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Filtered Renewal Processes with a two-sided Impact Function

by

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Abstract.

By means of a renewal theorem proved by Karlin (1955) an expression for the characteristic function of a filtered renewal process with a two-sided impact function is derived and formulae for the moments are proved.

Key Words:

Characteristic function, filtered renewal process, moments, renewal equation, traffic noise, two-sided filter.

1. Introduction and Summary.

Filtered renewal processes have various applications as models for certain physical phenomena (cf. Takács (1956) for examples) and in these examples a one-sided filter gives a sufficiently good description. When using a filtered renewal process as a model for the noise emitted from the stream of vehicles on a highway (Marcus (1973), (1974)) the one-sided filter corresponds to the unrealistic situation where a vehicle cannot be heard before it has passed the observer, so a process with a two-sided impact function must be used for the description.

In this paper we shall derive an expression for the characteristic function for such a process, and by means of the characteristic function formulae for the moments, covariance function and spectral density are easily obtained. The main tool in our derivation is a version of a theorem proved by Karlin (1955) concerning the solutions of a renewal equation that involves two-sided functions, i.e. functions with support on the entire real axis. In the next section we shall quote this theorem in the form we need, and finally in section 3 the filtered renewal processes are discussed.

2. The renewal equation.

A renewal equation is a convolution equation of the form

$$Z = z + Z * G, \tag{1}$$

where Z is the unknown function, z is a known function and G is a known distribution function. Throughout we assume that G satisfies the following conditions

(2) $G(t) = 0$ when $t \leq 0$

(3) G is a non-lattice distribution

(4) $$\theta = \int_0^\infty t\,dG(t) < \infty.$$

The *renewal function* U is defined by

$$U(t) = \sum_{n=1}^{\infty} G^{*n}(t),$$

where G^{*n} denotes the n-fold convolution of G with itself. This function obviously defines a measure $U(dt)$ on $[0,\infty[$.

Karlin's theorem in the desired form is then

Theorem 1.

Suppose that z is a directly Riemann integrable complex-valued function. Then (1) for every complex number q has a unique bounded solution Z satisfying

$$\lim_{t\to -\infty} Z(t) = q.$$

This solution is given by

$$Z(t) = q + z(t) + (U*z)(t),$$

and the limit

$$\lim_{t\to +\infty} Z(t) = q + \int_{-\infty}^{\infty} z(s)ds/\theta$$

holds.

Notes on the theorem:

1^0 The definition of the concept of direct Riemann integrability of real-valued functions on $[0,\infty[$ (see e.g. Feller (1971)) extends immediately to functions such as z in the theorem.

2^0 There is an ambiguity in the set of bounded solutions to (1), but in a given context it frequently seems to be possible to pick out a single natural solution as we shall see in the next section.

3^0 Karlin's proof of the theorem is based upon Wiener's tauberian theorem, but it is possible to prove it by means of elementary techniques if you assume that G has a finite second order moment and use the fact proved by Smith (1954) that the measure $C(dt) = U(dt) - dt/\theta$ which we denote the <u>corrected renewal measure</u> is then finite.

3. Filtered renewal processes.

Let $\{T_n\}$ $n = 1,2,\ldots$ be a renewal process whose <u>interarrival-time distribution</u>

$$G(t) = Pr\{T_n + 1 - T_n \le t\}$$

possesses the properties (2), (3) and (4). Let $\{Z_n\}$ $n = 1,2,\ldots$ be a sequence of independent identical distributed random variables taking values in some finitedimensional space A and denote their common distribution function H. Assume that the sequences $\{T_n\}$ and $\{Z_n\}$ are independent and consider a non-negative valued function f, <u>the impact function</u> defined on $R \times A$ satisfying

$$(5) \qquad \int_{-\infty}^{\infty} \int_A f(t,z)\, dH(z)\, dt < \infty .$$

Then the <u>filtered renewal process</u> $X(t)$ is defined by

$$X(t) = \sum_{n=1}^{\infty} f(t - T_n, Z_n), \quad t \in \mathbb{R}$$

and (5) ensures that $X(t)$ is a.s. finite for every t.

Defining

$$(6) \qquad \phi(t,w) = \int_A e^{iwf(t,z)}\, dH(z), \quad w \in \mathbb{R}$$

it is easy to derive an integral equation for the characteristic function $\Phi(t,w) = Ee^{iwX(t)}$ as

$$(7) \qquad \Phi(t,w) = \int_0^\infty \Phi(t-y,w)\, \phi(t-y,w)\, dG(y).$$

In fact (7) is a special case of the formula given by Smith (1973).

If we define

$$\psi(t,w) = \phi(t,w)\, \Phi(t,w)$$

and rewrite (7), we obtain the equation

$$(8) \qquad \psi(t,w) = \psi(t,w) - \Phi(t,w) + \int_0^\infty \psi(t-y,w)\, dG(y),$$

which for fixed $w \in \mathbb{R}$ is a <u>renewal equation</u>. To solve (8) we must find conditions to assure direct Riemann integrability of the function $\psi(t,w) - \Phi(t,w) = \Phi(t,w)(\phi(t,w) - 1)$.

Since Φ is bounded it suffices to find conditions under which the function $\phi(\cdot,w) - 1$ is directly Riemann integrable, and Taylor expansion yields

$$\phi(t,w) - 1 = \int_A iwf(t,z)\cos(w_1 f(t,z))\, dH(z)$$
$$- \int_A wf(t,z)\sin(w_2 f(t,z))\, dH(z),$$

where w_1 and w_2 lie between 0 and w. Thus both real and imaginary part of $\phi(t,w) - 1$ is dominated by the function $|w|\lambda_1(t)$, where we have defined

$$(9) \qquad \lambda_k(t) = \int_A f(t,z)^k\, dH(z), \quad k = 1,2,\ldots,$$

and hence the condition we look for is direct Riemann integrability of $\lambda_1(t)$. Assuming that this condition is fulfilled we obtain from theorem 1 the solutions of (8) for fixed w as

$$(10) \qquad \psi(t,w) = q(w) + \psi(t,w) - \Phi(t,w) + \int_0^\infty (\psi(t-x,w) - \Phi(t-x,w))\,dU(x)$$

where $q(w) = \lim_{t\to-\infty} \psi(t,w)$, i.e. for every w we must have $q(w) = 1$, and we conclude from (10) that

$$\Phi(t,w) = 1 + \int_0^\infty (\psi(t-x,w) - \Phi(t-x,w))\,dU(x).$$

Actually our main interest is not in the distribution of $X(t)$ but in the weak limit appearing as $t\to\infty$. This limit can be defined by considering a stationary renewal process $\{T_n\}$, = 0, ± 1, ± 2,... with interarrivaltime distribution G and a sequence $\{Z_n\}$, n = 0, ± 1, ± 2,... of independent random variables with distribution function H and assuming independency of those sequences. Then we define the stationary filtered renewal process as

$$X^*(s) = \sum_{n=-\infty}^{\infty} f(s-T_n, Z_n), \quad s \in \mathbb{R}.$$

This process is obiously stationary and it is easy to show that for every $s_0 \in \mathbb{R}$ the distribution of $X(t)$ will converge weakly to the distribution of $X^*(s_0)$ as $t\to\infty$.

By theorem 1 the limit function $\Phi^*(w) = \lim_{t\to\infty} \psi(t,w)$ exists and it equals

$$\lim_{t\to\infty} \Phi(t,w) = 1 + \int_{-\infty}^{\infty} (\psi(y,w) - \Phi(y,w))\,dy/\theta,$$

and because of the weak convergence Φ^* must be the characteristic function of $X^*(s)$ for every s. Let us summarize the result in the following

Theorem 2.

Assume that the function λ_1 given by (9) is directly Riemann integrable. Then the characteristic function $\Phi^*(w) = Ee^{iwX^*(s)}$ for every s is given by

$$\Phi^*(w) = 1 + \int_{-\infty}^{\infty} \Phi(y,w)(\phi(y,w) - 1)\ dy/\theta,$$

where $\Phi(y,w)$ is the characteristic function of $X(y)$ and $\phi(y,w)$ is given by (6).

Let us look at an example:

Example 1.

The only case in which the characteristic function of a stationary filtered renewal process was known previously is when the interarrivaltime distribution is exponential (i.e. when we are dealing with a filtered Poisson process). In this case we have (see e.g. Andersen et. al. (1977)).

$$(11) \quad \Phi^*(w) = \exp\left(\int_{-\infty}^{\infty} (\phi(y,w) - 1)\ dy/\theta\right).$$

The expression for the characteristic function of the corresponding non-stationary process $X(t)$ is

$$(12) \quad \Phi(t,w) = \exp\left(\int_{-\infty}^{t} (\phi(y,w) - 1)dy/\theta\right),$$

and it is easy to show that (11) and (12) are in accordance with theorem 2. □

Formulae for the moments $EX^*(s)^k$ are given by Takács (1956) in the special case where the impact function f is defined on $[0,\infty[\times \mathbb{R}$ and by Marcus (1974), who considered a filtered Markov renewal process which degenerates to our case when the state space of the Markov chain consists of a single point. Neither of these authors actually prove the correctness of the formulae, but nevertheless this is rather straightforward using theorems 1 and 2 as we shall see.

By differentiating k times under the integral sign in (7) and letting w = 0 we obtain the equation

$$(13) \qquad M_k(t) = \int_0^\infty \sum_{j=0}^{k} \binom{k}{j} M_j(t-y)\lambda_{k-j}(t-y)\,dG(y),$$

where $M_k(t) = EX(t)^k$ and $\lambda_1,\ldots,\lambda_k$ are given by (9). Rewriting (13) we obtain a <u>renewal</u> <u>equation</u>:

$$(14) \qquad M_k + h_k = h_k + (M_k + h_k)*G,$$

where

$$h_k(t) = \sum_{j=0}^{k-1} \binom{k}{j} M_j(t)\lambda_{k-j}(t).$$

Now (14) can be solved under the assumption that $\lambda_1\ldots,\lambda_k$ are directly Riemann integrable, and thus we get from theorem 1 that

$$M_k(t) = (h_k*U)(t) + q,$$

where the relevant solution clearly corresponds to $q = 0$, so the result is

$$(15) \qquad M_k(t) = \int_0^\infty \sum_{j=0}^{k-1} \binom{k}{j} M_j(t-y)\lambda_{k-j}(t-y)\,dU(y).$$

By theorem 1 $M_k(t)$ has a limit as $t \to \infty$:

$$(16) \qquad \lim_{t\to\infty} M_k(t) = \int_{-\infty}^{\infty} \sum_{j=0}^{k-1} \binom{k}{j} M_j(y)\lambda_{k-j}(y)\,dy/\theta,$$

but this does not prove that $EX^*(s)^k$ is given by the r.h.s. of (16) since weak convergence does not imply convergence of moments. We can, however, use theorem 2 to obtain the result, because differentiating k times under the integral sign, which is legal assuming that $\lambda_1,\ldots,\lambda_k$ are integrable gives the final

Theorem 3.

Suppose that $\lambda_1,\ldots,\lambda_k$ *given by* (9) *are directly Riemann integrable. Then the moments of* $X^*(s)$ *for every* s *is given by*

$$EX^*(s)^k = \int_{-\infty}^{\infty} \sum_{j=0}^{k-1} \binom{k}{j} M_j(y)\lambda_{k-j}(y)\,dy/\theta,$$

where $M_j(y)$ *are given by* (15).

The conclusion from theorem 3 is that the formulae given by Takâcs (1956) and Marcus (1974) are in fact correct.

Having established a formula for var $X^*(s)$ it is not difficult to find the *covariance function*

$$\begin{aligned} R(h) &= \operatorname{cov}(X^*(s),\ X^*(s+h)) \\ &= \int_{-\infty}^{\infty} \int_A f(t,z)f(t+h,z)\,dH(z)\,dt/\theta \\ &+ \int_0^{\infty} (\lambda_{11}(x-h)+\lambda_{11}(x+h))\,dC(x), \end{aligned}$$

where

$$\lambda_{11}(y) = \int_{-\infty}^{\infty} \lambda_1(t)\lambda_1(t+y)\,dt.$$

(see also Marcus (1974)) and if R is continuous and integrable the *spectral density*

$$(17)\qquad \hat{R}(x) = \frac{2\Pi}{\theta}\int_A \hat{f}(-x,z)\ \hat{f}(x,z)\,dH(z) + \frac{2\Pi}{\theta}\ \hat{\lambda}_{11}(x)\ \hat{c}(x),$$

where $\hat{f}(\circ,z)$ is the Fourier transform of $f(\cdot,z)$, $\hat{\lambda}_{11}$ is the Fourier transform of λ_{11} and $\hat{c}$ is the characteristic function of the corrected renewal measure. The spectral density is in the case of a one-sided filter given by Smith (1958) whereas the two-sided version does not seem to have been published before.

Let us finally look at the traffic noise example:

Example 2.

When studying models for traffic noise the impact function f is chosen as

$$f(t,z) = z/(a^2 + t^2 v^2),$$

where $a > 0$ is the distance from the highway to the microphone and $v > 0$ is the velocity of the vehicles, which is assumed constant. Finally z is proportional to the noise power emitted by a vehicle (cf. Andersen et. al. (1977)). In this case (17) becomes

$$\hat{R}(x) = \frac{\Pi}{\theta a^2 v^2} e^{-2a|x|/v} (EZ^2/2 + (EZ)^2 \Pi \hat{c}(x))$$

assuming that $EZ^2 < \infty$. This implies that

$$\int_{-\infty}^{\infty} \log \hat{R}(x)/(1+x^2)\, dx = -\infty,$$

so the process $X^*(s)$ is deterministic (Rozanov (1967)) as demonstrated by Andersen et.al. (1977) in the case of an underlying Poisson process. Actually the heuristic argument we gave for this fact will also apply in the case of a filtered renewal process.□

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