theta_n\omega \in C_{ab} \subset C_{xy}$, it follows that

(3.29) $$M = (F_n^*, Y_n = (a,b), \theta_n \in C_{ab}^*)$$

for a suitable $F_n^* \in \mathcal{F}_n$ and with C_{ab}^* the coterminal event

(3.30) $$C_{ab}^* = C_{ab} \cap \bigcap_{k\geqq 1} (C_{ab} \supset C_{Y_k}) .$$

Since clearly $C^*_{\omega_{n-1}\omega_n} \supset C^*_{\omega_{k-1}\omega_k}$ whenever $\tau(\omega) = n$ and $k > n$, it follows from Proposition 3.15 that τ is a transition reproducing CI-birth time for all Markov probabilities P. □

<u>Remark</u>. It follows from (3.29) that if $\tau \in \mathcal{BO}$ is given by (3.23)

and P is Markov, then conditionally on $\mathcal{F}_\tau$ and $(Y_\tau = (a,b))$, the P - law of the post - τ process is $P^b(\cdot \mid C^*_{ab})$ with C^*_{ab} defined by (3.30).

It is possible to give an alternative description of the class $\mathcal{B}0$, using as one of the parameters a transitive relation on J^2 (cf. Proposition 3.14).

First however a comment on the structure of coterminal events. If $C = C_V C_\infty$ is coterminal, where $V \subset J$, $C_\infty \in \mathcal{F}$ is invariant it may of course be possible to use other V, C_∞ for representing the given C. A canonical representation may be obtained in the following manner: define

$$V_0 = \{(x,y) \in J^2 : Y_n(\omega) = (x,y) \text{ for some } \omega \in C,\ n \in N_+\},$$

$$C_{\infty,0} = \bigcup_{n \geq 0} (\theta_n \in C).$$

Then evidently $V_0 \subset V$ and (by the invariance of C_∞) $C_{\infty,0} \subset C_\infty$. But the definition of V_0 and the fact that $C \subset (\theta_n \in C)$ for all n shows that $C \subset C_{V_0} C_{\infty,0}$ and consequently

$$C = C_{V_0} C_{\infty,0}.$$

Furthermore $C_{\infty,0}$ is invariant: since $(\theta_n \in C)$ increases with n, $C_{\infty,0} = \limsup_{n \to \infty} (\theta_n \in C)$.

Finally, if $C = C_V C_\infty$, $C' = C_{V'} C'_\infty$ are coterminal events in their canonical representations, then $C \subset C'$ iff $V \subset V'$, $C_\infty \subset C'_\infty$: the 'if' part is evident, so suppose that $C \subset C'$; then from the definition of V, V' necessarily $V \subset V'$, while if $\omega \in C_\infty$, then

$\theta_n \omega \in C \subset C'$ for some n and so $\omega \in C'_\infty$.

Now let $\tau \in \mathcal{BO}$ be given by (3.23) and let $C_{ab} = C_{V_{ab}} C_{\infty,ab}$ be the canonical representation of C_{ab}. Defining the relation $\succ$ on J^2 by

$$(a,b) \succ (c,d) \Leftrightarrow (c,d) \in V_{ab},\ C_{ab} \supset C_{cd}$$

it is readily checked using the remarks above that $\succ$ is transitive and that $(a,b) \succ (c,d) \Rightarrow C_{\infty,ab} \supset C_{\infty,cd}$, while it is clear that τ satisfies (3.32) below. The proof of the other half of the following proposition is straightforward and is omitted, but it should be noted, that for this part of the argument the compatibility condition in the statement of the proposition is required.

(3.31). Proposition. A random time τ belongs to $\mathcal{BO}$ if and only if for every $n \in N_+$ and $(a,b) \in J^2$ there exists $F_n \in \mathcal{F}_n$ and invariant $C_{\infty,ab} \in \mathcal{F}$ respectively and a transitive relation $\succ$ on J^2 compatible with the $C_{\infty,ab}$ in the sense that $(a,b) \succ (c,d) \Rightarrow C_{\infty,ab} \supset C_{\infty,cd}$ such that for $n_0 \in N_+$

(3.32) $\tau(\omega) = n_0 \Leftrightarrow$

$$n_0 = \inf\{n\in N_+ : \omega\in F_n, \omega\in C_{\infty,\omega_{n-1}\omega_n}, (\omega_{n-1},\omega_n) \succ (\omega_{k-1},\omega_k), k > n\}.$$

Suppose that $\tau \in \mathcal{BO}$ is given by (3.32) and consider the algebraic analogue $\underset{\tau}{\succ}$ of the birth time relation for τ with respect to P defined by

$$(a,b) \underset{\tau}{\succ} (c,d) \Leftrightarrow \exists\ \omega \in \Omega, n \in N_+ \text{ with } Y_\tau(\omega) = (a,b), Y_{\tau+n}(\omega) = (c,d).$$

Obviously $\underset{\tau}{\succ}$ is contained in $\succ$ (thinking of $\underset{\tau}{\succ}$ and $\succ$ as subsets of $J^2 \times J^2$), hence compatible with the $C_{\infty,ab}$, and it is easily checked that $\succ_\tau$ is transitive and that τ is given by (3.32) using $\succ_\tau$ instead of $\succ$. Thus $\succ_\tau$ is the smallest relation which can be used when representing τ by (3.32).

(3.33). Example. Suppose $f : J^2 \to \mathbb{R}$ and consider the random times $\underline{\tau}$ and $\overline{\tau}$ given as the first, respectively the last time that the sequence $(f(Y_n),\ n \in N_+)$ attains its ultimate minimum. Then $\underline{\tau},\ \overline{\tau} \in \mathcal{BO}$.

If P is Markov, the P - law of the conditional post - τ process given $\mathcal{F}_\tau$ and $f(Y_\tau) = u$ is

$$P^{X(\tau)}(\cdot \mid f(Y_n) \geqq u,\ n \in N_+)$$

if $\tau = \underline{\tau}$ and

$$P^{X(\tau)}(\cdot \mid f(Y_n) > u,\ n \in N_+)$$

if $\tau = \overline{\tau}$.

These random times were studied by Millar [5] who proved that they are CI - birth times for a wide class of Markov probabilities in continuous time.

The class $\mathcal{BO}$ possesses a nice closure property. Assume that $\sigma > 0$ and $\tau > 0$ are exact CI - birth times so that for $F_n,\ G_n \in \mathcal{F}_n$ and $C_{ab},\ D_{ab}$ coterminal the identities

$$(3.34)\qquad \begin{aligned} (\sigma = n, Y_\sigma = (a,b)) &= (F_n, Y_n = (a,b), \theta_n \in C_{ab}), \\ (\tau = n, Y_\tau = (c,d)) &= (G_n, Y_n = (c,d), \theta_n \in C_{cd}) \end{aligned}$$

hold exactly for $n \in N_+$, $(a,b),\ (c,d) \in J^2$, cf. (3.9). Now consider the random time $\rho = \sigma + \tau \circ \theta_\sigma$. Then

$(\rho = n, Y_\sigma = (a,b), Y_\rho = (c,d)) = (H_{n,ab}, Y_n = (a,b), \theta_n \in C_{ab} D_{cd})$

for suitable $H_{n,ab} \in \mathcal{F}_n$ and consequently for P Markov

(3.35) $$P(\theta_\rho \in \cdot \mid \mathcal{F}_\rho, Y_\sigma = (a,b), Y_\rho = (c,d)) = P^d(\cdot \mid C_{ab} D_{cd}) .$$

Thus ρ will not be a CI - birth time for P unless for all (a,b), (c,d) with $P(Y_\sigma = (a,b), Y_\rho = (c,d)) > 0$ the inclusion $C_{ab} \supset D_{cd}$ holds P^d - a.s. However with Proposition 3.15 and Theorem 3.24 in mind, the following result is not surprising and easily proved.

(3.36). Proposition. If $\sigma > 0$, $\tau > 0$ belong to $\mathcal{BO}$ and both satisfy (3.23) with the same family of coterminal events, then also $\rho \in \mathcal{BO}$ where $\rho = \sigma + \tau \circ \theta_\sigma$.

I shall conclude this section with some examples of random times which are CI - birth times for all Markov probabilities P, but do not in general belong to the class $\mathcal{BO}$. The times are described in an explicitly algebraic fashion.

(3.37). Example. Suppose that $(C_{ab}, (a,b) \in J^2)$ is a given family of coterminal events with

$$C_{ab} = (Y_k \in V_{ab}, 1 \leqq k \leqq n, \theta_n \in C_{ab})$$

for some $V_{ab} \subset J^2$ and all $n \in N_+$. It is then clear that the sets

$$A_n = [\bigcap_{k=1}^{n} \bigcup_{\ell=k+1}^{n} (Y_\ell \in V^c_{Y_k})] \cap (\theta_n \in C_{Y_n})$$

are mutually disjoint for $n \in N_+$ (for any $\omega \in \Omega$ there can be at most one $n \in N_+$ with $\theta_n\omega \in C_{\omega_{n-1}\omega_n}$), and hence any random time τ satisfying

$$(\tau = n) = A_n \qquad (n \in N_+)$$

is by (3.9) a CI - birth time for any Markov probability P.

(3.38). Example. With the (C_{ab}) as in the previous example, take

$$A_n = (C_{Y_k} C_{Y_n} = \emptyset, 1 \leqq k < n, \theta_n \in C_{Y_n}) \quad \text{or}$$

$$A_n = (\theta_n \in C_{Y_n}, C_{Y_k} C_{Y_n} = \emptyset, k > n) .$$

Again the A_n are disjoint and any τ with $(\tau = n) = A_n$ for $n \in N_+$ is a CI - birth time for all P, in the second case because for all (a,b)

$$C^*_{ab} = (C_{ab}, C_{Y_k} C_{ab} = \emptyset, k \in N_+)$$

is a coterminal event.

Throughout this section I have discussed CI - birth times τ which by definition obey a strong Markov property involving conditional independence of the pre - τ and post - τ processes given Y_τ. Other authors have studied birth times where this conditional independence occurs when conditioning not only on Y_τ but also on some auxiliary $\mathcal{F}_\tau$ - measurable variable.

Thus, in [4] Millar has introduced (for processes in continuous time) randomised coterminal times and shown that if τ is such a time, then conditionally on $\mathcal{F}_\tau$ the post - τ process is Markov with a transition function depending on X_τ and a $\mathcal{F}_\tau$ - measurable variable Z. Therefore, unless Z is a function of $(X_{\tau-}, X_\tau)$, τ will not be a CI - birth time.

Millar's definition of randomised coterminal times is implicitly algebraic and quite complicated. Simpler examples of birth times with a conditional independence property involving an extra variable can be found in Getoor [1].

From (3.35) it follows that with σ, τ given by (3.34), the random time $\rho = \sigma + \tau \circ \theta_\sigma$ will be a birth time with the kind of conditional independence discussed by Millar and Getoor, provided Y_σ is $\mathcal{F}_\rho$ - measurable. Finally it may be remarked that Millar points out that the class of randomised coterminal times is closed under the addition $(\sigma,\tau) \to \sigma + \tau \circ \theta_\sigma$.

4. CI - death times and the class $\mathcal{DO}$

This section contains characterisation results for classes of CI-death times (see Definition 2.3) for a Markov probability P.

As pointed out in [3], the results on regular birth times and the corresponding results on death times are duals. This duality is prevalent also in the theory of CI - birth times and CI - death times, so the death time results will be presented in the same order as their analogues in Section 3.

Recall the definition of a regular death time (see [3] or Section 2 above) and Definition 5.1 in [3] of the class $\mathcal{D}$.

The main result, Theorem 5.2, in Section 5 of [3] states that a random time τ is a regular death time for the Markov probability P^x iff τ is P^x - equivalent to a random time in $\mathcal{D}$.

A remark in [3] shows how this result may be generalised to Markov probabilities with a non-degenerate start. Since I shall need this generalisation here, I shall redefine $\mathcal{D}$ and restate the regular death time theorem.

If $H \subset J$, $V \subset J^2$, let τ_{HV} denote the modified terminal time

$$\tau_{HV} = \begin{cases} 0 & \text{if } X_0 \in H \\ \inf\{n \in N_+ : Y_n \in V\} & \text{otherwise}. \end{cases}$$

The class $\mathcal{D}$ is now defined to comprise all random times τ of the form

(4.1) $$\tau = \sup\{n : 1 \leqq n \leqq \tau_{HV}, \theta_{n-1} \in F\}$$

for some $H \subset J$, $V \subset J^2$, $F \in \mathcal{F}$. (By the usual convention $\tau = 0$ if the set in brackets is empty; in particular $\tau = 0$ on $(X_0 \in H)$).

Then the following is true: τ is a regular death time for the Markov probability P iff τ is P-equivalent to a random time in $\mathcal{D}$.

(4.2). <u>Lemma</u>. A finite random time τ belongs to $\mathcal{D}$ iff there exists $H \subset J$, $V \subset J^2$, $F \in \mathcal{F}$ such that

(4.3) $$(\tau = n) = (X_0 \in H, Y_k \in V, 1 \leqq k < n, \theta_{n-1} \in F) \qquad (n \in N_+).$$

(For $n = 1$, (4.3) reads $(\tau = 1) = (X_0 \in H, F)$).

<u>Proof</u>. Comparing with Proposition 5.3 (c) in [3] it is easy to see that if τ is finite and given by (4.1), then τ satisfies (4.3) with $F = (\tau' = 1)$, where using the notation from [3], $\tau' = \tau_{V^c F}$.

If conversely τ is finite and satisfies (4.3), there is for each ω at most one $n \in N_+$ such that $\omega_0 \in H$, $(\omega_{k-1}, \omega_k) \in V$, $1 \leqq k < n$,

$\theta_{n-1}\omega \in F$. Describing this n in particular as the last n for which $1 \leqq n \leqq \tau_{H^c V^c}(\omega)$, $\theta_{n-1}\omega \in F$ shows that $\tau \in \mathcal{D}$. □

There is a switch in notation from (4.1) to (4.3). For instance (4.1) forbids transitions in V prior to τ, while (4.3) demands that all pre - τ transitions belong to V. Of course (4.1) is modelled upon the definition of $\mathcal{D}$ from [3], but it does seem more reasonable to denote the set of possible pre - τ transitions by V rather than V^c.

The characterisation (4.3) of finite times in $\mathcal{D}$ corresponds to the following description of strictly positive times in $\mathcal{B}$: a random time $\tau > 0$ belongs to $\mathcal{B}$ iff there exists a coterminal C and for every $n \in N$, $F_n \in \mathcal{F}_n$ such that $(\tau = n) = (F_n, \theta_n \in C)$, cf. (3.16) of [3]. In particular the counterpart of the coterminal event C is the sequence $T = (T^n, n \in N_+)$ of <u>terminal events</u>

(4.4) $$T^n = (X_0 \in H, Y_k \in V, 1 \leqq k < n) .$$

Of course the invariant part of C matches the initial part $(X_0 \in H)$ of each T^n. Notice that $T^n \in \mathcal{F}_{n-1}$.

The next two results are the duals of Propositions 3.7 and 3.8. The proofs are combined into one. For $H_{ab} \subset J$, $V_{ab} \subset J^2$ the notation T^n_{ab} is used for the event in $\mathcal{F}_{n-1}$ given by (4.4) using H_{ab}, V_{ab} instead of H, V, and T_{ab} for the sequence (T^n_{ab}).

(4.5). <u>Proposition</u>. A random time τ is a CI - death time for the Markov probability P if and only if for every $(a,b) \in J^2$ the random time τ_{ab} defined by

(4.6) $$\tau_{ab} = \begin{cases} \tau & \text{on } (Y_\tau = (a,b)) \\ 0 & \text{otherwise} \end{cases}$$

is a regular death time for P.

(4.7). Proposition. A random time τ is a CI - death time for P if and only if there exists $F \in \mathcal{F}$ and for every $(a,b) \in J^2$ subsets $H_{ab} \subset J$, $V_{ab} \subset J^2$ such that

(4.8) $$(\tau = n, Y_\tau = (a,b)) = (T^n_{ab}, Y_n = (a,b), \theta_{n-1} \in F) \quad P\text{-a.s.}$$

for $n \in N_+$.

Proofs. With τ arbitrary and the τ_{ab} given by (4.6),

(4.9) $$P(K_\tau \in \cdot \mid \mathcal{G}_{\tau-1}) = P(K_{\tau(ab)} \in \cdot \mid \mathcal{G}_{\tau(ab)-1})$$

on $(Y_\tau = (a,b)) = (\tau_{ab} > 0)$, where (for an arbitrary random time τ) $\mathcal{G}_{\tau-1}$ is the sub σ-algebra of $\mathcal{F}$ generated by the sets $(\tau = 0)$, $(\tau = \infty)$, $(0 < \tau < \infty, \theta_{\tau-1} \in F)$ for $F \in \mathcal{F}$.

Suppose τ is a CI - death time for P. Then by (4.9) the pre - τ_{ab} and post - τ_{ab} processes are independent given $(Y_\tau = (a,b)) = (\tau_{ab} > 0) = (X_{\tau(ab)-1} = a)$. Therefore τ_{ab} has the conditional independence property required of regular death times. Furthermore, (4.9) implies that the pre - τ_{ab} process given $(\tau_{ab} > 0)$ is Markov, and therefore, by the definition of the pre - τ_{ab} process as $(\Delta, \Delta, \ldots)$ on $(\tau_{ab} = 0)$, so is the unconditional pre - τ_{ab} process.

Thus each τ_{ab} is a regular death time for P and hence in view of the modified version of Theorem 5.2 of [3] quoted above and Lemma 4.2, there exists $H_{ab} \subset J$, $V_{ab} \subset J^2$, $F_{ab} \in \mathcal{F}$ such that

$$(\tau_{ab} = n) = (T^n_{ab}, \theta_{n-1} \in F_{ab}) \qquad P\text{-a.s.}$$

for $n \in N_+$. Taking $F = \bigcup_{(a,b)} (Y_1 = (a,b), F_{ab})$ it is seen that (4.8) holds.

If conversely (4.8) holds, the definition of τ_{ab} and Lemma 4.2 shows each τ_{ab} to be a regular death time for P.

Finally, if all τ_{ab} are regular death times for P, (4.9) implies that τ is a conditional independence time and that the pre-τ process given $(Y_\tau = (a,b))$ is Markov with transitions equal to those of the unconditional pre-τ_{ab} process. □

Suppose now that τ is a CI-death time for P satisfying (4.8). Let J_{ab} denote the range in J of the conditional pre-τ process given $(Y_\tau = (a,b))$ so that

$$J_{ab} = \{x \in J : \sum_{n \geq 0} P(X_n = x, \tau > n, Y_\tau = (a,b)) > 0\},$$

in particular $J_{ab} = \emptyset$ if $P(Y_\tau = (a,b)) = 0$. (If this probability is strictly positive, the full range is of course $J_{ab} \cup \{\Delta\}$).

For (a,b) such that $P(Y_\tau = (a,b)) > 0$ define $g_{ab} : J \to \mathbb{R}$ by

(4.10) $$g_{ab}(z) = \sum_{n>0} P^z(Y_k \in V_{ab}, 1 \leq k < n, X_{n-1} = a).$$

(For $n = 1$ the condition $Y_k \in V_{ab}$, $1 \leq k < n$ is vacuous and the term in the sum is just $P^z(X_0 = a) = 1$ if $z = a$ and $= 0$ otherwise). Now

(4.11) $$P(X_n{=}x, \tau{>}n, Y_\tau{=}(a,b)) = \sum_{k>n} P(X_n{=}x, T^k_{ab}, Y_k{=}(a,b), \theta_{k-1} \in F)$$

$$= [P(X_n=x, T_{ab}^{n+1})][\sum_{k>n} P^x(Y_\ell \in V_{ab}, 1 \leq \ell < k-n, X_{k-n-1}=a)][P^a(X_1=b,F)]$$

$$= P(X_n = x, T_{ab}^{n+1}) \; g_{ab}(x) \; P^a(X_1 = b,F)$$

and consequently, if $P(Y_\tau = (a,b)) > 0$ necessarily $P^a(X_1 = b,F) > 0$ and in that case

(4.12) $$x \in J_{ab} \Leftrightarrow \sum_{n \geq 0} P(X_n = x, T_{ab}^{n+1}) > 0, \quad g_{ab}(x) > 0 .$$

Also, if μ is the initial law of P, $\mu(H_{ab}) > 0$.

The aim of the next result is to describe the law of the pre - τ process given Y_τ. The remarks just made ensure that the denominators in (4.14), (4.15) below are strictly positive.

(4.13). Lemma. If τ is a CI - death time for P^μ satisfying (4.8), then for every (a,b) with $P^\mu(Y_\tau = (a,b)) > 0$ the (stochastic) initial law ν_{ab} and (substochastic) transition function r_{ab} of the pre - τ process given $(Y_\tau = (a,b))$ are determined by

(4.14) $$\nu_{ab}(x) = \mu(x) \; 1_{H_{ab}}(x) \; g_{ab}(x) \Big/ \sum_{y \in H_{ab}} \mu(y) \; g_{ab}(y) \quad (x \in J),$$

(4.15) $$r_{ab}(x,y) = p(x,y) \; 1_{V_{ab}}(x,y) \frac{g_{ab}(y)}{g_{ab}(x)} \quad (x \in J_{ab}, \; y \in J).$$

Proof. Clearly

$$\nu_{ab}(x) = P^\mu(X_0 = x \mid Y_\tau = (a,b))$$

and ν_{ab} is stochastic on J_{ab} since given $(Y_\tau = (a,b))$ the pre - τ process is alive certainly at time 0. But

$$P^\mu(X_0 = x, Y_\tau = (a,b)) = \mu(x) \; 1_{H_{ab}}(x) \; g_{ab}(x) \; P^a(X_1 = b,F)$$

by (4.11) and from this (4.14) follows by normalisation.

For the proof of (4.15) it is clear that

$$r_{ab}(x,y) = P(X_{n+1} = y, \tau > n+1 \mid X_n = x, \tau > n, Y_\tau = (a,b))$$

for $x \in J_{ab}$, $y \in J$ and any $n \in \mathbb{N}$ such that the event conditioned upon has probability > 0. Now by a derivation similar to that of (4.11)

$$P(X_n=x, X_{n+1}=y, \tau>n+1, Y_\tau=(a,b)) = P(X_n=x, X_{n+1}=y, T_{ab}^{n+2})\, g_{ab}(y) P^a(X_1=b,F)$$

and from this (4.15) follows since

$$P(X_n = x, X_{n+1} = y, T_{ab}^{n+2}) = P(X_n = x, T_{ab}^{n+1})\, p(x,y)\, 1_{V_{ab}}(x,y) . \qquad \square$$

With this lemma and the remarks preceding it in mind, a mapping δ which is the dual of the β from (3.3) may be defined as follows. Suppose given a probability μ on J, a (in general substochastic) transition function p on J, a state $a \in J$ and a sequence $T = (T^n)$ of terminal events given by (4.4).

Define $g : J \to \mathbb{R}$ by

$$g(x) = \sum_{n>0} P^x(Y_k \in V, 1 \leq k < n, X_{n-1} = a)$$

and write

$$\delta(\mu,p,a,T) = (J_T, \nu, r)$$

for the triple consisting of the set

$$J_T = \{x \in J : \sum_{n\geq 0} P^\mu(X_n = x, T^{n+1}) > 0,\ g(x) > 0\} ,$$

the probability

(4.16) $$\nu(x) = \mu(x)\, 1_H(x)\, g(x) \Big/ \sum_{y\in H} \mu(y)\, g(y)$$

on J_T, and the transition function r on J_T given by

$$(4.17) \qquad r(x,y) = p(x,y)\, 1_V(x,y)\, \frac{g(y)}{g(x)}\,.$$

Naturally (4.16) and (4.17) only make sense and are only of interest when $J_T \neq \emptyset$. In that case one should think of ν as defined on all of J and r(x,y) as defined for all $x \in J_T$, $y \in J$. The assertions that ν is a probability on J_T and that r is substochastic on J_T are then proved easily. It is also seen that for $x \in J_T$, $r(x,y) > 0$ only if $y \in J_T$ so that no relevant probability mass is discarded by restricting r to J_T.

With the notation from Lemma 4.13 in particular

$$\delta(\mu,p,a,T_{ab}) = (J_{ab},\nu_{ab},r_{ab})\,.$$

The role of δ is therefore to identify the law of the conditional pre-τ process given Y_τ, and δ picks out the three ingredients describing that law as a probability on Ω_Δ: the range space, the initial measure and the transition function.

Remark. For τ a CI-death time for P^μ, Lemma 4.13 provided the law of the conditional pre-τ process given Y_τ run in the forward direction of time. Following the discussion in Section 2 on time reversal one might of course as well have found the law of the conditional pre-τ process run backwards. Retaining the notation from Lemma 4.13 and conditioning on $(Y_\tau = (a,b))$, it is clear from (2.4) that the reversed pre-τ process (which of course starts at a) has transition function

$$\hat{r}_{ab}(x,y) = r_{ab}(y,x)\,\frac{e(y)}{e(x)}$$

on J_{ab}, where

$$e(z) = \sum_{n\geq 0} P^{\mu}(X_n = z, \tau > n \mid Y_\tau = (a,b)) .$$

Invoking (4.15) and (4.12) it is seen that

$$\text{(4.18)} \qquad \hat{r}_{ab}(x,y) = p(y,x)\, 1_{V_{ab}}(y,x)\, \frac{f_{ab}(y)}{f_{ab}(x)}$$

where

$$f_{ab}(z) = \sum_{n\geq 0} P^{\mu}(X_n = z, T_{ab}^{n+1}) .$$

In accordance with the remarks in Section 2, $\hat{r}_{ab}$ depends on μ while r_{ab} does not. This is the reason for considering the forward direction of time when defining δ.

Consider again a CI - death time τ for P and define the <u>death time relation for τ with respect to P</u> as the relation $\prec_{\tau,P}$ on J^2 given by

$$(c,d) \prec_{\tau,P} (a,b) \Leftrightarrow \sum_{n>0} P(Y_n = (c,d), \tau > n, Y_\tau = (a,b)) > 0 ,$$

i.e. $(c,d) \prec_{\tau,P} (a,b)$ if it is possible to have an (a,b) - transition at time τ with a (c,d) - transition preceding it. Usually I shall write $\prec$ instead of $\prec_{\tau,P}$.

(4.19). <u>Definition</u>. A CI - death time τ for P^{μ} is called <u>transition reproducing</u> if

$$(c,d) \prec_{\tau,P} (a,b) \Rightarrow \delta(\nu_{ab}, r_{ab}, c, T_{cd}) = (J_{cd}, \nu_{cd}, r_{cd})$$

whenever $P^{\mu}(Y_\tau = (c,d)) > 0$.

For later reference it is useful to write out in detail what the definition means in terms of range spaces, initial measures and transition functions. It is seen that τ is transition reproducing iff $(c,d) \prec (a,b)$ implies (for $P^{\mu}(Y_{\tau} = (c,d)) > 0$)

$$(4.20) \quad J_{cd} = \{x \in J : \sum_{n \geq 0} R_{ab}^{\nu(ab)}(X_n = x, T_{cd}^{n+1}) > 0,\ \tilde{g}_{cd}(x) > 0\},$$

$$(4.21) \quad \nu_{cd}(x) = \nu_{ab}(x)\, 1_{H_{cd}}(x)\, \tilde{g}_{cd}(x) \Big/ \sum_{y \in H_{cd}} \nu_{ab}(y)\tilde{g}_{cd}(y) \qquad (x \in J),$$

$$(4.22) \quad r_{cd}(x,y) = r_{ab}(x,y)\, 1_{V_{cd}}(x,y) \frac{\tilde{g}_{cd}(y)}{\tilde{g}_{cd}(x)} \qquad (x \in J_{cd}, y \in J),$$

where $\tilde{g}_{cd} : J \to \mathbb{R}$ is defined by

$$(4.23) \qquad \tilde{g}_{cd}(x) = \sum_{n>0} R_{ab}^{x}(Y_k \in V_{cd}, 1 \leq k < n, X_{n-1} = c)\ .$$

From these considerations it is easily checked that the definition is consistent so that although the sequence T_{cd} is not uniquely determined from P, $\delta(\nu_{ab}, r_{ab}, c, T_{cd})$ does not depend on the choice of T_{cd}.

(4.24). <u>Proposition.</u> If τ is a transition reproducing CI - death time for P, then the relation $\prec_{\tau,P}$ is transitive.

<u>Proof.</u> Suppose τ satisfies (4.8). In terms of the conditional pre - τ process the condition that $(c,d) \prec (a,b)$ is equivalent to requiring that $P(Y_{\tau} = (a,b)) > 0$ and that

$$\sum_{n>0} R_{ab}^{\nu(ab)}(Y_n = (c,d)) > 0\ .$$

To show that $\prec$ is transitive it is therefore enough to show that $(c,d) \prec (a,b)$, $(e,f) \prec (c,d)$ implies

(4.25) $$\sum_{n>0} R_{ab}^{\nu(ab)}(Y_n = (e,f)) > 0 .$$

But because τ is transition reproducing, (4.20) – (4.22) show that $J_{cd} \subset J_{ab}$ and that for $x, y \in J_{cd}$, $\nu_{cd}(x) > 0$ only if $\nu_{ab}(x) > 0$ and $r_{cd}(x,y) > 0$ only if $r_{ab}(x,y) > 0$. Since

$$\sum_{n>0} R_{cd}^{\nu(cd)}(Y_n = (e,f)) > 0$$

by the assumption $(e,f) \prec (c,d)$, (4.25) now follows. □

(4.26). Proposition. A random time τ is a transition reproducing CI - death time for the Markov probability P if and only if there exists $F \in \mathcal{F}$ and for every $(a,b) \in J^2$ subsets $H_{ab} \subset J$, $V_{ab} \subset J^2$ such that

(4.27) $$(\tau = n, Y_\tau = (a,b)) = (T_{ab}^n, Y_n = (a,b), \theta_{n-1} \in F) \quad P\text{-a.s.}$$

for $n \in N_+$, and

(4.28) $$(c,d) \underset{\tau,P}{\prec} (a,b) \Rightarrow H_{cd} \subset H_{ab}, \quad V_{cd} \subset V_{ab} .$$

Proof. Suppose that τ satisfies (4.27) with respect to $P = P^\mu$ and is transition reproducing. Define

$$H_{ab}^* = H_{ab} \cap \{x \in J_{ab} : \mu(x) > 0\} ,$$

$$V_{ab}^* = V_{ab} \cap \{(x,y) \in J_{ab}^2 : p(x,y) > 0\} .$$

From (4.14), (4.15) it follows immediately that (4.27) holds with the H_{ab}, V_{ab} replaced by H_{ab}^*, V_{ab}^* and of course that (4.14), (4.15) themselves are not altered by this replacement. For the 'only if' assertion it therefore remains to show that (4.28) is valid after this replacement.

To see this assume that $(c,d) \prec (a,b)$ and $P(Y_\tau = (c,d)) > 0$. By assumption (4.20) - (4.22) hold (with H^*_{ab}, V^*_{ab} inserted throughout) and the argument is then completed by verifying the following implications using also (4.12), (4.14) and (4.15),

$$x \in H^*_{cd} \Rightarrow \nu_{cd}(x) > 0 \Rightarrow \nu_{ab}(x) > 0 \Rightarrow x \in H^*_{ab},$$

$$(x,y) \in V^*_{cd} \Rightarrow r_{cd}(x,y) > 0 \Rightarrow r_{ab}(x,y) > 0 \Rightarrow (x,y) \in V^*_{ab}.$$

The 'if' assertion of the proposition is proved by showing that for a τ satisfying (4.27) and (4.28), (4.20) - (4.22) are valid when $(c,d) \prec (a,b)$, $P(Y_\tau = (c,d)) > 0$. By (4.28), $T^{n+1}_{cd} \subset T^{n+1}_{ab}$ for every n so that certainly $J_{cd} \subset J_{ab}$. But then the three desired identities follow easily when appealing to (4.14), (4.15) and observing that the $\tilde{g}_{cd}$ from (4.23) because of (4.28) satisfies

$$\tilde{g}_{cd}(x) = g_{cd}(x)\,\frac{g_{ab}(c)}{g_{ab}(x)} \qquad (x \in J_{ab}). \qquad \square$$

(4.29). <u>Definition</u>. Let $\mathcal{DO}$ be the class of random times τ such that for $n_0 \in N_+$

(4.30) $\quad \tau(\omega) = n_0 \Leftrightarrow n_0 =$

$$\sup\{n\in N_+ : \omega\in T^n_{\omega_{n-1}\omega_n}, H_{\omega_{k-1}\omega_k} \subset H_{\omega_{n-1}\omega_n}, V_{\omega_{k-1}\omega_k} \subset V_{\omega_{n-1}\omega_n}, 1\leq k<n, \theta_{n-1}\omega\in F\},$$

where $F \in \mathcal{F}$ is arbitrary and for each transition $(a,b) \in J^2$, $H_{ab} \subset J$, $V_{ab} \subset J^2$ are arbitrary with

$$T^n_{ab} = (X_0 \in H_{ab}, Y_k \in V_{ab}, 1 \leqq k < n) \qquad (n \in N_+)$$

determining the associated sequence of terminal events.

See also Proposition 4.32 below.

The main result on transition reproducing CI - death times can now be stated. The proof is structurally identical to that of Theorem 3.24 and is therefore omitted.

(4.31). Theorem. A random time τ is a transition reproducing CI - death time for the Markov probability P if and only if τ is P - equivalent to a random time in $\mathcal{DO}$.

If τ satisfies (4.30) and P^μ is Markov with $P^\mu(Y_\tau = (a,b)) > 0$, the range space, initial measure and transition function for the conditional pre - τ process given $(Y_\tau = (a,b))$ are determined by (4.12), (4.14) and (4.15) respectively when substituting H^*_{ab}, V^*_{ab} for H_{ab}, V_{ab} there and in (4.10), where

$$H^*_{ab} = H_{ab}, \qquad V^*_{ab} = \{(x,y) \in V_{ab} : H_{xy} \subset H_{ab}, V_{xy} \subset V_{ab}\} .$$

The dual of Proposition 3.31 is the following result.

(4.32). Proposition. A random time τ belongs to $\mathcal{DO}$ if and only if there exists $F \in \mathcal{F}$ and for every $(a,b) \in J^2$ a subset $H_{ab} \subset J$ and a transitive relation $\prec$ on J^2 compatible with the H_{ab} in the sense that $(c,d) \prec (a,b) \Rightarrow H_{cd} \subset H_{ab}$, such that for $n_0 \in N_+$

$$\tau(\omega) = n_0 \Leftrightarrow$$

$$n_0 = \sup\{n \in N_+ : \omega_0 \in H_{\omega_{n-1}\omega_n}, (\omega_{k-1},\omega_k) \prec (\omega_{n-1},\omega_n), 1 \leqq k < n, \theta_{n-1}\omega \in F\} .$$

As in the birth time case there is a minimal relation $\underset{\tau}{\prec}$ which can be used to represent $\tau \in \mathcal{DO}$ in this manner, viz.

$(c,d) \underset{\tau}{\prec} (a,b) \Leftrightarrow$

$\exists\ \omega\in\Omega,\ n\in N_+$ with $n<\tau(\omega)$ such that $Y_n(\omega) = (c,d)$, $Y_\tau(\omega) = (a,b)$.

(4.33). Example. The random times $\underline{\tau}$, $\overline{\tau}$ from Example 3.33 both belong to $\mathcal{DO}$.

I shall conclude this section with some further remarks on time reversal and duality. Suppose τ is a CI - death time for P^μ satisfying (4.8) so that the law of the pre - τ chain given $(Y_\tau = (a,b))$ is determined by (4.12), (4.14), (4.15) with g_{ab} given by (4.10). Also, (4.18) yields the transition function for the reversed pre - τ process given $(Y_\tau = (a,b))$.

It is clear from (4.8) that without loss of generality it may and henceforth shall be assumed that

(4.34) $$\sum_{n\geq 0} P^\mu(X_n = x) > 0 \qquad (x \in J).$$

The sequence (T^n_{ab}) of terminal events when read backwards corresponds to the coterminal event

$$\hat{C}_{ab} = (Y_k \in \hat{V}_{ab},\ 1 \leq k < \zeta,\ X_{\zeta-1} \in H_{ab})$$

$$= (Y_k \in \hat{V}_{ab} \cup (H_{ab} \times \{\Delta\}) \cup \{(\Delta,\Delta)\},\ k \geq 1)$$

in the space $\Omega(\zeta < \infty)$ to which the paths of the reversed conditional pre - τ process belong. (Of course $\hat{V}_{ab} = \{(x,y) \in J^2 : (y,x) \in V_{ab}\}$). It is therefore natural to attempt to interpret the law of the reversed pre - τ process given

$(Y_\tau = (a,b))$ as the conditional Markov probability

(4.35) $$\hat{P}^a(\cdot \mid \hat{C}_{ab})$$

for some transition function $\hat{p}$ (not depending on (a,b)) in duality to p and depending (cf. (4.18)) on μ.

More precisely, I want to find $\xi : J \to \mathbb{R}$ strictly positive and subinvariant such that with

(4.36) $$\hat{p}(x,y) = p(y,x)\,\frac{\xi(y)}{\xi(x)},$$

the identity (cf. (3.4))

(4.37) $$\hat{p}(x,y)\, 1_{\hat{V}_{ab}}(x,y)\,\frac{\hat{P}^y(\hat{C}_{ab})}{\hat{P}^x(\hat{C}_{ab})} = \hat{r}_{ab}(x,y)$$

holds for all (a,b) with $P^\mu(Y_\tau = (a,b)) > 0$, all $x \in J_{ab}$, $y \in J$ with in particular

(4.38) $$\hat{P}^x(\hat{C}_{ab}) > 0 \qquad (x \in J_{ab}).$$

I shall not do this in complete generality but restrict myself to the case where all states $x \in J$ are transient (with respect to p). Then the series (4.34) converges for all $x \in J$ and I claim that with the natural choice

$$\xi(x) = \sum_{n \geq 0} P^\mu(X_n = x)$$

(which by (4.34) is > 0), the $\hat{p}$ given by (4.36) is a substochastic transition function generating a Markov chain with finite life time such that the requirements (4.37) and (4.38) are met.

It is easy and standard that

$$\sum_{z \in J} p(z,y)\ \xi(z) = \xi(y) - \mu(y) \qquad (y \in J). \tag{4.39}$$

Therefore ξ is subinvariant for p and $\hat{p}$ is substochastic.

Since $\hat{p}(y,\Delta) = 1 - \sum_{z \in J} \hat{p}(y,z)$, (4.39) may also be written

$$\xi(y)\ \hat{p}(y,\Delta) = \mu(y) \qquad (y \in J) \tag{4.40}$$

and using this it is easy to see that each $\hat{P}^x$ has finite life time:

$$\begin{aligned}\hat{P}^x(\zeta < \infty) &= \sum_{n \geq 1} \sum_{y \in J} \hat{P}^x(X_{n-1} = y, X_n = \Delta) \\ &= \sum_{n \geq 1} \sum_{y \in J} P^y(X_{n-1} = x)\ \frac{\xi(y)}{\xi(x)}\ \hat{p}(y,\Delta) \\ &= 1 .\end{aligned}$$

It remains to check that (4.37) and (4.38) are valid. But comparing (4.18) with (4.36) and (4.37) it is seen that for this it is sufficient that (4.38) holds and that

$$\frac{\xi(y)\ \hat{P}^y(\hat{C}_{ab})}{\xi(x)\ \hat{P}^x(\hat{C}_{ab})} = \frac{f_{ab}(y)}{f_{ab}(x)} \qquad (x \in J_{ab}, y \in J).$$

It is therefore enough even that for some $K > 0$

$$\xi(x)\ \hat{P}^x(\hat{C}_{ab}) = K\ f_{ab}(x) \qquad (x \in J), \tag{4.41}$$

since (4.38) then follows in particular because $f_{ab} > 0$ on J_{ab}, (cf. (4.12)).

Now by the definition of f_{ab},

$$f_{ab}(x) = \sum_{y \in H_{ab}} \mu(y) \sum_{n \geq 1} P^y(Y_k \in V_{ab}, 1 \leqq k < n, X_{n-1} = x),$$

while

$$\xi(x)\ \hat{P}^x(\hat{C}_{ab}) = \xi(x) \sum_{n \geq 1} \sum_{y \in H_{ab}} \hat{P}^x(Y_k \in V_{ab}, 1 \leqq k < n, X_{n-1} = y, X_n = \Delta)$$

$$= \sum_{y \in H_{ab}} \hat{p}(y,\Delta) \sum_{n \geq 1} \xi(y)\ P^y(Y_k \in V_{ab}, 1 \leqq k < n, X_{n-1} = x)$$

which by (4.40) reduces to the expression for $f_{ab}(x)$ and (4.41) is proved.

5. The class $\mathcal{BO} \cap \mathcal{DO}$

In [3] Theorem 6.2, it was shown that if the transition function p is irreducible, a random time τ is both a regular birth time and a regular death time for P iff τ is P - equivalent to a terminal or a coterminal time.

As a purely algebraic analogue of this result, I shall in this section describe the intersection of the classes $\mathcal{BO}$ and $\mathcal{DO}$. A characterisation result similar to Theorems 3.24 and 4.31 may then be obtained by showing (which is not too difficult) that a random time τ is a transition reproducing CI - birth and CI - death time for a Markov probability P iff τ is P - equivalent to a random time in $\mathcal{BO} \cap \mathcal{DO}$.

(5.1). Definition. Two transitive relations $\succ$ and $\prec$ on J^2 are said to be nearly disjoint if it is impossible to find $n \geqq 2$ and $x_0, \ldots, x_n \in J$ such that

$$(5.2) \qquad \begin{aligned} (x_0,x_1) &\succ (x_{i-1},x_i)\,, \qquad (1 < i \leqq n) \\ (x_{i-1},x_i) &\prec (x_{n-1},x_n)\,, \qquad (1 \leqq i < n)\,. \end{aligned}$$

(5.3). Example. If $\succ$ and $\prec$ are disjoint (as subsets of $J^2 \times J^2$) they are nearly disjoint: for (5.2) to hold it is required in particular that $(x_0,x_1) \succ (x_{n-1},x_n)$ and $(x_0,x_1) \prec (x_{n-1},x_n)$.

The desired result is the following proposition.

(5.4). Proposition. If a random time τ belongs to the intersection $\mathcal{BO} \cap \mathcal{DO}$, then there exists two nearly disjoint relations $\succ$ and $\prec$ on J^2 and for every $(a,b) \in J^2$ there exists invariant $C_{\infty,ab} \in \mathcal{F}$ and subsets $H_{ab} \subset J$ such that the relation $\succ$ and the $C_{\infty,ab}$ are compatible and the relation $\prec$ and the H_{ab} are compatible and

(5.5) $(\tau=n, Y_\tau=(a,b)) = (X_0 \in H_{ab}, Y_k \prec (a,b), 1 \leqq k<n, Y_n=(a,b), (a,b) \succ Y_k, k>n)$

for $n \in N_+$, $(a,b) \in J^2$.

If conversely the relations $\succ$ and $\prec$ and the $C_{\infty,ab}$, H_{ab} are given with $\succ$ and $\prec$ nearly disjoint and the compatibility conditions above satisfied, then the sets on the right hand side of (5.5) are mutually disjoint and (5.5) defines a random time τ belonging to $\mathcal{BO} \cap \mathcal{DO}$.

Remark. The compatibility requirements are of course those appearing in Propositions 3.31 and 4.32.

Proof. One half is very easy. With $\succ$ and $\prec$ nearly disjoint it is immediate that the sets, from now on denoted by M, on the right of (5.5) are disjoint for n and (a,b) varying, and then it is clear from Propositions 3.31 and 4.32 that $\tau \in \mathcal{BO} \cap \mathcal{DO}$.

Suppose now that $\tau \in \mathcal{BO} \cap \mathcal{DO}$. From Propositions 3.31 and 4.32 two descriptions of τ are available so that

$$(5.6) \quad (\tau = n, Y_\tau = (a,b)) = (F_n, Y_n = (a,b), (a,b) \succ Y_k, k > n, C_{\infty,ab})$$
$$= (X_0 \in H_{ab}, Y_k \prec (a,b), 1 \leq k < n, Y_n = (a,b), \theta_{n-1} \in F)$$

where we may and shall assume that $\succ = \underset{\tau}{\succ}$ and $\prec = \underset{\tau}{\prec}$ are the minimal relations associated with τ as described following the two propositions. (But $F_n \in \mathcal{F}_n$, $F \in \mathcal{F}$ are sets which are not the same as the ones appearing in the representations of τ in the propositions).

Intersecting the two sets on the right of (5.6) it is seen that $(\tau = n, Y_\tau = (a,b)) \subset M$. For the converse, suppose that $\omega \in M$. Then of course $(a,b) \succ (\omega_{k-1}, \omega_k)$ for $k > n$ which in view of the definition of $\underset{\tau}{\succ}$ is enough to guarantee that $(Y_\tau = (a,b)) \neq \emptyset$. But if $\omega' \in (Y_\tau = (a,b))$ with $\tau\, \omega' = m$, the first half of (5.6) shows that the path

$$\omega'' = (\omega_0', \ldots, \omega_{m-2}', \omega_{n-1}, \omega_n, \ldots)$$

belongs to $(\tau = m, Y_\tau = (a,b))$ and therefore, by the second half of (5.6), it is true that $\theta_{n-1}\omega \in F$. Because $\omega \in M$ another application of the second half of (5.6) shows that $\omega \in (\tau = n, Y_\tau = (a,b))$.

Thus (5.5) holds and it remains only to see that $\succ$ and $\prec$ are nearly disjoint. Therefore assume that $n \geq 2$ and $x_0, \ldots, x_n \in J$ exist so that (5.2) holds. Then in particular $(x_0,x_1) \gtrless (x_{n-1},x_n)$ so by the minimality of $\succ$ and $\prec$, $(Y_\tau = (x_0,x_1)) \neq \emptyset$, $(Y_\tau = (x_{n-1},x_n)) \neq \emptyset$ and (5.5) implies the existence of $m \in N_+$ and $\alpha_0, \ldots, \alpha_m$ with $\alpha_0 \in H_{x_0x_1}$, $(\alpha_{i-1},\alpha_i) \prec (x_0,x_1)$ for $1 \leqq i < m$, $(\alpha_{m-1},\alpha_m) = (x_0,x_1)$ and of $\beta = (\beta_0,\beta_1, \ldots)$ with $(\beta_0,\beta_1) = (x_{n-1},x_n)$, $(x_{n-1},x_n) \succ (\beta_{i-1},\beta_i)$ for $i > n$, $\beta \in C_{\infty,x_{n-1}x_n}$. But now consider the path

$$\omega = (\alpha_0, \ldots, \alpha_{m-2}, x_0, x_1, \ldots, x_{n-1}, x_n, \beta_2, \beta_3, \ldots)$$

and verify that because $\succ$ and $\prec$ are transitive and $H_{x_0x_1} \subset H_{x_{n-1}x_n}$, $C_{\infty,x_0x_1} \supset C_{\infty,x_{n-1}x_n}$, (5.5) shows that

$$\omega \in (\tau = m, Y_\tau = (x_0,x_1)) \cap (\tau = m+n-1, Y_\tau = (x_{n-1},x_n))$$

which is impossible. □

It should be emphasized that it is necessary to consider nearly disjoint rather than disjoint relations in the proposition. Due to the minimality of $\underset{\tau}{\succ}$ and $\underset{\tau}{\prec}$, to see this it is enough to give an example of a $\tau \in \mathcal{BO} \cap \mathcal{DO}$ and transitions (a,b), (c,d) such that $(a,b) \underset{\tau}{\gtrless} (c,d)$. But let $e \neq f$ denote two states in J and define $(0 < \tau < \infty)$ as the union of the two disjoint sets

$$(\tau = 1, Y_\tau = (a,b)) = (Y_1 = (a,b), X_2 = e, X_3 = c, X_4 = X_5 = \ldots = d),$$

$$(\tau = 4, Y_\tau = (c,d)) = (Y_1 = (a,b), X_2 = f, X_3 = c, X_4 = X_5 = \ldots = d).$$

Then $\underset{\tau}{\succ}$ and $\underset{\tau}{\prec}$ are given by

$$(a,b) \underset{\tau}{\succ} (b,e), (e,c), (c,d), (d,d), \qquad (c,d) \underset{\tau}{\succ} (d,d),$$

$$(a,b), (b,f), (f,c) \underset{\tau}{\prec} (c,d).$$

Clearly both relations are transitive so $\tau \in \mathcal{BO} \cap \mathcal{DO}$, but $(a,b) \underset{\tau}{\gtrless} (c,d)$.

These considerations incidentally show that $\mathcal{BO} \cap \mathcal{DO}$ contains other random times than the $\underline{\tau}$, $\overline{\tau}$ discussed in Examples 3.33 and 4.33, since for those the associated two relations are disjoint.

I shall conclude with a brief description of the path decompositions valid for random times in $\mathcal{BO} \cap \mathcal{DO}$.

Suppose τ is given by (5.5) and let P^μ be Markov. If $P^\mu(Y_\tau = (a,b)) > 0$, then conditionally on $(Y_\tau = (a,b))$ the pre - τ and post - τ processes are independent and both Markov with stationary transitions.

If ν_{ab} is the initial law and r_{ab} the transition function of the conditional pre - τ process, then ν_{ab} and r_{ab} are given by (4.14) and (4.15) with

$$V_{ab} = \{(x,y) \in J^2 : (x,y) \prec (a,b)\} .$$

Finally, the transition function q_{ab} of the conditional post - τ process is given by (3.4) with

$$V = \{(x,y) \in J^2 : (a,b) \succ (x,y)\} ,$$

$$C = (Y_n \in V, n \in N_+, C_{\infty,ab}) .$$

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