Steen A. Andersson

Distribution of Maximal Invariants Using Proper Action and Quotient Measures

Preprint
March
1979
2

Institute of Mathematical Statistics
University of Copenhagen

Steen A. Andersson

DISTRIBUTION OF MAXIMAL INVARIANTS USING PROPER ACTION AND QUOTIENT MEASURES*

Preprint 1979 No. 2

INSTITUTE OF MATHEMATICAL STATISTICS
UNIVERSITY OF COPENHAGEN

March 1979

Summary

This paper generalizes results of Wijsman concerning representation of the distribution of a maximal invariant by densities. The idea is to use proper action and quotient measure directly defined on the space of orbits. This notion provides the results without the usual complicated assumptions and concepts.

AMS 1970 subject classification. Primary 62A05, 62H10; Secondary 62H15, 28A70.

Key words and phrases. Maximal invariant, probability ratio, proper action, quotient measure.

Preface

This note was carried out while I was visiting Department of Statistics, Stanford University in the academic year 1977-78.

The influence of the surrounding compelled me into some interest in the field of non-centrality distributions of maximal invariants. It turned out that the notion of proper action and quotient measure, which I together with two of my colleagues Hans Brøns and Søren Tolver Jensen have used for years in a development of an algebraic theory for normal statistical models, was suited to express the representation of non-central distribution of a maximal invariant. In this way most of the arguments were turned into simple consequences of the theory of quotient measures.

1. Introduction

Consider a statistical model with sample space X, parameter set Θ and $\theta \to P_\theta$; $\theta \in \Theta$ as the parametrization of the unknown probability measures on X. For a subset Θ_o of Θ we have a statistical testing problem of testing the hypothesis $H_o : \theta \in \Theta_o$ versus the hypothesis $H : \theta \in \Theta$. Let now $t : X \to Y$ be a function related to the testproblem, for example an estimator under H_o or a function with relation to teststatistics. For the testing problem above the transformed measure $t(P_\theta)$ could be or is called a central distribution if $\theta \in \Theta_o$ and a non-central distribution if $\theta \in \Theta \setminus \Theta_o$.

For some problems in multivariate statistical analysis there exists a huge litterature in which one tries to represent the non-central distributions. These representations are given in terms of what we could call the correctionfactor, which is the density of $t(P_\theta)$, $\theta \in \Theta$ with respect to $t(P_{\theta_o})$, $\theta_o \in \Theta_o$ that is the ratio $\frac{dt(P_\theta)}{dt(P_{\theta_o})}$. In the description of the correctionfactor in some special cases group actions play a fundamental role.

The special studies have created the theory of hypergeometric functions of matrix arguments and their expansion in zonal polynomials. The theory goes back to A.T. James [10]. Several authors since then have concentrated on the study of these functions with the aim of obtaining asymptotic results and optimality properties of estimators and teststatistics. The review paper by R.T. Muirhead [12] gives the relevant references.

The application of group actions often forces one into the study of a maximal invariant function, for example an estimator can be a maximal invariant function or if the statistical problem is invariant under a group action all invariant teststatistics has a unique factorization through a maximal invariant function. The use of invariance and Haar measures goes back to Stein [13]. For this reason there also exists litterature which concentrates on the problem of finding the distribution of a maximal invariant function from a general point of view, R. Berk [3], J. Bondar [4], V. Koehn [11], Wijsman [15], and [16].

Roughly speaking the problem is the following: Let G be a group acting on X and let P be a probability measure on X which has a density f with respect to a relatively invariant measure λ on X, that is $P = f\lambda$. With $\pi : X \to X/G$ we denote the orbit projection, that is, X/G is the space of orbits and π maps an $x \in X$ into the orbit through x. Different representations of X/G and π usually goes under the name of a maximal invariant function. Often one wants to find maximal invariant functions $t : X \to Y$, where Y has a nice extra structure, which allows one to define a "natural" measure ν on Y such that t(P) can be represented as a density with respect to ν. One representation is to choose Y to be a global cross-section and t to be a projection on Y along orbits. If one furthermore supposes that G together with another group H are subgroups in a third group K, such that $K = HG$ and K acts transitively on X one has under complicated conditions on the isotropic subgroups in K, H and G that H act transitively on Y and ν becomes the unique (up to a positive constant) invariant measure on Y. Furthermore this is mixed up with conditions and concepts from the theory of differential geometry, as Lie groups, analytic manifolds and mappings. In the above set up one has to concern oneself about the existence of K, H and global-cross sections and a lot of unnecessary structure.

When global cross sections and extra groups fail to exist one then studies the existence of local-cross sections, which together with the additional assumptions that X be an open subset of a Euclidian space and a Cartan space under G, that λ be the restriction of the Lebesgue measure and G be a Lie subgroup of the general linear group, provide an expression for the "probability ratio" $\frac{dt(P_1)}{dt(P_2)}$, where $P_1 = f_2\lambda$ and $P_2 = f_2\lambda$.

Using the notation of the quotient measure we shall show how most of the complicated assumptions and existence problems disappear. In particular we shall derive more general results in an almost trivial way.

2. The decomposition of a measure

We shall use the notion of (positive) Radon measures (see Bourbaki [6]). A measure on a locally compact Hausdorff space X is a positive linear form $\mu : K(X) \to \mathbb{R}$ where $K(X)$ is the vector-space of continuous realvalued functions on X with compact support. The integration theory is the extension of μ to a larger class of functions called the integrable functions. The relation to the abstract measure - and integration theory on a σ-ring generated by the compact sets in X is obtained through Riesz's representation theorem. When X is small, that is, has denumerable basis for the topology, the difference between the two approaches is only formal. Let $M(X)$ be the space of (Radon) measures on X equipped with the weak topology.

For $\mu \in M(X)$ we denote the support of μ by $\text{supp}(\mu)$. The integral of a μ-integrabel function f is denoted by $\int_X f(x)d\mu(x)$ or $\int_X f d\mu$. For $f \in K(X)$ we have in addition the expression $\mu(f)$. The definition of measurability with respect to μ of a mapping from X into a topological space T can be found in Bourbaki [6] and in the cases where T has a denumerable basis for the topology the measurability with respect to μ is the classical one, that is the inverse image of a Borel-set in T is μ-measurable. otherwise the condition is stronger than the classical one.

Let now ν be a measure on Y and let $(\mu_y)_{y \in Y}$ be a family of measures on X indexed by Y. Suppose that

(1) for every $k \in \mathcal{K}(X)$ the real valued function

$y \to \mu_y(k)$; $y \in Y$ is ν-integrable.

In this case we are able to define a measure λ called the mixture of the family $(\mu_y)_{y \in Y}$ with respect to ν by the definition

(2) $$\lambda(k) = \int_Y \mu_y(k) d\nu(y) \quad ; \quad k \in \mathcal{K}(X).$$

(see Bourbaki [7] and Tjur [14]).

The measure λ is also denoted by $\int_Y \mu_y d\nu(y)$. To ensure the extension of (2) to integrable functions we shall suppose that the mapping

(3) $$\begin{aligned} Y &\to \mathcal{M}(X) \\ y &\to \mu_y \end{aligned}$$

is measurable with respect to ν and that all spaces are σ-compact.

In this case the relation (2) is extended in the following way. Let f be a λ-integrable function, then for ν-almost all $y \in Y$ we have that f is μ_y-integrable and the ν-almost everywhere defined realvalued function $y \to \int_X f(x)\,d\mu_y(x)$ on Y is ν-integrable with the integral

(4) $$\int_Y \left(\int_X f(x)\,d\mu_y(x)\right) d\nu(y) = \int_X f(x)\,d\lambda(d).$$

Let now $t : X \to Y$ be λ-measurable.

If furthermore for ν-almost all $y \in Y$ we have that

(5) $$\operatorname{supp}(\mu_y) \subseteq t^{-1}(y)$$

then we call the pair $((\mu_y)_{y \in Y}, \nu)$ a decomposition of λ with respect to t (see Bourbaki [7]).

Strictly speaking it is not necessary for our purpuse to define mixtures and decompositions as generally as above. The following definition will do:

The assumption (1) can be replaced by the following condition

(1') For every $k \in \mathcal{K}(X)$ the function
$y \to \mu_y(k)$; $y \in Y$ is an element in $\mathcal{K}(Y)$.

Since (1') does not depend on ν we are able to define the mixture of $(\mu_y)_{y \in Y}$ with respect to every $\nu \in \mathcal{M}(Y)$ and we get a continuous mapping

(6) $$\mathcal{M}(Y) \to \mathcal{M}(X)$$
$$\nu \to \int_Y \mu_y\,d\nu(y)$$

defined by the family $(\mu_y)_{y \in Y}$.

When all spaces are σ-compact then the extension of (2) to the λ-integrable functions is ensured. To define a decomposition we can suppose that t is continuous and (5) is valid for all $y \in Y$. Again this more restrictive version of (5) does not contain ν and it will ensure that (6) becomes injective. We shall furthermore point out that $t(\lambda)$ is not defined since t generally is not transforming the measure λ. Nevertheless if t is transforming $\lambda = \int_Y \mu_y d\nu(y)$ that is for every $k \in K(Y)$ $k \circ t$ is λ-integrable, then we have $t(\lambda)(k) = \lambda(k \circ t) = \int_Y \int_X k(t(x)) d\mu_y(x) d\nu(y) = \int_Y k(y) \mu_y(X) d\nu(y)$, which shows that $t(\lambda) = g\nu$, where $g(y) = \mu_y(X)$ for ν-almost all $y \in Y$. If all measures are probability measures we have $t(\lambda) = \nu$ and $(\mu_y)_{y \in Y}$ is a version of the conditional distributions given t.

Suppose that we have a decomposition. If $P = f\lambda$ (f is a non negative λ-integrable function with integral 1) then allthough t is not transforming λ then it is easy to represent $t(P)$, as $t(P) = g\nu$, where

$$(7) \qquad g(y) = \int_X f(x) d\mu_y(x) \text{ for } \nu\text{-almost all } y \in Y.$$

The statement follows from the following calculations: For $h \in K(Y)$ we have $t(P)(h) = \int_X h \circ t \, dP = \int_X f \cdot (h \circ t) d\lambda =$ $\int_Y \int_X f(x) h(t(x)) d\mu_y(x) d\nu(y) = \int_Y h(y) \int_X f(x) d\mu_y(x) d\nu(y) = \int_Y hg d\nu$, since $\mathrm{supp}(\mu_y) \subseteq t^{-1}(y)$ ν-almost all $y \in Y$.

It is seen from the above considerations that the problem of describing the non-central distribution is reduced or rather changed to the problem of existence and characterization of a decomposition of the measure λ.

In the case where X and Y are Riemann manifolds, t is a (surjectively) regular transformation and λ the geometric measure λ_X on X such a decomposition $((\mu_y)_{y\in Y}, \lambda_Y)$ exists and is characterized in the following way, λ_Y is the geometric measure on Y and $\mu_y = F\lambda_{X_y}$, where λ_{X_y} is the geometric measure on the sub-Riemann manifold $t^{-1}(y)$ of X and the density F is a differentiable function on X defined by means of the differential of t. Nevertheless we shall not concentrate on this case but on an other case namely when a group action is present. Under "nice" conditions, group actions ensure a decomposition through the so-called quotient measure.

3. The quotient measure

The relevant refences in this section is Bourbaki [8] and [5] and for a comprehensive treatment Andersson [1].

Let G be a σ-compact locally compact Hausdorff group and suppose that G acts properly on X ($G \times X \to X$, $(g,x) \to gx$). Proper action means that the mapping

$$G \times X \to X \times X \qquad (8)$$
$$(g,x) \to (gx,x)$$

is proper (the inverse image of a compact set is compact). The condition ensures that the final topology under the orbit projection $\pi : X \to X/G$ is locally compact and Hausdorff too (and of course also σ-compact) such that the notion of Radon measures on X/G can be applied. Furthermore the orbit mapping $\pi_x : G \to X$ ($g \to gx$) for $x \in X$ becomes proper. Since a proper mapping transform every measure we have that $\beta_x = \pi_x(\beta)$ is welldefined for every measure β on X.

The mapping

$$(9) \qquad X \to M(X)$$
$$x \to \beta_x$$

becomes continuous and $\mathrm{supp}(\beta_x) \subseteq \pi^{-1}(\pi(x))$, $x \in X$. If β is a right Haar measure on G (9) will be a G-invariant function, which then has a unique (continuous) factorization through X/G. Then we have the continuous mapping

$$(10) \qquad X/G \to M(X)$$
$$u \to \beta_u$$

with $\mathrm{supp}(\beta_u) = \pi^{-1}(u)$.

For $k \in K(X)$ the realvalued function $\bar{k}$ on X/G defined by $\bar{k}(u) = \beta_u(k)$ becomes an element in $K(X/G)$ (see condition (1')) and the positive linear mapping

$$(11) \qquad K(X) \to K(X/G)$$
$$k \to \bar{k}$$

becomes positive linear and onto. This defines an injective "linear" mapping

$$(12) \qquad M(X/G) \to M(X)$$
$$\mu \to \mu^{\#}$$

with an image defined by the condition that λ is in the image iff

$$(13) \qquad g\lambda = \Delta_G(g)\lambda; \quad \forall g \in G,$$

where Δ_G is the modul of the group G. The condition (13) means that λ is relarively invariant with multiplicator Δ_G^{-1} and then if G is unimodular (13) means invarians. The above considerations then show that for a measure λ on X, which satisfies (13) there exists one and only one measure denoted by λ/β, called the quotient-measure, such that $(\lambda/\beta)^{\#} = \lambda$, that is for every λ-integrable function we have the relation

(14) $$\int_X f(x)\,d\lambda(x) = \int_{X/G}\left(\int_X f(x)\,d\beta_u(x)\right)d\nu/\beta(u)$$

or, since

$$\int_X f(x)\,d\beta_u(x) = \int_G f(gx)\,d\beta(g) \quad \text{for } \pi(x) = u$$

we also have

(15) $$\int_X f(x)\,d\lambda(x) = \int_{X/G}\left(\int_G f(gx)\,d\beta(g)\right)d\nu/\beta(u)$$

The measure λ then have the decomposition $((\beta_u)_{u \in X/G}, \lambda/\beta)$ characterized above.

4. The application of the quotient measure to the distribution of a maximal invariant.

Let G act properly on X and let P be a probability with density f with respect to λ where λ satisfies (13) in 3^o, that is $P = f\lambda$. Then it follows directly from the consideration in 2^o and 3^o that the distribution $Q = \pi(P)$ of the orbit-projection π (the maximal invariant function) is given by $Q = h\lambda/\beta$ where the non-negative λ/β-integrable function h with λ/β-integral 1 is given λ/β-almost everywhere by

(16) $$h(u) = \int_G f(gx)\,d\beta(g) ,$$

where $\pi(x) = u$. Another way to write (16) is that for λ-almost all $x \in X$ we have

(17) $$\tilde{h}(\pi(x)) = \int_G f(gx)\,d\beta(g)$$

(see Bourbaki [5], §2, n°3 proposition 3 a). If μ is another measure on X which is relatively invariant under G with multiplicator χ_o, that is $g^{-1}\mu = \chi_o(g)\mu$ for every $g \in G$, where $\chi_o : G \to \mathbb{R}_+ \setminus \{o\}$ is continuous and $\chi_o(g_1 g_2) = \chi_o(g_1)\chi_o(g_2)$ for every $g_1, g_2 \in G$, we can in the case where $P = f\mu$ use the following facts to obtain a representation of $\pi(P)$. Since X/G is σ-compact it is also paracompact and it follows from Proposition 7 in §2, 4° in Bourbaki [5], that for every continuous multiplicator χ there exists a continuous positive function denoted by n on X with the property

(18) $$n(gx) = \chi(g)n(x) \quad \forall x \in X, \ \forall g \in G.$$

Let n be a continuous positive function on X which satisfies (18) with $\chi = \chi_o \Delta_G^{-1}$, then the measure $\lambda = n^{-1}\mu$ satisfies (13) and $P = nf\lambda$. Therefore $\pi(P) = \tilde{h}\lambda/\beta$, where now (16) and (17) are replaced by

(19) $$\tilde{h}(u) = n(x)\int_G f(gx)\chi_o(g)\,d\alpha(g) \qquad \text{and}$$

(20) $$\tilde{h}(\pi(x)) = n(x)\int_G f(gx)\chi_o(g)\,d\alpha(g),$$

Where $\alpha = \Delta_G^{-1}\beta$ becomes a left Haar-measure on G.

For two probability measures P_1 and P_2 on X with densities f_1 and f_2 with respect μ we then have the function ρ on X/G defined by

(21) $$\rho(\pi(x)) = \frac{\int_G f_1(gx)\chi_o(g)\,d\alpha(g)}{\int_G f_2(gx)\chi_o(g)\,d\alpha(g)}$$

for those $x \in X$ for which the denominator is positive. Then ρ becomes a version of $\dfrac{d\pi(P_1)}{d\pi(P_2)}$.

If one replaces the linear Cartan assumption in Wijsman [15] with the proper action assumption the result (21) is more general than the similar one (3) in [15] in the sense that it does not require that X is an (open) subset of an Euclidian space $\mathbb{R}^n$, that μ is the restriction of the Lebesgue measure to X and that G is a Lie subgroup of the group $Gl(n)$ of $n \times n$ non-singular matrices. Furthermore the concept of local cross-section and the existence of local cross-section (almost everywhere with respect to the Lebesgue measure) seems to disappear into the concept of proper action. Wijsman's result (3) in [15] now follows from (21) and the remark that the Lebesgue measure is relatively invariant under G with $g \to |\det(g)|$ as the multiplicator.

The proper action assumption is nice to work with since we have the following almost trivial result, which together with the remarks below can be considered as a very useful generalization of theorem 2 in [15].

Proposition 1: Let G act properly on X, let H be a closed subgroup of G and let $Y \subseteq X$ be closed with the property $HY = Y$. Then the restriction of the proper action $G \times X \to X$ to $H \times Y \to Y$ is proper.

Proof: First remark that Y and H are locally compact in the trace topology since they are closed. In fact the locally compact subgroups in G are precisely the closed subgroups. Since the mapping $\delta : H \times Y \to Y \times Y$ $((h,y) \to (hy,y))$ is the restriction of the proper mapping (8) and $H \times Y$ respectively $Y \times Y$ is closed in $G \times X$ respectively $X \times X$ δ is proper, that is the action of H on Y is proper.

Remark 1: Every continuous action of a compact group is proper.

Remark 2: The classical (transitive) action of the group of $n \times n$ non-singular matrices on the set of $n \times n$ positive definite matrices is proper, since one only has to show that the inverse image of a bounded set by the mapping (8) is bounded.

Remark 3: The (transitive and free) action of the translation group on an affine space is trivially proper.

The combination of the remarks and the proposition above gives the proper action condition without difficulties in invariance consideration in multivariate analysis. A new application is given in Andersson and Perlman [2].

5. Characterization of the quotient-measure by invariance

In Wijsman [15] and [16] the case of global cross-section is treated, that is a homeomorfic spliting of X into a G-orbit Y and a global cross-section Z, which is a nice subset of X crossing every orbit in exactly one point. A "projection" along orbits on Z plays the role of a maximal invariant function. First one runs into the problem of the existence of a global cross-section and secondly one runs into the problems of the definition of a measure on Z, which can help to represent the probability measure by densities. Furthermore a global-cross section does not exist in many application. To solve these problems one assumes that there is a "bigger" group K including G and another group H as subgroups, which acts transitively on X. Under complicated assumptions which among other things include the concepts of manifolds and Lie groups and conditions on the isotropic subgroups in K,H and G depending on a point x_o in X, one proves that a global cross-section Z exists and furthermore that H acts transitively on Z. The last assertion helps to define measures on Z by invarians and the distribution in terms of this measure. Let us show in the following that in the case where K is the semidirect product of H and G we can define directly on the orbitspace a transitive action and we can define the quotient measure by invariance.

Suppose that the locally compact group K acts transitively and properly on X and let H and G be closed subgroups such that for every $h \in H$ and $g \in G$ there exists $g' \in G$ such that

$$hg = g'h \tag{22}$$

and $K = HG$. That is K is the semidirect product of H and G. The formula (22) determinates a grouphomomorphism

$$H \to \mathrm{Aut}(G)$$
$$h \to \phi_h = (g \to hgh^{-1}),$$

where Aut(G) is the group of automorphism of G. For every $h \in H$ we have the mapping $x \to \pi(hx)$ from X into X/G, which is seen to be G-invariant because of (22). Then $h \in H$ uniquely defines $\bar{h} : X/G \to X/G$, and it is easily seen that we have a continuous transitive action of H on X/G given by

$$\begin{aligned} H \times X/G &\to X/G \\ (h,u) &\to \bar{h}(u). \end{aligned} \tag{23}$$

The continuity follows from the facts that the diagram

(24)
$$\begin{tikzcd} H \times X \arrow[r] \arrow[d, "1 \times \pi"] & X \arrow[d, "\pi"] \\ H \times X/G \arrow[r] & X/G \end{tikzcd},$$

where the horizontal mappings are the actions, is commutative and that $1 \times \pi$ and π are open and onto.

For $f \in K(X)$, $h \in H$, $u \in X/G$ and $x \in \pi^{-1}(u)$ we have (see (11))
$(\overline{hf})(u) = \beta_u(hf) = \int_G f(h^{-1}gx)\,d\beta(g) = \int_G f(\phi_h^{-1}(g)h^{-1}x)\,d\beta(g) =$
$(\mathrm{mod}\phi_h)\int_G f(gh^{-1}x)\,d\beta(g) = (\mathrm{mod}\phi_h)(h\bar{f})(u)$, where $\mathrm{mod}\phi_h$ is the modul of ϕ_h, that is

(25) $$(\overline{hf}) = (\mathrm{mod}\phi_h)(h\bar{f}), \quad h \in H, \ f \in K(X)$$

From (24) it follows that (see(11))

(26) $$(h\mu)^{\#} = (\mathrm{mod}\phi_h)\ h\mu^{\#}$$

Furthermore a measure $\mu \in M(X/G)$ is relatively invariant under H with multiplicator χ iff $\mu^{\#}$ is relatively invariant under H with multiplicator $h \to \chi(h)(\mathrm{mod}\phi_h)^{-1}$. In that case $\mu^{\#}$ must be relatively invariant under K with multiplicator $\chi_1(k) = \chi(h)(\mathrm{mod}\phi_h^{-1})\Delta_G(g)$ for $k = gh$. The uniqueness up to a positive constant of a measure on X which is relatively invariant under K (the action is proper) together with the sujectivity of the mapping (12) and formula (26) shows that a relatively invariant measure on X/G under H is unique up to a positive constant.

Finally we only have to remark that the quotient measure λ/β for a λ, which have the property (13) then must be relatively invariant under H with multiplicator $h \to \mathrm{mod}\ \phi_h$, since $(\lambda/\beta)^{\#} = \lambda$. This last remark defines the quotient measure by invariance.

Since this section takes care of the global cross-section case let us end up with some remarks about the global cross-section situation. Suppose that G acts proper and transitive on Y and let Z be a locally compact, σ-compact and Hausdorff, then G is also acting proper on $Y \times Z$ by

$$(27) \qquad \begin{array}{c} G \times (Y \times Z) \to (Y \times X) \\ (g,(y,z)) \quad (gy,z) \end{array}$$

To see that the action is proper one only have to remark that a product of proper mappings is proper. Let now μ be a measure on $Y \times Z$ which is relatively invariant with multiplicator χ_o. Let n be a positiv continuous function on Y with the property (18) with $\chi = \chi_o \cdot \Delta_G^{-1}$. Since the action on Y is transitive n is unique up to a positive constant. The measure $\lambda = n^{-1}\mu$ satisfies (13) and for $k \in K(Y)$ and $h \in K(Z)$ we have $\int_{Y\times Z} k(y)h(z)d\mu(y,z) =$ $\int_{Y\times Z} k(y)n(y)h(z)d\lambda(y,z) = h(z)\int_Z\int_G k(gy)n(gy)d\beta(g)\ d\lambda/\beta(z) =$ $\lambda/\beta(h) \cdot \nu(k)$, where ν becomes a unique up to positive constant relative invariant measure on Y with multiplicator χ_o. Then we have $\lambda = \nu \otimes \lambda/\beta$. This remark was already stated by Farrell [9].

References.

[1] Andersson, S.A. (1978). Invariant Measures, Technical report 129. Stanford University, Department of Statistic, Stanford.

[2] Andersson, S.A. and Perlman (1979). Two testing problems relating the real and complex multivariate normal distribution. In preparation.

[3] Berk, Robert H. (1967). A special group structure and eqivariant estimation. Ann. Math. Statist. **38**, 1436-1445.

[4] Bondar, James V. (1976). Borel cross-sections and maximal invariants. Ann. Statist. 4, 866-877.

[5] Bourbaki, N. (1963). Éléments de Mathématique. Integration. Chap. 7 á 8. Hermann, Paris.

[6] Bourbaki, N. (1965). Éléments de Mathématique. Integration. Chap. 1-4. Hermann, Paris.

[7] Bourbaki, N. (1956). Éléments de Mathématique. Integration. Chap. 5. Hermann, Paris.

[8] Bourbaki, N. (1960). Éléments de Mathématique. Topologie general, Chap. 3 á 4. Hermann, Paris.

[9] Roger, H. Farrel (1976). Techniques of Multivariate Calculation, Springer-Verlag.

[10] James, Alan T. (1964). Distributions of matrix variates and latent roots derived from normal samples. Ann. Math. Statist. 35, 475-501.

[11] Koehn, Uwe (1970). Global cross sections and the densities of maximal invariants. Ann.Math.Statist. 41, 2045-2056.

[12] Muirhead, R.T. (1978). Latent roots and matrix variates: a review of some asymptotic results. Ann.Statist. 6, 5-33.

[13] Stein, C.M. (1956). Some problems in multivariate analysis, Part. 1. Technical Report No. 6, Department of Statistics, Stanford University, Stanford.

[14] Tjur, T. (1974). Conditional probability distribution, Lecture Notes 2, Institute of Mathematical Statistics, University of Copenhagen.

[15] Wijsman, R.A. (1967). Cross-sections of orbits and their appkication to densities of maximal invariants. Proc. Fifth Berkeley Symp. Math. Statist.

and Prob., University of California Press, Berkeley and Los Angeles, 1, 389-400.

[16] Wijsman, R.A. (1978). Distribution of Maximal invariant using global and local cross-sections. University of Illinois at Urbana-Champaign.

PREPRINTS 1978

COPIES OF PREPRINTS ARE OBTAINABLE FROM THE AUTHOR OR FROM THE INSTITUTE OF MATHEMATICAL STATISTICS, UNIVERSITETSPARKEN 5, 2100 COPENHAGEN Ø, DENMARK.

No. 1 Tjur, Tue: Statistical Inference under the Likelihood Principle.

No. 2 Hering, Heinrich: The Non-Degenerate Limit for Supercritical Branching Diffusions.

No. 3 Henningsen, Inge: Estimation in M/G/1-Queues.

No. 4 Braun, Henry: Stochastic Stable Population Theory in Continuous Time.

No. 5 Asmussen, Søren: On some two-sex population models.

No. 6 Andersen, Per Kragh: Filtered Renewal Processes with a Two-Sided Impact Function.

No. 7 Johansen, Søren & Ramsey, Fred L.: A Bang-Bang Representation for 3x3 Embeddable Stochastic Matrix.

No. 8 Braun, Henry: A Simple Method for Testing Goodness of Fit in the Presence of Nuisance Parameters.

No. 9 Lauritzen, S.L. & Speed, T.P. & Vijayan, K.: Decomposable Graphs and Hypergraphs.

No. 10 Hald, Anders: On the Statistical Theory of Sampling Inspection by Attributes.

No. 11 Darroch, J.N. & Lauritzen, S.L. & Speed, T. P.: Markov Fields and Log-linear Interaction Models for Contingency Tables.

No. 12 Speed, T.P.: A Note on Nearest-Neighbour Gibbs and Markov Probabilities.

No. 13 Andersson, Steen A.: Canonical Correlations with Respect to a Complex Structure.

PREPRINTS 1979

COPIES OF PREPRINTS ARE OBTAINABLE FROM THE AUTHOR OR FROM THE INSTITUTE OF MATHEMATICAL STATISTICS, UNIVERSITETSPARKEN 5, 2100 COPENHAGEN Ø, DENMARK.

No. 1 Edwards, David: Large Sample Tests for Stationarity and Reversibility in Finite Markov Chains.

No. 2 Andersson, Steen A. : Distribution of Maximal Invariants Using Proper Action and Quotient Measures.