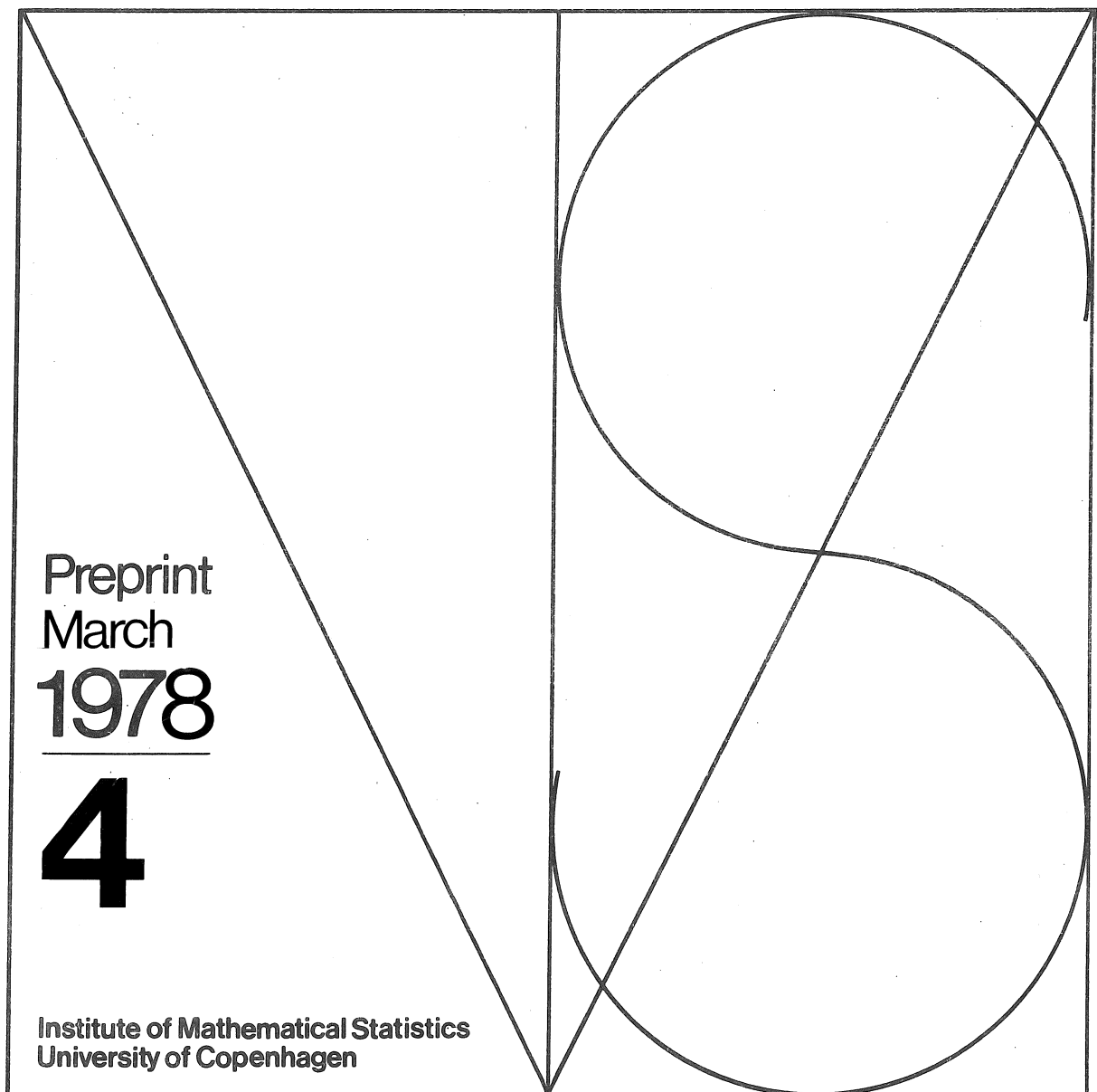Henry Braun

Stochastic Stable Population

Theory in Continuous Time

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STOCHASTIC STABLE POPULATION THEORY
IN CONTINUOUS TIME**

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Abstract

This paper presents the stochastic stable population theory for two families of continuous time models, (i) those for which the asymptotics can be derived from a single renewal relation and (ii) those for which a system of coupled renewal equations must be specified. Some effort is devoted to a characterization of these two families and, for purposes of illustration, both a uniregional age and parity dependent model and a multiregional age dependent model are analyzed in detail. In each case, the question of the proper definition of the stable age distribution and the reproductive value is addressed and it is shown how the distribution of the age at childbearing can be interpreted in terms of instantaneous changes in the reproductive value of the population.

0.   INTRODUCTION

In recent years substantial interest has focussed on giving a stochastic treatment in continuous time of the development of populations. A principal tool in attacking this problem has been the theory of generalized age-dependent branching processes, for which the seminal references are Crump and Mode [1968, 1969] and Jagers [1969]. In his recent text, Jagers [1975] offers a more complete treatment of the subject.

To demonstrate that the results of classical deterministic stable population theory could be rigorously derived within a stochastic framework has been one area of particular concern. Excellent reviews of this and related work, emphasizing the interplay between the mathematical theory and the demographic applications, can be found in Keiding [1973], Mode and Littman [1975], and Keiding and Hoem [1976]. The latter, in particular, give a compact but lucid derivation of stochastic stable popu- lation theory in which many of the objects familiar to demogra- phers, such as the Lotka equation and the reproductive value, appear naturally in a mathematical context.

Although the model employed by Keiding and Hoem (hereafter referred to as KH) supposes that the forces of fertility and mortality are dependent only on age, the deterministic literature (cf. Goodman [1969], Feeney [1970], Rogers [1975]) treats more complex models in which such factors as marriage, parity and place of residence are also considered. The chief aim of this paper is to demonstrate that these more complex models are in fact amenable to stochastic analysis.

We begin this study by exploring the range of models that can be treated by simple extensions of the methods presented in KH, and then by characterizing a crucial property they must all possess. To make these ideas more concrete, a full development of the age and parity model is carried out.

In section 2, with the aid of multivariate renewal theory, we study the so-called multiregional models, in which the population is divided into groups distinguished by differential fertility, mortality, and migration rates. Whereas these groups will often be identified with different geographical regions, this need not be the case in other applications. Rogers and Willekens [1976a] give an excellent review of the non-stochastic work in this area, and Rogers [1975] deals with the subject in greater detail.

The following section discusses a hybrid model of some interest, while a new approach to the age at childbearing problem is given in section 4. A number of convolution formulas for some net maternity functions appearing in section 1 are found in the Appendix.

Throughout, questions of rigour are not dealt with directly. This seems reasonable inasmuch as the rather carefully formulated arguments of KH apply to the analyses presented here without essential change.

Finally, our theorems are usually phrased in terms of expected population size and expected age distributions. Results involving almost sure convergence are omitted, and the interested reader should consult Jagers [1975] or Athreya and Rama Murthy [1976].

1. GENERAL SINGLE REGION MODELS

A. Preliminaries

In this section we propose to show how the mathematical structures developed in KH can accomodate more general models. We begin by carrying through the analysis of one such model in which birth intensities are functions not only of age but also of parity. This is a case of practical importance and its elucidation brings to the fore the essentials of the method. The reader will profit most by comparing the sequence of theorems presented below with the analagous sequence in KH; for his benefit, appropriate references are given in the text. At the conclusion of the section, a rather general model is formulated and discussed.

A model is specified once the forces of transition acting on an individual at any point in her history have been described. In KH, these forces were determined solely by age, whereas in our model both age and parity are involved. This pair of characteristics henceforth will be denoted by the word "status", and the added wrinkles in our analysis derive from the fact that we have a two-dimensional, rather than a one-dimensional, status variable.

With an individual of status (x,j), i.e. of age x and parity j, we associate a birth process assumed to be an inhomogeneous Poisson process having a bounded continuous intensity function $\lambda_j(x)$, whose value depends on both age and parity. This process generates the $(j+1)^{\underline{th}}$ birth to the parent individual. Let $\{K(t):t\geq o\}$ denote the process counting births to the individual. $K(\cdot)$ is a fairly complicated function of the $\{\lambda_j(\cdot):j=1,2,\dots\}$.

(cf. the Appendix). Let L denote a random variable (the life length) independent of $K(\cdot)$ with support on $[o,\omega]$ (for $0<\omega<\infty$) and distribution function

$$q(x) = P\left\{L\leq x\right\} = 1-\exp\left\{-\int_0^x \mu(s)\,ds\right\}.$$

We call $\mu(x)$ the force of mortality. Note that $\mu(\cdot)$ is a function of age alone. Let $p(x) = 1-q(x)$ and $_tp_x = p(x+t)/p(x)$.

Now let $E_{u,j}$ denote the expectation operator, conditional on the event that the individual is of status (u,j) and that $L>u$. Define

$$\Phi_{u,j}(t) = E_{u,j}\left\{K[(u+t)\wedge L]-K(u)\right\}$$

and

$${}^k\Phi_{u,j}(t) = P\left\{K(u\wedge L) <k\leq K[(u+t)\wedge L] \mid K(u)=j, L>u\right\}.$$

Then the usual computation shows that

$$\Phi_{u,j}(t) = \sum_{k=j+1}^{\infty} {}^k\Phi_{u,j}(t).$$

In the appendix we shall derive formulas for $\Phi_{u,j}(\cdot)$ and ${}^k\Phi_{u,j}(\cdot)$ which will show that it is permissible to work with the following derivatives:

$$\phi_{u,j}(t) = \frac{\partial}{\partial t}\,\Phi_{u,j}(t)$$

and

$${}^k\phi_{u,j}(t) = \frac{\partial}{\partial t}\,{}^k\Phi_{u,j}(t).$$

Note that $\Phi_{0,0}(\cdot)$ is the net maternity function and $\Phi_{0,0}(x) = EK(x\wedge L)$. We can carry out our calculations in terms of $\Phi_{u,j}(\cdot)$, etc. and need not express our results in terms of $p(\cdot)$ and $m_j(\cdot)$, as is done explicitly in KH.

Employing the notation of KH for family trees and family histories, let

$$Z_i^{a,j}(t) = \chi\left\{<i>\epsilon \text{ J is born, and at time t is alive of status } (x,j), \text{ some } x\leq a\right\},$$

$$Z^{a,j}(t) = \sum_{i\epsilon I(\omega)} Z_i^{a,j}(t),$$

and

$$Z^a(t) = \sum_j Z^{a,j}(t).$$

We have here introduced the notational convention that a missing index denotes summation over that index. Suppose that the population of interest begins at time t=0 with a single ancestor of status (u,j). The proof of the next result makes use of the fact that in our set-up we can still make an additive decomposition of the population into the lines begun by the offspring of the original ancestor. $Y_n(t)$ denotes the number of individuals born before time t in the nth generation and $^kY_n(t)$ denotes the number of those counted in $Y_n(t)$ descended from $<k>$ in I_1. Let S_k be the time at which $<k>$ is born.

Proposition 1.1.   (confer KH, Theorem 3.1).

$$E_{u,j} Y_n(t) = \Phi_{u,j} * \Phi_{0,0}^{*(n-1)}(t) \quad \text{for } n\geq 1.$$

Proof: From the definitions we have $\Phi_{u,j}(t) = E_{u,j}Y_1(t)$.
Assume the result true for $n\leq N$. Then

$$E_{u,j} Y_{N+1}(t) = \sum_{k=j+1}^{\infty} E_{u,j} {}^kY_{N+1}(t)$$

$$= \sum_k \int_o^t E_{u,j}[{}^kY_{N+1}(t) | S_k=s] {}^k\Phi_{u,j}(ds)$$

$$= \sum_k \int_0^t E_{0,0} \, Y_N(t-s) \, {}^k\Phi_{u,j}(ds)$$

$$= \int_0^t \Phi_{0,0}^{*(n)}(t-s) \, \Phi_{u,j}(ds). \quad \square$$

Now let

$$\begin{aligned}
{}_t w_{u,j}(k) = P\{&\text{individual has parity } k \text{ at time } t \mid \\
&\text{she is alive at time } t \text{ and was} \\
&(u,j) \text{ at time } 0\},
\end{aligned}$$

$$N_{u,j}^{a,k}(t) = E_{u,j} \, Z^{a,k}(t),$$

and

$$N_{u,j}^a(t) = E_{u,j} \, Z^a(t).$$

We are now ready to present a basic (backward) renewal relation, and the reader should note the role of the term ${}_t w_{u,j}(k)$.

<u>Proposition 1.2.</u> (confer KH, Theorem 4.1)

$$(1.1) \qquad N_{u,j}^{a,k}(t) = \chi_{[o,a]}(u+t) \, {}_t p_u \, {}_t w_{u,j}(k)$$

$$+ \int_0^t N_{0,0}^{a,k}(t-s) \, \Phi_{u,j}(ds).$$

<u>Proof:</u> $Z^{a,k}(t) = Z_0^{a,k}(t) + \sum_{\ell=j+1}^{\infty} {}^\ell Z^{a,k}(t),$

where $\quad {}^\ell Z^{a,k}(t) = \sum_{i \in {}^\ell J} Z_i^{a,k}(t).$

$$E_{u,j} \, Z_0^{a,k}(t) = \chi_{[o,a]}(u+t) \, P\{L_0 \geq u+t \mid L_0 > u\} \cdot$$

$$P\{K(u+t) = k \mid L_0 \geq u+t, (u,j) \text{ at time } 0\}$$

$$E_{u,j} \, {}^\ell Z^{a,k}(t) = \int_0^t E_{u,j}\{{}^\ell Z^{a,k}(t) \mid S_\ell = s\} \, {}^\ell\Phi_{u,j}(ds)$$

$$= \int_0^t N_{0,0}^{a,k}(t-s) \, {}^\ell\Phi_{u,j}(ds).$$

Recall now that

$$\Phi_{u,j}(s) = \sum_{\ell=j+1}^{\infty} {}^{\ell}\Phi_{u,j}(s). \quad \Box$$

<u>Corollary</u>: $N_{u,j}^{a}(t) = \chi_{[0,a]}(u+t) {}_{t}p_{u} + \int_{0}^{t} N_{0,0}^{a}(t-s) \Phi_{u,j}(ds).$

<u>Proof</u>: Follows immediately once we notice that

$$\sum_{k=0}^{\infty} {}_{t}w_{u,j}(k) = 1. \quad \Box$$

<u>Remark</u>: In the derivation of (1.1) it was essential that each newborn begin life with status $(0,0)$ and, hence, subject to the same birth intensity $\lambda_{0}(\cdot)$. This is, in fact, the justification for the term $N_{0,0}^{a,k}(t-s)$ in the integral.

With the crucial Proposition 1.2 in hand we are ready to derive the classical results on the growth of stable populations. Readers who wish to maintain the logical progression should go directly to section 1.C.

In section B below we indulge in a small digression in order to derive Lotka's integral equation for births.

B.   <u>Lotka's Integral Equation</u>

Firstly, we require some notation. Let $Z_{0}^{*}(t) = 1$ and $Z_{i}^{*}(t) = \chi_{\{S_{i} \leq t\}}$ for $i \in J-\{0\}$ and, finally, $Z^{*}(t) = \sum_{i \in J} Z_{i}^{*}(t)$. Then $Z^{*}(t)$ denotes the number of births in $[0,t]$ plus one for the original ancestor. Also let $N_{u,j}^{*}(t) = E_{u,j}Z^{*}(t)$.

In analogy with Proposition 1.2, we can easily show that

$$N_{u,j}^{*}(t) = 1 + \int_{0}^{t} N_{0,0}^{*}(t-s) \Phi_{u,j}(ds),$$

or, alternately, (as in Theorem 4.3 of KH) that

$$(*) N^*_{u,j}(t) = 1 + \Phi_{u,j}(t) + \int_0^t [N^*_{u,j}(t-y) - 1] \, \Phi_{0,0}(dy).$$

Since $n^*_{u,j}(t) = \frac{\partial}{\partial t} N^*_{u,j}(t)$ is a well defined quantity (cf. Theorem 3.2, Corollary 2 of KH), we have by differentation:

$$n^*_{u,j}(t) = \phi_{u,j}(t) + \int_0^t n^*_{u,j}(t-y) \, \Phi_{0,0}(dy).$$

This is Lotka's equation in the case of a single ancestor.

We now go on to deal with the case of an arbitrary initial population. Suppose that $Z^{a,j}(0)$ denotes the number of indivi-duals of age $\leq a$ and parity j at time 0. We assume that the initial population is almost surely finite and that $N^{a,j}(0) = EZ^{a,j}(0)$ exists, is finite and absolutely continuous as a function of a. Let $n^{a,j}(0) = \partial N^{a,j}(0)/\partial a$.

Now multiply equation (*) by $n^{u,j}(0)$, carry out the approriate integration with respect to u and summation with respect to j, obtaining

$$N^*(t) = N(0) + \sum_j \int_0^\infty \Phi_{u,j}(t) n^{u,j}(0) \, du$$

$$+ \int_0^t [N^*(t-y) - N(0)] \, \Phi_{0,0}(dy),$$

where $\quad N^*(t) = \sum_j \int_0^\infty N^*_{u,j}(t) n^{u,j}(0) \, du.$

Finally, differentiate with respect to t, to get

$$n^*(t) = \sum_j \int_0^\infty \phi_{u,j}(t) n^{u,j}(0) \, du$$

$$+ \int_0^t n^*(t-y) \, \Phi_{0,0}(dy).$$

This is the usual form of Lotka's equation.

C.   Asymptotic Results

The formal analysis of our process will now resemble the one in KH, with the main change being the redefinition of the stable age distribution and the reproductive value in order to take account of the finer classification of the population. The stable age distribution is the limiting distribution of ages towards which the population, on average, converges. The reproductive value of an individual represents, in some sense, his worth from the standpoint of the growth of the population. That is to say, expected births due to the individual are exponentially discounted with time. Further discussion of these points can be found in Keyfitz [1968].

It should be recalled that the analysis below depends very heavily on the fact that each newborn is subject to the same forces of transition. Thus, the model can not be generalized to permit secular changes in these forces, nor their variation among the individuals in the population (except in a rather special formulation - see section 3)

Let us first establish some notation. Let

$$I(\rho) = \int_{0}^{\infty} e^{-\rho x} \Phi_{0,0}(dx).$$

The normal arguments show that the equation

$$I(\rho) = 1$$

has at most a single real solution. For all human populations the solution, call it r, certainly exists. The use of r mathematically is simply a normalization of the function $\Phi_{0,0}(\cdot)$ which

permits later application of a renewal theorem.

Now define,

$$h(x) = e^{-rx} p(x), \quad H(x) = \int_0^X h(s)\,ds$$

$$\psi(x) = e^{-rx} \phi_{0,0}(x)$$

$$c(x) = h(x)/H(\infty), \quad C(x) = H(x)/H(\infty)$$

$$A = \int_0^\infty x e^{-rx} \Phi_{0,0}(dx) = E_\psi(X)$$

$$V(x) = \int_0^\infty e^{-rt} \Phi_{x,o}(dt).$$

Also define,

$$h_k(x) = e^{-rx} p(x)\, {}_xw_{0,0}(k), \quad H_k(x) = \int_0^X h_k(s)\,ds$$

$$c_k(x) = h_k(x)/H_k(\infty)$$

$$V(x,j) = \int_0^\infty e^{-rt} \Phi_{x,j}(dt)$$

The quantities in the first group are direct analogues of those in KH (page 165). In particular, $c(\cdot)$ is the density of the stable age distribution, ψ the distribution of the age of childbearing in the stable population, A its mean, and $V(x)$ the reproductive value of an individual who is $(x,0)$. The quantities in the second set are required in order to deal with the question of parity. In particular, the stable age distribution and the reproductive value have been suitably modified. Note that $V(x) = V(x,0)$.

A population is called Malthusian if the solution r to the characteristic equation exists, $A < \infty$, $H(\infty) < \infty$, and $V(\cdot)$ is bounded. In what follows, we assume that there are no difficulties with boundedness and that direct Riemann integrability is available

where needed (and hence,that the usual renewal theorem applies).
This can be done without loss of generality, since the arguments
of KH apply here as well.

Proposition 1.3.   (confer KH, Theorem 5.2) As t - > ∞ in a
Malthusian process,

(1.2) $e^{-rt} N_{u,0}^{a,k}$ (t) → C_k (a) V(u) H_k (∞)/A

(1.3) $e^{-rt} N_{u,0}^{a}$ (t) → C (a) V(u) H(∞)/A

(1.4) $e^{-rt} N_{u,j}^{a}$ (t) → C(a) V(u,j) H(∞)/A.

Proof: Taking a special case of (1.1) gives

$$e^{-rt} N_{0,0}^{a,k} (t) = \chi_{[0,a]}(t) p(t) \; _t w_{0,0}(k) e^{-rt}$$

$$+ \int_0^t e^{-rt} N_{0,0}^{a,k}(t-s) \; \Phi_{0,0}(ds).$$

Applying the renewal theorem (see KH, Theorem 5.1),

$$e^{-rt} N_{0,0}^{a,k} (t) \rightarrow \int_0^a e^{-rx} p(x) \; _x w_{0,0}(k) \; dx/A = H_k(a)/A.$$

Equation (1.2) now follows by the same arguments as in 3° of the
proof of Theorem 5.2 in KH. Equation (1.3) is obtained by summa-
tion over k. For (1.4) we argue that

$$e^{-rt} N_{0,0}^{a} (t) \rightarrow H(a)/A.$$

From the Corollary to Proposition 1.2 we have

$$e^{-rt} N_{u,j}^{a}(t) = \chi_{[0,a]}(u+t) \; _t p_u e^{-rt}$$

$$+ \int_0^t e^{-rt} N_{0,0}^{a}(t-s) \Phi_{u,j}(ds)$$

$$\rightarrow \frac{H(a)}{A} \int_0^\infty e^{-rs} \Phi_{u,j}(ds)$$

$$= \frac{H(a)\ V(u,j)}{A} \qquad \text{as } t \rightarrow \infty. \quad \square$$

The following result is useful in establishing the properties of the stable age distribution and the reproductive value in this new setting.

<u>Proposition 1.4.</u>   (Confer KH, Theorem 6.1)

$$N_{u,0}^a(s+t) = \sum_j \int_0^\infty N_{u,0}^{dx,j}(s)\ N_{x,j}^a(t).$$

<u>Proof:</u>   Apply the argument of KH separately for each j. $\square$

<u>Proposition 1.5.</u>   (Confer KH, Theorem 6.2)

$$e^{rt} C_k(a) = \sum_j \int_0^\infty c_j(x)\ N_{x,j}^{a,k}(t)\,dx.$$

<u>Proof:</u>   As an immediate consequence of Proposition 1.4 we have

$$N_{u,0}^{a,k}(s+t) = \sum_j \int_0^\infty N_{u,0}^{dx,j}(s)\ N_{x,j}^{a,k}(t).$$

Apply Proposition 1.3 to get

$$\frac{N_{0,0}^{a,k}(s+t)}{N_{0,0}(s)} \rightarrow e^{rt} C_k(a) \quad \text{as } s \rightarrow \infty.$$

Finally as in KH we have

$$\sum_j \int_0^\infty \frac{N_{0,0}^{dx,j}(s)}{N_{0,0}(s)}\ N_{x,j}^{a,k}(t) \rightarrow \sum_j \int_0^\infty c_j(x)\ N_{x,j}^{a,k}(t)\,dx. \quad \square$$

<u>Corollary:</u> $\displaystyle \sum_j \int_0^\infty c_j(x)\ N_{x,j}^a(t)\,dx = e^{rt} C(a).$

Proposition 1.6. $\quad \sum\limits_{j} \int\limits_0^\infty V(u,j) \, N_{u,o}^{dx,j}(t) = e^{rt} V(u).$

Proof: From Proposition 1.4,

$$e^{rt} \, e^{-r(s+t)} \, N_{u,0}^a \, (s+t) = \sum\limits_j \int\limits_0^\infty e^{-rs} \, N_{x,j}^a(s) \, N_{u,0}^{dx,j}(t).$$

And,

$$LHS \to e^{rt} \, H(a) \, V(u)/A$$

$$RHS \to \sum\limits_j \int\limits_0^\infty \frac{H(a)}{A} \, V(u,j) \, N_{u,0}^{dx,j}(t). \quad \square$$

These last two propositions establish the fundamental proper-
ties of the stable age distribution and the reproductive value.
One way of interpreting the latter result is as follows:

On average, the reproductive value of the population at time
t, multiplied by e^{-rt}, is a constant - namely, the reproductive
value of the initial ancestor. In fact, this "martingale property"
lies at the core of more sophisticated treatments of the problem
(see Athreya & Rama Murthy [1976]). In order to obtain sensible
results, we must have a proper definition of the reproductive
value for each age-parity segment of the population.

D.   A GENERAL MODEL

Keeping in mind the constraints on our model described at
the beginning of section 1.C., we now formulate a fairly general
setup which can be dealt with by the simple extensions of the
methods in KH developed in sections 1.B and 1.C.

Suppose that the status of each individual is described by a vector $\underline{x} = (x_1, \ldots, x_n)$ of continuous parameters, and a vector $\underline{m} = (m_1, \ldots, m_n)$ of discrete parameters. For example, we might have x_1 = age, x_2 = marital duration,..., m_1 = indicator of marital status, m_2 = parity,... Each individual has associated with her a status vector $(\underline{x}, \underline{m})$, which changes to reflect the developing life history.

We make the crucial assumption that every newborn begins with the same status vector, denoted by $(\underline{0}, \underline{0})$, and is subject initially to the same forces of transition. These forces may depend on some or all of the parameters in the status vector.

As is usual in vector notation, we say that $\underline{x} \leq \underline{y}$ if and only if $x_i \leq y_i$ ($i = 1, 2, \ldots, n$). Now let $z^{\underline{y}, \underline{\ell}}(t)$ denote the size of the population alive at time t having status vector $(\underline{x}, \underline{m})$ such that $\underline{x} \leq \underline{y}$ and $\underline{m} \leq \underline{\ell}$. Conditioning on a single ancestor of status $(\underline{x}, \underline{m})$ let

$$N^{\underline{y}, \underline{\ell}}_{\underline{x}, \underline{m}}(t) = E_{\underline{x}, \underline{m}}\, z^{\underline{y}, \underline{\ell}}(t).$$

For convenience of notation, suppose that the vector variable $\underline{x}$ takes values in a subset $\mathcal{X}$ of n-dimensional Euclidean space, and the vector variable $\underline{m}$ takes values in a subset M of the n-dimensional Cartesian product of the positive integers. Assume also that the first component of $\underline{x}$, x_1, is age.

The theory developed so far carries over then quite directly, and there is no need to present new proofs of the results. For the sake of completeness, we list some of these results which, with the compact symbolism we have at hand, are almost the same

as their counterparts in sections 1.B and 1.C.

The basic renewal relation (Proposition 1.2) becomes

$$N_{\underline{u},\underline{j}}^{\underline{a},\underline{k}}(t) = \chi_{[0,a]}(u_1+t) \; {}_t v_{\underline{u},\underline{j}}(\underline{a},\underline{k})$$

$$+ \int_0^t N_{\underline{0},\underline{0}}^{\underline{a},\underline{k}}(t-s) \; \Phi_{\underline{u},\underline{j}}(ds),$$

where

$$_t v_{\underline{u},\underline{j}}(\underline{a},\underline{k}) = P\{\text{individual is alive and of status}$$

$$(\underline{a},\underline{k}) \text{ at time } t \mid \text{she was of status}$$

$$(\underline{u},\underline{j}) \text{ at time } 0\}.$$

Note that this probability is necessarily zero if $u_1 + t \neq a_1$. As before, the variable of integration is time.

Lotka's integral equation becomes

$$n^*(t) = \sum_M \int_{\mathbb{x}} \Phi_{\underline{u},\underline{j}}(t) \; n^{\underline{u},\underline{j}}(0) \; d\underline{u}$$

$$+ \int_0^t n^*(t-y) \; \Phi_{\underline{0},\underline{0}}(dy).$$

The Malthusian parameter r is the root of the following equation in the unknown ρ:

$$\int_0^\infty e^{-\rho x} \; \Phi_{\underline{0},\underline{0}}(dx) = 1.$$

Again, this is just for purposes of normalization. The asymptotics follow directly and we can find, for example, a stable age distribution $C(\cdot,\cdot)$ satisfying the relation:

$$\sum_M \int_{\mathbb{x}} C(d\underline{x},\underline{m}) \; N_{\underline{x},\underline{m}}^{\underline{y},\underline{\ell}}(t) = e^{rt} \; C(\underline{y},\underline{\ell}).$$

This setup should be sufficiently general to cover most needs. One major practical difficulty that has been passed over in the discussion is the actual calculation of the relevant net maternity functions. The appendix has a brief sketch of one set of calculations for a typical model which can incorporate considerations of fetal wastage, postpartum amenorrhea, and so on. These models have been discussed in Hoem [1970], Sheps and Menken [1973] and further developed by Mode [1975] and Mode and Littman [1975]. It should be emphasized that these elaborations do not affect the gross mathematical structure, but only how the status-dependent birth intensities are constructed.

2.   MULTIREGIONAL MODELS

A.   Preliminaries

Consider a population dispersed over a domain composed of a finite number of regions. In a given region all individuals are assumed subject to the same age-dependent forces of fertility and mortality. However, since these forces may vary from region to region, the mathematical apparatus of the previous section proves inadequate. Movement from region to region is permitted, with migration intensities that depend on the region of residence, the destination and the age of the individual. For the sake of simplicity, duration dependence is disregarded; thus, an individual upon migration is immediately subject to all the forces belonging to her new home.

The status of an individual is determined by her age and region of residence. The age variable can, in fact, be replaced

easily in the present analysis by a more general status variable
(cf. section 1.D), upon which the various forces of transition
would depend. Because such an approach would complicate the mathe-
matics only by rendering the notation unwieldy, we confine our-
selves to the use of the age factor alone. Lastly, to make the
notation more transparent, the Greek alphabet will be used only
for the region variable. The letters $\alpha, \beta, \gamma, \ldots$ denoting particular
regions, with η and ξ serving as indices of summation.

Let $Z^{a,\beta}(t)$ denote the number of individuals living in region
β at time t, aged $\leq$ a and descended from a single ancestor at
time 0. $E_{u,\alpha}$ denotes expectation conditional on the event that
the individual (usually the ancestor) is of status (u,α) and
$L > u$, at some given time. We then define:

$$N_{u,\alpha}^{a,\beta} = E_{u,\alpha} \; Z^{a,\beta}(t) ,$$

$$\Phi_{u,\alpha}^{\eta}(t) = E \; [\text{number of children born in region}$$
$$\eta \text{ by time t, to an individual who}$$
$$\text{is } (u,\alpha) \text{ at time 0]},$$

and

$$_t v_{u,\alpha}(\beta) = P\{\text{individual is alive and in region } \beta$$
$$\text{at time t } | \text{ she was } (u,\alpha) \text{ at time 0}\}.$$

This last term is an analogue of a conditional probability de-
fined in section 1.A. (using "w" rather than "v"), but including
the probability of being alive at time t. We then have the
following analogy of Proposition 1.2.

Proposition 2.1.

(2.1) $\quad N_{u,\alpha}^{a,\beta}(t) = \chi_{[0,a]}(u+t) \, _t v_{u,\alpha}(\beta) + \sum_\eta \int_0^t N_{0,\eta}^{a,\beta}(t-s) \, \Phi_{u,\alpha}^\eta(ds)$.

__Proof:__ $\quad Z^{a,\beta}(t) = Z_{<0>}^{a,\beta}(t) + \sum_\eta \sum_k \, {}^{k,\eta}Z^{a,\beta}(t)$

where ${}^{k,\eta}Z^{a,\beta}(t)$ denotes the number of individuals alive at time t in region β and aged $\leq$ a, descended from the kth child born to $< 0 >$ in region η.

The desired result now follows by taking expectations, for clearly

$$E_{u,\alpha} \, Z_{<0>}^{a,\beta}(t) = \chi_{[0,a]}(u+t) \, _t v_{u,\alpha}(\beta)$$

by definition, and using exactly the argument of Proposition 1.2 separately for each η gives the last term in (2.1). $\square$

Equation (2.1) is a multivariate renewal equation. A discussion of this topic can be found in Çinlar [1975] pages 323 ff. We give here only a brief sketch of some of the mathematical considerations. As in the univariate case, the main problem is the proper normalization of the functions (s) with respect to which the integration is carried out, in this case $\Phi_{0,\alpha}^\eta(\cdot)$.

We begin by writing

$$R_{\xi\eta} = \Phi_{0,\xi}^\eta(\infty)$$

and

(2.2) $$\Xi_{\xi\eta}(x) = \Phi_{0,\xi}^\eta(x)/R_{\xi\eta}.$$

Then $\{\Xi_{\xi\eta}(\cdot)\}$ are probability measures by construction and are assumed nonatomic. This is realistic in view of their role

as normalized net maternity functions. Interest now centers on
the matrix $R = (R_{\xi\eta})$. Note that each $R_{\xi\eta}$ can be interpreted as
a net reproduction rate.

The mathematical theory requires that R be irreducible and
of spectral radius 1. In demographic applications this latter
requirement is not fulfilled in general. This parallels the uni-
variate case where the function $\Phi_{0,0}(\cdot)$ is not a probability
measure and must be normalized. Such a normalization was carried
out by introducing the Malthusian parameter as the solution of a
characteristic equation (see section 1.C.). A similar approach
can be carried out in this case.

For each real ρ define

$$R_{\xi\eta}(\rho) = \int_0^\infty e^{-\rho x}\, \Phi_{0,\xi}^{\eta}\,(dx),$$

and let

$$R(\rho) = (R_{\xi\eta}(\rho)).$$

Clearly, there exists a unique value of ρ, call it r, such that
R(r) is a matrix with spectral radius 1. Alternatively, we may
say that the matrix

$$\left(\int_0^\infty e^{-rx}\, \Phi_{0,\xi}^{\eta}\,(dx) \right)$$

has 1 as its dominant characteristic root. In this form, there is
an obvious analogy between the univariate and multivariate cases
(see also Rogers and Willekens [1976a], pages 13-15). Using the
Malthusian parameter r, we can obtain asymptotic results having
the characteristic exponential form.

B.   Asymptotics

Proposition 2.2.   In the setup described above, suppose that a real solution r to the generalized Lotka equation exists. Let

$$\Xi^*_{\xi\eta}(t) = \frac{1}{R_{\xi\eta}(r)} \int_0^t e^{-rx} \Phi^\eta_{0,\xi}(dx)$$

and

$$\mu_{\xi\eta} = \int_0^\infty x \, \Xi^*_{\xi\eta}(dx).$$

We assume that $\mu_{\xi\eta} < \infty$ $\forall \xi, \eta$ and that

$$\int_0^\infty e^{-rt} \, t \, v_{0,\eta}(\xi) < \infty \quad \forall \xi, \eta.$$

(These are just the usual finiteness assumptions). Let $\underline{\ell}$ and $\underline{h}$ be left and right eigenvectors of $R(r)$. Since $R(r)$ is positive and has spectral radius 1, $\underline{\ell}$ and $\underline{h}$ both positive can be found. We then have as $t \to \infty$.

(2.3) $$e^{-rt} N^{a,\beta}_{0,\alpha}(t) \to h_\alpha \sum_\eta c_\eta \int_0^a e^{-rs} \, s v_{0,\eta}(\beta) \, ds,$$

where

$$c_\eta = \frac{\ell_\eta}{\sum_{\xi,K} h_\eta \ell_\xi R_{\xi K}(r) \mu_{\xi K}}$$

Proof: The particular form of the result follows directly from some work of Asmussen & Hering [1977]. Similar expressions can also be found in Çinlar [1975] pages 328-335.

Remark: The components h_α of the vector $\underline{h}$ will play an important role in defining a single measure for the reproductive value of an individual. Note that the initial region of residence α of

the ancestor enters the limit on the RHS of (2.3) only through h_α.
In fact we may write

$$(2.4) \qquad e^{-rt} N_{0,\alpha}^{a,\beta} (t) \to h_\alpha \, f(a,\beta)$$

where

$$f(a,\beta) = \sum_\eta c_\eta \int_0^a e^{-rs} \, {}_s v_{0,\eta} (\beta) \, ds.$$

Making a formal comparison bewteen (2.4) and (1.4), leads
one to suspect that $f(a,\beta)$ must be involved in the definition
of the stable age distribution. To see that this is so, we re-
quire a simply established analogue to Proposition 1.4:

$$(2.5) \qquad N_{u,\alpha}^{a,\beta} (s+t) = \sum_\eta \int_0^\infty N_{u,\alpha}^{dx,\eta} (t) \, N_{x,\eta}^{a,\beta} (s).$$

Consequently, it is an easy matter to derive the result that in
the stable age distribution, the proportion of the population
aged $\leq$ a in region β is given by

$$\frac{f(a,\beta)}{\sum_\eta f(\infty,\eta)}.$$

C. The Reproductive Value

In deriving an index of reproductive value, we will let the
mathematics lead us naturally to an appropriate expression. From
equations (2.1) and (2.4) we find that as $t \to \infty$,

$$e^{-rt} N_{u,\alpha}^{a,\beta} (t) \to \sum_\xi f(a,\beta) \, h_\xi \int_0^\infty e^{-rs} \, \phi_{u,\alpha}^\xi (ds).$$

We now argue exactly as in Proposition 1.6. That is, we obtain
appropriate limiting forms for both sides of equation (2.5) as
$s \to \infty$:

$$e^{rt} \, e^{-r(s+t)} \, N_{u,\alpha}^{a,\beta}(s+t) \to e^{rt} \sum_{\xi} f(a,\beta) \, h_{\xi} \int_0^{\infty} e^{-rs} \, \Phi_{u,\alpha}^{\xi}(ds)$$

and

$$e^{-rs} \sum_{\eta} \int_0^{\infty} N_{u,\alpha}^{dx,\eta}(t) \, N_{x,\eta}^{a,\beta}(s)$$

$$\to \sum_{\eta} \int_0^{\infty} N_{u,\alpha}^{dx,\eta}(t) \, [\sum_{\xi} f(a,\beta) \, h_{\xi} \int_0^{\infty} e^{-rs} \, \Phi_{x,\eta}^{\xi}(ds)].$$

Equating the two sides, we have

$$(2.6) \quad \sum_{\eta} \int_0^{\infty} N_{u,\alpha}^{dx,\eta}(t) \, [\sum_{\xi} h_{\xi} \int_0^{\infty} e^{-rs} \, \Phi_{x,\eta}^{\xi}(ds)]$$

$$= e^{rt} \, [\sum_{\xi} h_{\xi} \int_0^{\infty} e^{-rs} \, \Phi_{u,\alpha}^{\xi}(ds)].$$

In view of (2.9), we are forced to define the reproductive value of an individual aged u residing in region α to be

$$V(u,\alpha) = \sum_{\xi} h_{\xi} \int_0^{\infty} e^{-rs} \, \Phi_{u,\alpha}^{\xi}(ds).$$

We may, therefore, rewrite (2.6) to parallel Proposition 1.6:

$$\sum_{\eta} \int_0^{\infty} N_{u,\alpha}^{dx,\eta}(t) \, V(x,\eta) = e^{rt} \, V(u,\alpha).$$

Relative to a region ξ, the reproductive value of an individual of status (u,α) is given by the quantity

$$v_{u,\alpha}(\xi) = \int_0^{\infty} e^{-rs} \, \Phi_{u,\alpha}^{\xi}(ds).$$

Because children born in different regions have different net maternity functions, the $\{v_{u,\alpha}(\xi)\}$ can not be simply added together. Rather, they must be weighted by the factors $\{h_{\xi}\}$ before being combined into a single expression. These weights are quite

natural since we have from (2.4) as $t \to \infty$

$$\frac{N_{0,\xi}^{a,\beta}(t)}{N_{0,\eta}^{a,\beta}(t)} \to \frac{h_\xi}{h_\eta} \ .$$

Thus, the components $\{h_\xi\}$ measure, from the standpoint of total eventual population, the relative worth of newborn in different regions. Rogers and Willekens [1976b] contains a discussion related to these ideas.

D.   The Lotka equation

The derivation of the Lotka equation for the multiregional model is a fairly straightforward generalization of the technique used in Section 1.B. Let

$$N_{u,\alpha}^{*,\beta}(t) = \text{number of births by time } t \text{ in region } \beta,$$

starting from a single ancestor with

status (u,α) at time 0.

In analogy to equation (*) of Section 1.B we have

$$(2.7) \qquad N_{u,\alpha}^{*,\beta}(t) = 1 + \Phi_{u,\alpha}^{\beta}(t) + \sum_\eta \int_0^t \Phi_{0,\eta}^{\beta}(t-s) \, N_{u,\alpha}^{*,\eta}(ds)$$

$$= 1 + \Phi_{u,\alpha}^{\beta}(t) + \sum_\eta \int_0^t [N_{u,\alpha}^{*,\eta}(t-y) - 1] \, \Phi_{0,\eta}^{\beta}(dy) \ .$$

In the middle expression, each term in the summation represents the contribution of those children of the ancestor born in region η.

Suppose that the density of individuals of status (u,α) at time 0 is denoted by $n^{u,\alpha}(0)$. Multiplying (2.7) by $n^{u,\alpha}(0)$, integrating with respect to u and summing over α, yields

$$(2.8) \quad N^{*,\beta}(t) = N(0) + \sum_{\alpha} \int_0^{\infty} \Phi_{u,\alpha}^{\beta}(t) \, n^{u,\alpha}(0) \, du$$

$$+ \sum_{\eta} \int_0^t [N^{*,\eta}(t-y) - N(0)] \, \Phi_{0,\eta}^{\beta}(dy) \, ,$$

where
$$N^{*,\beta}(t) = \sum_{\alpha} \int_0^{\infty} N_{u,\alpha}^{*,\beta} \, n^{u,\alpha}(0) \, du.$$

Differentiating (2.8) with respect to t, we have

$$(2.9) \quad n^{*,\beta}(t) = \sum_{\alpha} \int_0^{\infty} \Phi_{u,\alpha}^{\beta}(t) \, n^{u,\alpha}(0) \, du$$

$$+ \sum_{\eta} \int_0^t n^{*,\eta}(t-y) \, \Phi_{0,\eta}^{\beta}(dy).$$

The LHS of (2.9) is the density of births at time t in region β only. The first sum on the RHS represents the contribution of those alive at time 0. The second sum represents the contribution of those born after time 0, with the contribution of each region counted separately (hence the summation over η). The Lotka equation for the population as a whole would be a vector of equations like (2.9), with each component being an equation for another value of β. Understandably, the number of components would equal the number of regions in the model.

3.   A HYBRID MODEL

There is a clear contrast between the models of Sections 1 and 2. In the former, all individuals are initially subject to the same forces of fetility and mortality, though these forces will change in different ways as the life histories develop. In the latter, individuals begin life on different footings, depending on the region of birth. Because only a finite number of

different regions are permitted, the model is somewhat primitive.
Below will be found a brief sketch of a model which at first
sight is more advanced than the multiregional models, but the
analysis of which ultimately reduces to that of Section 1, rather
than Section 2. Understanding why this is so will help us to
modify the conclusions of the first section.

Suppose then we have a parameter η taking values in a space E,
such as [0,1]. At birth, each individual is assigned a value of η
at random according to some distribution G ($\cdot$), independently of
the parameter values of the parent, any previous offspring, etc.
The individual retains this value of η through her lifetime. Now
assume that with each value of η is associated a function Φ_η ($\cdot$)
where

$$\Phi_\eta (x) = E \text{ [number of offspring by time x to an}$$
$$\text{individual aged 0 at time 0, and}$$
$$\text{having the particular value of } \eta \text{ as}$$
$$\text{her parameter value].}$$

Thus, we have an infinite multiregional model, each value of η
corresponding to a different region.

Furthermore, let

$$N^a_{u,\eta} (t) = E \text{ [number of individuals alive and}$$
$$\text{aged } \leq a \text{ at time t } | \text{ population}$$
$$\text{begins with a single individual at}$$
$$\text{time 0 of age u and parameter } \eta].$$

In what follows, we only consider u = 0, and so, drop this
subscript entirely. Let also

$$M^a(t) = \int_E N^a_\xi(t) \, G(d\xi).$$

Arguing formally, it is easy to see that

$$N^a_\xi(t) = \int_0^t \left[\int_E N^a_\eta(t-s) \, G(d\eta) \right] \Phi_\xi(ds).$$

Now, if the parameter value ξ of the ancestor is also chosen according to the distribution $G(\cdot)$, then we have

$$M^a(t) = \int_E N^a_\xi(t) \, G(d\xi)$$

$$= \int_E \left[\int_0^t M^a(t-s) \, \Phi_\xi(ds) \right] G(d\xi)$$

$$= \int_0^t M^a(t-s) \, \Phi(ds),$$

where $\Phi(s) = \int_E \Phi_\xi(s) \, G(d\xi)$.

This is, of course, a simple renewal equation of the type met in Section 1, except that the role of the net maternity function is played here by a weighted average of individual net maternity functions.

This last result suggests that it is not heterogeneity, per se, which precludes an analysis by the methods of Section 1. Rather, it is only when the heterogeneity is determined (at least in part) by the history of the process up to the time of birth, that the methods of Section 2 are required. In the multiregional models, for example, the history of the individual is affected by the status of the mother (i.e. region of residence) at the time of birth. On the other hand, in the present section the individual's history is quite independent of the past. Put more

crudely, at birth each individual has the same chance as any other
of obtaining a particular value of η and the associated net ma-
ternity function.

These considerations make clear that the group of models of
Section 1 is very large and can accomodate various family planning
schemes. For example, we could allow each individual, upon reach-
ing a certain parity, to decide with probability one-half to have
no more births and with probability one-half to have another. Of
course, the more complicated the model, the more involved the cal-
culations.

4. THE AGE AT CHILDBEARING PROBLEM

The function

$$\psi(x) = e^{-rx} \Phi(x)$$

was introduced in Section 1 as the (density of the) distribution
of the age at childbearing in the stable population. A justifi-
cation for this term from the point of view of the stochastic
theory was given by Jagers [1975], who computed the expected
number of individuals born by time t to mothers aged $\leq$ a. As
t → ∞, the ratio of this quantity to the expected number of indi-
viduals born by time t converges, as a function of a, to the age
at childbearing.

Keiding [1973] has given a heuristic derivation from another
point of view. He examined the probability that a child born at
time t had a mother aged $\leq$ a and argued that this should converge
to $\int_0^a \psi(a)da$ as t → ∞. A direct proof seems to involve measur-
ability difficulties, but probably can be worked out using Palm

functions. We present here a more elementary approach involving
the reproductive value.

Working with the model in KH, let $\lambda(\cdot)$ be the intensity func-
tion of the Poisson process generating births. Define for $x \geq y$,

$$V_y(x) = \int_0^\infty e^{-r[s+(x-y)]} {}_{s+(x-y)}P_y \; \lambda(y+[s+(x-y)])\,ds$$

$$= \text{discounted value to an individual aged } y$$

$$\text{of births from age } x \text{ onwards.}$$

Then

$$V_x(x) = \int_0^\infty e^{-rs} {}_sP_x \; \lambda(x+s)\,ds$$

is the ordinary reproductive value.

Define

$$W_h(x) = V_x(x) - V_x(x+h)$$

$$= \int_0^h e^{-rs} {}_sP_x \; \lambda(x+s)\,ds.$$

Then $W_h(x)$ is that portion of the reproductive value "spent"
between ages x and $x+h$.

Consider now the quantity

$$\Delta(h,a) = \frac{1}{h} \int_0^a W_h(x) \, z^{dx}(t).$$

That is, $\Delta(h,a)$ is the normalized change in the reproductive
value of the population from t to $t+h$ due to those individuals
aged $\leq a$ at time t. Then

$$E_0[\lim_{h \to 0} \Delta(h,a)] = E_0[\int_0^a \lambda(x) \, z^{dx}(t)]$$

$$= \int_0^a \lambda(x) \, N_0^{dx}(t).$$

As $t \to \infty$,

$$\frac{E_0[\lim_{h \to 0} \Delta(h,a)]}{E_0[\lim_{h \to 0} \Delta(h,\infty)]} = \frac{\int_0^a \lambda(x) \, N_0^{dx}(t)}{\int_0^\infty \lambda(x) \, N_0^{dx}(t)}$$

$$\to \int_0^a \psi(x) \, dx = \Psi(a), \quad \text{as } t \to \infty.$$

Thus $\Psi(a)$ measures for large t the proportion, on average, of the instantaneous change in the reproductive value of the population due to individuals aged $\leqq a$.

The same calculation can be carried out for the model in Section 1. Define

$$W_h(x,j) = V(x,j) - V(x+h,j).$$

Then

$$E_{00}\left[\lim_{h \to 0} \int_0^a \frac{1}{h} \sum_j V_h(x,j) \, Z^{dx,j}(t)\right]$$

$$= \int_0^a \sum_j \phi_{x,j}(0) \, N^{dx,j}(t).$$

As $t \to \infty$,

$$\frac{\int_0^a \sum_j \phi_{x,j}(0) \, N^{dx,j}(t)}{\int_0^\infty \sum_j \phi_{x,j}(0) \, N^{dx,j}(t)}$$

$$\to \frac{\int_0^a e^{-rx} \, p(x) \sum_j \phi_{x,j}(0) \, _xw_{0,0}(j) \, dx}{\int_0^\infty e^{-rx} \, p(x) \sum_j \phi_{x,j}(0) \, _xw_{0,0}(j) \, dx}$$

$$= \int_0^a e^{-rx} \phi_{0,0}(x) \, dx,$$

where we have used the identity

$$\Phi_{0,0}(x+h) - \Phi_{0,0}(x) = \sum_j \Phi_{x,j}(h) \, {}_x w_{0,0}(j) \, p(x).$$

APPENDIX

We now develop some formulas for the cohort net maternity function. To facilitate comparisons, we use the notation of Mode [1975].

Consider a cohort aged 0 at time 0. Let T_0 be the (random) time of marriage and T_k $(k \geq 1)$ be the time of the kth birth. Note that time and age coincide in this scheme. Define

$$H_n(x,t) = P\{T_n \leq t \mid T_0 = x\}$$

$$P_{n,n+1}(y,t) = P\{T_{n+1} - T_n \leq t \mid T_n = y\}.$$

Clearly, we have

(A.1) $$H_{n+1}(x,t) = \int_0^t P_{n,n+1}(s,t-s) \, H_n(x,ds),$$

which is called a Markovian convolution by Mode. Note that our formula differs slightly from his (4.2.8). It should be remembered that the conditional distributions $P_{n,n+1}(y,t)$, since they take account of mortality, are defective in the sense that $P_{n,n+1}(y,\infty) < 1$. The connection between these expressions and the functions $p(\cdot)$, $\lambda(\cdot)$ introduced in Section 1 is obvious:

$$P_{n,n+1}(y,t) = \int_0^t p_y \left[1 - e^{-\int_y^s \lambda_n(u)\,du} \right] \lambda_n(s)\,ds.$$

Finally,

$$E[K(t \wedge L) \mid T_0 = x] = \sum_{n=1}^{\infty} H_n(x,t),$$

so that

(A.2)
$$\Phi_{0,0}(t) = \int_0^t \sum_{n=1}^{\infty} H_n(x,t) \, G_0(dx)$$

where $G_0(x)$ is the cumulative marriage curve; i.e., $G_0(x) = P\{T_0 \leq x\}$. In constructing the functions $P_{n,n+1}(\cdot,\cdot)$ one can take account of such features as fetal wastage, postpartum amenorrhea, etc. Finally, the $P_{n,n+1}(\cdot,\cdot)$ are generally taken to be continuous functions of their arguments. Together with (A.1) and (A.2), this implies the differentiability of the net maternity function.

Note added in proof: Two further references have been brought to the author's attention. "Master Equation Treatment of Aging Populations" by N.G. van Kampen (Reports on Mathematical Physics $\underset{\sim}{11}$ [1977] pp. 111-122) develops the asymptotics of the Bellman-Harris process from a point of view slightly different from the usual one. It is claimed that this approach can easily be applied to the analysis of models involving interaction among individuals. The second, "Reconciliation of the Integral Equation and the Projection Techniques in the Age-Parity Specific Stable Population Model" by P. Das Gupta (Reprint No. 463, Institute of International Studies, U. of California at Berkeley) studies a particular discrete time deterministic population model, and contains some interesting historical references.

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