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Four Simple Characterizations of Standard Borel Spaces without any reference to a topology or a metric

The theory behind the usual definition of a standard Borel space (given below) is certainly not elementary [4]. These spaces are of great interest, e.g. the conditional distribution of a measurable mapping from a standard Borel space onto another such space admits a regular version ([4], p. 147). Our results are of some theoretical interest [5] and may also be of some educational value to those who don't want to coat their theorems in too technical terms.

Definition

A countably generated Borel space is a standard Borel space X if there exists a complete separable metric space Y (i.e. a Polish space) such that the σ-algebra on X and the Borel σ-algebra on Y are σ-isomorphic.

Theorem

Let $(X,\mathcal{F})$ denote a countably generated Borel space, and $\mathcal{A}$ denote a countable algebra over X generating $\mathcal{F}$, i.e. $\mathcal{F} = \sigma(\mathcal{A})$.

Then

a) $(X,\mathcal{F})$ is standard iff there exist an $\mathcal{A}$ such that every finite and finitely additive measure on $\mathcal{A}$ is σ-additive on $\mathcal{A}$;

b) $(X,\mathcal{F})$ is standard iff there exists an $\mathcal{A}$ such that every $\mathcal{A}$-measurable partition of X is finite. [$\mathcal{D} \subseteq \mathcal{A}$ is a $\mathcal{A}$-measurable partition of X iff $X = \cup\mathcal{D}$, $\emptyset \notin \mathcal{D}$, and the sets in $\mathcal{D}$ are pairwise disjoint];

c) $(X,\mathcal{F})$ is standard iff there exist an $\mathcal{A}$ such that every covering $\mathcal{C} \subseteq \mathcal{A}$ of $A \in \mathcal{A}$ contains a finite subcovering;

d) $(X,\mathcal{F})$ is standard iff there exists an $\mathcal{A}$ which is semicompact [$\mathcal{A}$ is semicompact iff $A_n \in \mathcal{A}$, $n \in \mathbb{N}$, $\bigcap_{n=1}^{n} A_n \neq \emptyset$ for $\forall n$ implies $\bigcap_{n\geq 1} A_n \neq \emptyset$].

Remark

It will follow from the proof of the theorem, that if $(X,\mathcal{F})$ is standard and $\mathcal{A}$ satisfies one of the conditions stated in a),b),c), and d) then $\mathcal{A}$ satisfies all four.

For the proof of the theorem we recall the following: Let X_1 and X_2 denote metric spaces. Two Borel sets $B_1 \subseteq X_1$ and $B_2 \subseteq X_2$ are called isomorphic, if there exists a bijection $\phi : B_1 \to B_2$ such that both ϕ and ϕ^{-1} are measurable. Then the celebrated isomorphism theorem [4, p. 14] states: Suppose X_1 and X_2 are complete and separable. Then B_1 and B_2 are isomorphic iff they have the same cardinality.

Proof of the Theorem

We shall first prove that a),b),c), and d) are equivalent, then "if" in b), and finally "only if" in c).

"a)⇐b)⇔c)". Obvious.

"a)⇒b)". Suppose every finite and finitely additive measure on $\mathcal{A}$ is σ-additive. Suppose also that there exists an infinite partition $\{A_n : n \in \mathbb{N}\} \subseteq \mathcal{A}$ such that $X = \bigcup_{n=1}^{\infty} A_n$. Choose a $\mathcal{F}$-atom $e_n \subseteq A_n$ for $\forall n \in \mathbb{N}$. Define $E = \bigcup_{n=1}^{\infty} e_n$. From Prop. 1. (stated below!) we conclude that there exists a finitely additive measure $Q : E \cap \sigma(\{e_n : n \in \mathbb{N}\}) \to \mathbb{R}$ such that $1 = Q\ E) > \sum_{n=1}^{\infty} Q(e_n)$, i.e. Q is not σ-additive.

Define a mapping $P : \mathcal{A} \to \mathbb{R}$ such that $P(A) = Q(E \cap A)$ for $\forall A \in \mathcal{A}$. Obviously P is finitely additive on $\mathcal{A}$, but $1 = P(X) = Q(E) > \sum_{n=1}^{\infty} Q(e_n) = \sum_{n=1}^{\infty} Q(E \cap A_n) = \sum_{n=1}^{\infty} P(A_n)$ - a contradiction. Hence a) implies b).

"b)⇔d)". Suppose $A_n \in \mathcal{A}(n \in \mathbb{N})$ and $\bigcap_{n=1}^{m} A_n \neq \emptyset$ for all m. It is sufficient to consider the case $(A_n)_{n \geqq 1}$ strictly decreasing. If $\bigcap_{n \geqq 1} A_n = \emptyset$ then $X = \bigcup_{n \geqq 1} A_n^c = A_1^c \cup \bigcup_{n \geqq 1} (A_{n+1}^c \setminus A_n^c)$. Since $A_{n+1}^c \setminus A_n^c \neq \emptyset$ this contradicts the finite paratition property. Hence b)⇒d).

If $\{D_n : n \in \mathbb{N}\} \subseteq \mathcal{A}$ is a partition of X, then consider $A_m = \bigcup_{n \geqq m} D_n, m \in \mathbb{N}$: Obviously $\bigcap_{m \geqq 1} A_m = \emptyset$. From this follows d)⇒b).

"if" in b). This follows from Prop. 2.

"only if" in c). We break the proof up into three cases.

Case 1. Suppose $\mathcal{F}$ has only finitely many atoms. Then $\mathcal{A} = \mathcal{F}$ serves our purpose.

Case 2. Suppose $\mathcal{F}$ has infinitely but countably many atoms. Then the statement follows from Prop. 1.

Case 3. Suppose $\mathcal{F}$ has uncountably many atoms. From the isomorphism theorem ([4], p. 14. See also p. 147) it follows that the usual σ-algebra over a complete separable metric space with uncountably many atoms is σ-isomorphic to the usual σ-algebra over M, where M is the countable product of {o,1} with the natural product topology. It is therefore sufficient to prove that there exists an algebra $\mathcal{A}$ with the required properties. We note that M is a separable, totally disconnected, and compact topological space [2]. Consider the algebra $\mathcal{A}_M$ over M generated by all sets of the form $\{(x_1,x_2,\ldots)\in M : x_i=1\}$. These sets are countably many in number and clopen (i.e. both closed and open). Hence the same holds for the algebra $\mathcal{A}_M$. Since M is compact it is clear that $\mathcal{A}_M$ satisfies the requirments in c).

Remarks

Suppose $(X_i,\mathcal{F}_i)$ is standard and $\mathcal{A}_i$ has the properties stated in a),b), c), and d) above, i = 1,2,3,... . Then $\mathcal{A}_1$ contains at most finitely many $\mathcal{F}_1$-atoms. $\mathcal{A}_1$ is in general not unique and certainly not in Case 2.

If $F \in \mathcal{F}_1$, then $(F,\mathcal{F}_1\cap F)$ is standard but $\mathcal{A}_1\cap F$ will not nesessarily have the required properties, if $F \notin \mathcal{A}_1$ or F is not a $\mathcal{F}_1$-atom.

$\prod_{i=1}^{\infty}(X_i,\mathcal{F}_i)$ is standard and the algebra $\prod_{i=1}^{\infty}\mathcal{A}_i$ (this is in general not a σ-algebra!) will have the required properties.

On finitely additive measures

In this section we shall prove the existence of a finitely additive probability measures [In short p-measure], which are not σ-additive, on $(X,\mathcal{F})$ where $\mathcal{F}$ contains infinitely but countably many atoms. Of course $(X,\mathcal{F})$ is standard. The existence of such p-measures has probably been known for a long time ([2], ex. p.73; [3]), but we shall give our own proof because we want to digress a little from our main subject and look at the simplest structure of these measures.

Proposition 1

$(X,\mathcal{F})$ is a Borel space with infinite but countably many atoms e_i, $i = 0,1,2,\ldots$. Hence $X = \bigcup_{i=0}^{\infty} e_i$, $e_i \neq \emptyset$, $e_i \cap e_j = \emptyset$ $(i \neq j)$.

(i) The algebra $\mathcal{A}$ generated by the sets $e_0 \cup \bigcup_{m=n}^{\infty} e_m = A_n$, $n \in \mathbb{N}$ is countable, generates $\mathcal{F}$, and every $\mathcal{A}$-measurable partition of X is finite.

(ii) There exists a finitely additive p-measure on $\mathcal{F}$ which is not σ-additive.

Proof

(i) It is clear that $\mathcal{A}$ is countable. $\mathcal{F} = \sigma(\mathcal{A})$, since $e_n = A_n \setminus A_{n+1}$, $n \in \mathbb{N}$, and $e_0 = \bigcap_{n=1}^{\infty} A_n$. If $\{F_i\}$ is a partition of X, then for some i say i_0, $e_0 \subseteq F_{i_0}$. If $F_{i_0} \in \mathcal{A}$, then $X \setminus F_{i_0}$ contains only finitely many atoms, and $\# \{F_i\} < +\infty$.

(ii) The proof will follow from the considerations below, if we note the obvious fact, that all Borel spaces with infinitely, but countably many atoms have σ-isomorphic σ-algebras,

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We shall use the following version of Riesz's well known representation theorem quoted from [4] (p. 35):

Let (X,ρ) be a metric space, $C(X)$ be the Banach space of all bounded real valued continuous functions with the sup-norm and $\mathfrak{A}$ the algebra generated by all the open subsets of X.

If Λ is a nonnegative linear functional on $C(X)$ such that $\Lambda(1) = 1$, then there exists a unique finitely additive regular p-measure μ on $\mathfrak{A}$ such that

$$\Lambda(f) = \int f d\mu \quad \text{for } \forall f \in C(X).$$

Conversely, if μ is a finitely additive p-measure on $\mathfrak{A}$ then the map: $f \to \int f d\mu$ is nonnegative, linear, and $\Lambda(1) = 1$.

Suppose $x_0 \in X$ and any neighbourhood of x_0 contains infinitely many points.Define a nonnegative linear functional Λ on $C(X)$:

$$\Lambda(f) = f(x_0) \quad \text{for } \forall f \in C(X).$$

$\Lambda(1) = 1$, and there exists a unique finitely additive regular p-measure μ on $\mathfrak{A}$. Obviously $\mu A = 1_A(x_0)$ for $\forall A \in \mathfrak{A}$, so μ is σ-additive.

Consider the space $(\tilde{X},\tilde{\rho})$, where $\tilde{X} = X \setminus \{x_0\}$ and $\tilde{\rho} = \rho|_{\tilde{X} \times \tilde{X}}$.

Define a mapping $p : C(\tilde{X}) \to \mathbb{R}$ such that

$$p(f) = \lim_{r\to 0} \sup_{x\in\tilde{B}(r)} f(x)$$

where $\tilde{B}(r) = \{x \in \tilde{X} : \rho(x,x_0) < r\}$.

Then for $f,g \in C(\tilde{X})$,

$$p(\alpha f + \beta g) \leqq \alpha p(f) + \beta p(g), \quad \alpha,\beta \in \mathbb{R}.$$

The set $\tilde{C} = \{f \in C(\tilde{X}) : f = g|_{\tilde{X}}$ for some $g \in C(X)\}$ is a linear subspace of $C(\tilde{X})$ and $p|_{\tilde{C}}$ is a nonnegative linear functional on $\tilde{C}$. Hence by the Hahn-Banach theorem there exists a nonnegative linear functional Λ on $C(\tilde{X})$ such that $\Lambda(f) = p(f)$ for $\forall f \in \tilde{C}$. Hence $\Lambda(1) = 1$.

According to the quoted theorem there exists a unique finitely additive regular p-measure ν on $\tilde{\mathfrak{A}}$. Obviously $\nu(\tilde{B}(n^{-1}) = 1$ for $\forall n \in \mathbb{N}$. Since $\bigcap_{n=1}^{\infty} \tilde{B}(n^{-1}) = \emptyset$ ν is not σ-additive.

Consider now the triple $(\tilde{X}, \tilde{\mathfrak{A}}, \nu)$. It is possible to extend it to $(X, \mathfrak{A}, \bar{\nu})$ where

$$\bar{\nu}(A) = \nu(A \setminus \{x_0\}) \quad \text{for } \forall A \in \mathfrak{A}.$$

Then $\bar{\nu}(\{x_0\}) = 0$ and ν is finitely additive but it is not regular (and of course not σ-additive!). Consider $\{x_0\}$; If G is an open set containing x_0 then $\bar{\nu}(G) = 1$.

It is interesting to note that $\bar{\nu}$ and μ are singular in the sense that $\bar{\nu}(\{x_0\}) = 0$ and $\mu(\{x_0\}) = 1$, although they represent the same functional on C(X), i.e. $\int f d\bar{\nu} = \int f d\mu$ for all $f \in C(X)$.

Suppose that X is infinite but countable. Denote the distinct elements in X by x_n, $n = 0,1,2,\ldots$. Suppose also that $\lim_n \rho(x_n, x_0) = 0$. Then it is seen that the conciderations above applied to (X,ρ) prove Prop 1.(ii) and reveal the simplest structure of finitely additive p-measure on $(X,\mathcal{F})$ since in this case $\mathfrak{A} = \mathcal{F}$ = the powerset of X.

Example

Suppose X is a countably infinite metric space where the distinct elements are x_n, $n = 1,2,\ldots$. Let $\mathcal{A}_k$ denote the σ-algebra generated by the sets $\{x_k\} \cup \{x_n : n \geqq m\}, m = 1,2,\ldots$. Let $\hat{\mathcal{A}}$ denote the cofinite algebra over X and $\mathcal{F}$ the powerset of X. Then $\mathcal{A}_k \subseteqq \hat{\mathcal{A}} \subseteqq \mathcal{F}$, $\sigma(\mathcal{A}_k) = \sigma(\hat{\mathcal{A}}) = \mathcal{F}$ and $\mathcal{A}_k$ have the finite partition property stated in Prop. 1 (i). Suppose also that μ_k is the point measure determined by $\mu_k(\{x_k\}) = 1$. Then we have, when $k \to \infty$:

$$\mu_k \overset{w}{\to} \mu_m \quad \text{iff} \quad x_k \to x_m ;$$

$$\mu_k(\{x\}) \to 0 \quad \text{for } \forall x \in X;$$

$$\mu_k(A) \to \mu_m(A) \text{ for } \forall A \in \mathcal{A}_m;$$

$$\mu_k(A) \to \nu(A) \quad \text{for } \forall A \in \hat{\mathcal{A}}$$ where ν is finitely additive p-measure on $\hat{\mathcal{A}}$ determined by $\nu(A) = 0$ or 1 if $\# A < \infty$ or $\# (X \setminus A) < \infty$ respectively.

On totally disconnected compact spaces

Proposition 2

Let $(X,\mathcal{F})$ denote a Borel space and $\mathcal{A}$ denote a countable algebra over X generating $\mathcal{F}$. Suppose that any $\mathcal{A}$-measurable partition of X is finite (See Th. b)).

Then $\mathcal{F}$ is σ-isomorphic to the naturnal (or usual!) Borel-algebra of a totally disconnected compact metric space.

Proof:

It follows that $\mathcal{F}$ is contably generated and hence contains its atoms. We shall identify the points in each $\mathcal{F}$-atom and thus work with the canonical representation of $(X,\mathcal{F})$ denoted in the following by $(\tilde{X},\tilde{\mathcal{F}})$ ([4], p. 133).

This identification process will map $\mathcal{A}$ onto an algebra denoted by $\tilde{\mathcal{A}}$. Obviously $\mathcal{F}$ and $\tilde{\mathcal{F}}$ are σ-isomorphic. We note that $\tilde{\mathcal{A}}$ separates the points in X.

Let $\tilde{\mathcal{T}}$ denote the topology having $\tilde{\mathcal{A}}$ as a base. Then it is clear that $(\tilde{X},\tilde{\mathcal{T}})$ is a separable Hausdorff space and $\tilde{\mathcal{F}} = \sigma(\tilde{\mathcal{T}})$. Since the sets in $\tilde{\mathcal{A}}$ are clopen , i.e. both open and closed, the space $(\tilde{X},\tilde{\mathcal{T}})$ is totally disconnected [2]. The compactness follows from the finite partition property assumed. It follows from the Urysohn metrization theorem ([2], p. 24) that $(\tilde{X},\tilde{\mathcal{T}})$ is metrizable as a compact-and therefore as a complete metric space.

Remark

It is easily proved that $\tilde{A}$ contains every clopen set in $\tilde{T}$. We note also that $\tilde{A}$ is a determinating class in the theory of weak convergence ([1], Th 2 2), and every open set $\tilde{T}$ is the disjoint union of countably many clopen sets.

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