Inge Henningsen

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Preprint
February
1978

3

Institute of Mathematical Statistics
University of Copenhagen

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INSTITUTE OF MATHEMATICAL STATISTICS
UNIVERSITY OF COPENHAGEN

FEBRUARY 1978

H.C.Ø.-tryk
København

1. Introduction

In [1] Clarke studies estimation in the M/M/1-queue, observing the process until the busy time reaches som preassigned level τ. In this paper we study the M/G/I-queue under Clarke's sampling scheme. In section 3 we show that the maximum likelihood estimator for the arrival intensity λ is $\hat{\lambda} = \frac{N(\tau)}{R(\tau)}$, where $N(\tau)$ is the total number of arrivals and $R(\tau)$ is the total observation time. In the case with traffic intensity $\rho < 1$, we prove that under mild regularity conditions the maximum likelihood estimator for the parameters in the service time distribution will exist for $\tau \to \infty$ if the maximum likelihood estimator exists in the problem with fixed sample-size.

In section 4 we show that for $\tau \to \infty$ $(\sqrt{\tau}(\frac{N(\tau)}{R(\tau)} - \lambda), \sqrt{\tau}(\frac{M(\tau)}{\tau} - \mu))$ is asymptotically normally distributed with independent components. Here $M(\tau)$ is the total number of arrivals and $\frac{1}{\mu}$ is the mean service time.

In section 5 we take the M/M/1-queue as a special case and derive the likelihood ratio test for the hypothesis $\rho = \rho_0$ and prove that Q is asymptotically distributed as χ^2 with f=1.

The asymptotic results for the distribution of $(\hat{\lambda}, \hat{\mu})$ in the M/M/1-queue are well-known see e.g. Cox [2], Syski [9], Wolff [10].

In these papers the simple M/M/1-queue is considered as a special case of a Markovian or a birth and death queuing model, i.e. the authors assume exponential holding times for both the arrival and the service distribution.

The approach in the present paper is different. We do not assume the queue to be Markov. In stead we apply results from the theory of regenerative (and cumulative) processes. This imposes the restriction $\rho < 1$, since this condition is necessary to ensure that the queue returns to 0 infinitely often.

2. Description of the queue.

Consider the M/G/1-queue (in the notation of Kendall [4]). In this the interarrival distribution is exponential with intensity $\lambda > 0$ and we have a general service time distribution $B(v)$.

The queue discipline is "first come, first served" and we assume an infinite waiting capacity.

In the following section we consider the problem of deriving maximum likelihood estimates for (λ, θ) using the sampling method of Clarke [1], i.e. observing the queue until the busy time reaches some fixed level τ.

Let

$U_1, U_2, \ldots$ be time intervals between consecutive arrivals

and

$V_1, V_2, \ldots$ the service times.

$U_1, U_2, \ldots$ are i.i.d. random variables with density

$$f(u) = \lambda e^{-\lambda u} I(u>0) \tag{1}$$

$V_1, V_2, \ldots$ are i.i.d. and are independent of $U_1, U_2, \ldots$ We assume that $B(v)$ has a density $b(v)$.

Let $Q(t)$ be the queue length at time t and define

$$T_1 = \min\{t \mid Q(t)=0\}$$

$$T_n = \min\{t \mid \exists s > T_{n-1} : Q(s) > 0 \text{ and } Q(t)=0\}, n=2,3,\ldots$$

T_n is the length of the n'th busy period. $T_1, T_2, \ldots$ are independent and $T_2, T_3, \ldots$ are identically distributed.

Define

$$Z_n=\min(t>T_n \mid Q(t)>0) \quad n=1,2,...$$

Z_n is the length of the n'th empty period. $Z_1,Z_2,...$ are i.i.d. Because of the regeneration property of the exponential distribution the distribution of $Z_1,Z_2,...$ is again exponential with parameter λ and $Z_1,Z_2,...$ are independent of $T_1,T_2,...$.

Define for $\tau>0$

$$M(\tau)=\max(m \mid \sum_{i=1}^{m} V_i \leq \tau)$$

$$\nu(\tau)=\max(m \mid \sum_{i=1}^{m} T_i \leq \tau)$$

and let $M_n, n=1,2,...$ be the number of departures in the n'th busy period. Note that for $n\geq 2$ M_n is also the number of arrivals.

Since $M(\tau)$ is defined solely in terms of $V_1,V_2,...$ $M(\tau)$ is independent of $U_1,U_2,...$ and $Z_1,Z_2,...$

Define finally

$$R(\tau)=\tau+\sum_{i=1}^{\nu(\tau)} Z_i$$ i.e. $R(\tau)$ is the total observation time

and

$N(\tau)$ = the total number of arrivals in the interval $(0,R(\tau))$.

3. Estimation of (λ,θ).

To simplify the notation we will assume $Q(0-)=0$ and $Q(0)=1$, i.e. the observation period starts just after a customer arrives to an empty queue. The asymptotic results can easily be modified to cover the general situation. Under this assumption T_1 has the same distribution as $T_2,T_3,...$.

We will derive the likelihood function stepwise in the manner of Clarke.

The statistical model is

$U_1, U_2, \ldots$ are independent and exponentially distributed with $EU_1 = \frac{1}{\lambda}$, $\lambda > 0$.

$V_1, V_2, \ldots$ are independent and have distribution $B(v,\theta)$, where B has density b and $\theta \in \Theta$.

We shall further assume that θ and λ varies independently.

Let n and m be positive integers with $n \geq m$.

The density of $(M(\tau), V_1, \ldots, V_{M(\tau)})$ is

$$\prod_{j=1}^{m} b(v_j,\theta) P_0\{\sum_{i=1}^{m+1} V_i > \tau \mid V_1 = v_1, \ldots, V_m = v_m\} I\{\sum_{i=1}^{m} v_1 < \tau\}$$

$$= \prod_{j=1}^{m} b(v_j,\theta)(1 - B(\tau - \sum_{i=1}^{m} v_1, \theta)) I\{\sum_{i=1}^{m} v_i < \tau\}.$$

When $M(\tau) = m, V_1 = v_1, \ldots, V_m = v_m$ are given the density of $U_1, \ldots, U_m$ is

$$\lambda^m e^{-\lambda \sum_{i=1}^{m} u_i}.$$

Since $R(\tau)$ and $\nu(\tau)$ are functions of $V_1, \ldots, V_{M(\tau)}$ and $U_1, \ldots, U_M(\tau)$ only, the density of $N(\tau)$, $U_{M(\tau)+1}, \ldots, U_{N(\tau)}$ for given $M(\tau) = m$, $V_1 = v_1, \ldots, V_{M(\tau)} = v_m$, $U_1 = u_1, \ldots, U_{M(\tau)} = u_m$, $R(\tau) = r$ is

$$\lambda^{n-m} e^{-\lambda \sum_{i=m+1}^{n} u_i} e^{-\lambda(r - \sum_{i=1}^{n} u_i)} I\{\sum_{i=1}^{n} u_i < r\},$$

Hence

$$L_\tau(\lambda,\theta) = \prod_{j=1}^{M(\tau)} b(V_j,\theta)\,\lambda^{N(\tau)} e^{-\lambda R(\tau)} \Big(1-B\big(\tau-\sum_{i=1}^{M(\tau)} V_i,\theta\big)\Big)\cdot h, \tag{2}$$

where h does not depend on λ and θ.

From (2) follows readily

(i) θ and λ can be estimated separately

(ii) if $\lambda < \frac{1}{EV_1}$ the queue returns to 0 infinitely often and one would expect that some kind of regenerative process argument would suffice to ensure the existence of $\hat{\theta}$ in the cases where one had a maximum likelihood estimator of θ for fixed sample size.

That this is the case is seen from the following:

Theorem 1.

The M/G/1-queue has for $\tau\to\infty$ a unique maximum likelihood estimator $(\lambda_\tau,\theta_\tau)$ for (λ,θ) under the following conditions:

(i) $EV_1 < \infty$

(ii) $E|\log b(V_1,\theta)| < \infty$

(iii) $V_1, V_2, \ldots, V_n$ admits for $n\to\infty$ a unique maximum likelihood estimator $\hat{\theta}_n$ for $\theta \in \Theta$.

Proof:

From (2) we have

$$L_\tau(\lambda,\theta) = \prod_{i=1}^{M(\tau)} b(V_i,\theta)\Big(1-B\big(\tau-\sum_{i=1}^{M(\tau)} V_i,\theta\big)\Big)\lambda^{N(\tau)} e^{-\lambda R(\tau)}\cdot h.$$

It is immediate that

$$\hat{\lambda} = \frac{N(\tau)}{R(\tau)} \tag{3}.$$

To find $\hat{\theta}_\tau$ we maximize

$$L_\tau(\theta) = \prod_{i=1}^{M(\tau)} b(V_i,\theta)\left(1-B\left(\tau-\sum_{i=1}^{M(\tau)} V_i,\theta\right)\right)$$

or minimize

$$-\frac{1}{M(\tau)}\log L_\tau(\theta) = -\frac{1}{M(\tau)}\sum_{i=1}^{M(\tau)} \log b(V_i,\theta) - \frac{1}{M(\tau)}\log\left(1-B\left(\tau-\sum_{i=1}^{M(\tau)} V_i,\theta\right)\right) \quad (4).$$

We note, that $M(\tau)$ is a cumulative process in the sense of Smith [7] with $V_1, V_2, \ldots$ defining the associated renewal process (actually $M(\tau)$ is the number of renewals), and by Theorem 7 of [7] we have

$$\frac{M(\tau)}{\tau} \to \frac{1}{EV_1}, \quad \tau \to \infty \qquad (5)$$

with probability one.

Lemma 1: For $\tau \to \infty$

$$\left|\frac{1}{M(\tau)}\sum_{i=1}^{M(\tau)} \log b(V_i,\theta) - \frac{1}{\left[\frac{\tau}{EV_1}\right]}\sum_{i=1}^{\left[\frac{\tau}{EV_1}\right]} \log b(V_i,\theta)\right| \to 0$$

with probability one.

Proof: Since $E|\log b(V_1,\theta)| < \infty$ by condition (ii), we have

$$\frac{1}{\left[\frac{\tau}{EV_1}\right]}\sum_{i=1}^{\left[\frac{\tau}{EV_1}\right]} \log b(V_i,\theta) \to E\log b(V_1,\theta) \quad \text{for } \tau\to\infty$$

with probability one by the strong law of large numbers, and

$$\frac{1}{M(\tau)}\sum_{i=1}^{M(\tau)} \log b(V_i,\theta) =$$

$$\frac{\tau}{M(\tau)} \cdot \frac{1}{\tau}\sum_{i=1}^{M(\tau)} \log b(V_i,\theta) \to E\log b(V_1,\theta) \quad \text{for } \tau\to\infty$$

with probability one by (5) and theorem 7 of [7].

Lemma 2. For $\tau \to \infty$

$$-\frac{1}{M(\tau)} \log \left(1-B\left(\tau-\sum_{i=1}^{M(\tau)} V_i, \theta\right)\right) \to 0$$

with probability one.

Proof: We have

$$0 \leqq -\frac{1}{M(\tau)} \log \left(1-B\left(\tau-\sum_{i=1}^{M(\tau)} V_i, \theta\right)\right) \leqq -\frac{1}{M(\tau)} \log \left(1-B\left(V_{M(\tau)+1}, \theta\right)\right).$$

Since condition (ii) implies $-E \log (1-B(V_1)) < \infty$, the result follows from the strong law of large numbers together with (5).

Lemma 1 and 2 together with (4) shows that

$$\left|-\frac{1}{M(\tau)} \log L_\tau(\theta) + \frac{1}{[\frac{\tau}{EV_1}]} \sum_{i=1}^{[\frac{\tau}{EV_1}]} \log b\,(V_i,\theta)\right| \to 0 \text{ for } \tau\to\infty \quad (6).$$

with probability one.

Since $-\frac{1}{n}\sum_{i=1}^{n} \log b(V_i,\theta)$ has a unique minimum for $n \to \infty$ by condition (iii), the same obtains for $-\frac{1}{M(\tau)} \cdot \log L_\tau(\theta)$.

Remark 1. We have actually proved that the existence of the maximum likelihood estimator in the process-case and in the fixed sample size case is equivalent.

Remark 2. In most practical situations conditions (i) and (ii) are needed to establish the existence of the maximum likelihood estimator in the case with fixed sample size.

Corollary 1. From (6) and the uniqueness of the maximum likelihood estimator follows

$$\hat\theta_\tau - \hat\theta_{[\frac{\tau}{EV_1}]} \to 0 \quad \text{for } \tau \to \infty. \qquad (7)$$

with probability one.

4. Asymptotic distribution of $(\frac{M(\tau)}{\tau}, \frac{N(\tau)}{R(\tau)})$.

In many practical situations the service time distribution is parametrized by $\mu=\frac{1}{EV_1}$, and we might have $\hat{\mu}_\tau=\frac{M(\tau)}{\tau}$, i.e. the relative frequency of departures. This is f. in. the case in the M/M/1-queue.

For a more general discussion, see Keiding and Lauritzen [3].

In this section we derive the asymptotic distribution for $(\frac{M(\tau)}{\tau}, \frac{N(\tau)}{R(\tau)})$ using results from the theory of regenrative processes.

Let $\frac{1}{\mu}=EV_1$, and assume $\lambda<\mu$, i.e. assume the traffic intensity $\rho<1$.

The essential argument is the following:
The queue regenerates itself every time an arriving customer finds the server idle, (see e.g. Smith [8] p. 257). By the assumption $\lambda<\mu$, the queue is empty infinitely often. Hence the asymptotic results from the theory of regenerative processes apply.

With the notation from section 2 we have the following theorem:

Theorem 2. In the M/G/1-queue assume $VV_1=\sigma^2<\infty$. Then

$$\sqrt{\tau}\ (\frac{M(\tau)}{\tau} - \mu, \frac{N(\tau)}{R(\tau)} - \lambda) \Rightarrow N(0, \begin{pmatrix} \sigma^2\mu^3 & 0 \\ 0 & \frac{\lambda^2}{\mu} \end{pmatrix}) \text{ for } \tau\to\infty ,$$

where $\Rightarrow$ means weak convergence.

Proof: For $\lambda\leq\mu$ we have $P(T_1<\infty)=1$ and for $\lambda<\mu$ $ET_1<\infty$ ([6] p. 75), i.e. $\nu(\tau)\to\infty$ for $\tau\to\infty$ with probability one. In the rest of the proof we shall restrict ourself to the case $\Omega' = \{\omega \mid \nu(\tau)(\omega)\to\infty \text{ for } \tau\to\infty\}$.

We can write $N(\tau)=M(\tau)+N'-M'$, where N' and M' are the arrivals and departures respectively in the last uncompleted busy cycle.

We have $M'\leqq N'$, and since

$$R(\tau)-\sum_{i=1}^{\nu(\tau)}(T_i+U_i)<T_{\nu(\tau)+1}$$

we have

$$N'\leqq N_{\nu(\tau)+1}$$

and the relation

$$P\{\frac{N'}{\sqrt{\tau}}>\varepsilon\}<\frac{EN'}{\sqrt{\tau}\cdot\varepsilon}<\frac{EN_1}{\sqrt{\tau}\cdot\varepsilon}\to 0 \text{ for } \tau\to\infty$$

gives

$$\frac{N'}{\sqrt{\tau}}\overset{P}{\to}0. \qquad (9)$$

$R(\tau)$ is a cumulative process relative to $T_1, T_2, \ldots$, hence

$$\frac{R(\tau)}{\tau}\to\frac{E(T_1+U_1)}{ET_1}<\infty$$

with probability one.

From this result together with (9) follows

$$|\sqrt{\tau}(\frac{N(\tau)}{R(\tau)}-\lambda)-\sqrt{\tau}(\frac{M(\tau)}{R(\tau)}-\lambda)|\leqq\sqrt{\tau}\frac{N'}{R(\tau)}\overset{P}{\to}0.$$

Hence by Slutskys theorem $(\sqrt{\tau}(\frac{M(\tau)}{\tau}-\mu),\sqrt{\tau}(\frac{N(\tau)}{R(\tau)}-\lambda))$ has the same limiting distribution as $(\sqrt{\tau}(\frac{M(\tau)}{\tau}-\mu),\ \sqrt{\tau}(\frac{M(\tau)}{R(\tau)}-\lambda))$ if any.

We need the following

Lemma 3. Let T be the length of a busy period and let M be the number of customers served in T. We have

$$ET = \frac{1}{\mu-\lambda} \qquad EM = \frac{\mu}{\mu-\lambda}$$

$$VT = \frac{\mu^3\sigma^2\lambda}{(\mu-\lambda)^3} \qquad VM = \frac{\mu^3\lambda^2\sigma^2+\lambda\mu^2}{(\mu-\lambda)^3}$$

$$ETM = \frac{\lambda\mu^3\sigma^2+\mu^2}{(\mu-\lambda)^3}$$

Proof: Let $\psi(\theta) = Ee^{-\theta V}$, where V is the service time.

We have

$$-\psi'(0) = \frac{1}{\mu}, \tag{10}$$

$$\psi''(0)-(\psi'(0))^2 = \sigma^2. \tag{11}$$

Define $\beta(z,\theta) = Ez^M e^{-\theta T}$.

From [6] p. 153 and p. 158 follows, that β fulfils the functional equation

$$\beta = z\psi(\theta+\lambda-\lambda\beta) \tag{12}$$

Differentiation with respect to z and θ gives for $z=1$, $\theta=0$

$$EM = \frac{\psi(0)}{1+\lambda\psi'(0)} \tag{13}$$

$$ET = -\frac{\psi'(0)}{1+\lambda\psi'(0)} \tag{14}$$

$$EM(M-1) = \frac{1}{1+\lambda\psi'(0)}\lambda EM(\psi''(0)\lambda EM - 2\psi'(0)) \tag{15}$$

$$ET^2 = \frac{1}{1+\lambda\psi'(0)} \; (\psi''(0)(1+\lambda ET)^2 \tag{16}$$

$$ETM = - \frac{1}{1+\lambda\psi'(0)} \; (1+\lambda ET)(\psi'(0)-\psi''(0)\lambda EM) \tag{17}$$

Substitution of the values from (10) and (11) in (13)-(17) completes the proof.

Next we note that $M(\tau)$ is also a cumulative process relative to $T_1, T_2, \ldots$ and we have

Lemma 4. For $\tau \to \infty$

$$\left(\sqrt{\tau}\left(\frac{M(\tau)}{\tau} - \mu\right), \sqrt{\tau}\left(\frac{R(\tau)}{\tau} - \frac{\mu}{\lambda}\right)\right) \Rightarrow N(0,\Gamma),$$

$$\text{where} \qquad \Gamma = \begin{pmatrix} \mu^3\sigma^2 & \frac{\mu^3\sigma^2}{\lambda} \\ \frac{\mu^3\sigma^2}{\lambda} & \frac{\mu^3\sigma^2+\mu}{\lambda^2} \end{pmatrix}$$

Proof: The result follows from [7] theorem 10, and lemma 3, since

$$\frac{E\Delta_n M(\tau)}{ET} = \frac{EM}{ET} = \mu$$

$$\frac{E\Delta_n R(\tau)}{ET} = \frac{ET+EU}{ET} = \frac{\mu}{\lambda}$$

$$V\Delta_n\left(\frac{M(\tau)}{\tau} - \mu\right) = V(M-\mu T) = \frac{\mu^3\sigma^2}{\mu-\lambda}$$

$$V\Delta_n\left(\frac{R(\tau)}{\tau} - \frac{\mu}{\lambda}\right) = V\left(U+\frac{\lambda-\mu}{\lambda}T\right) = \frac{\mu^3\sigma^2+\mu}{\lambda^2(\mu-\lambda)}$$

$$V\left(\Delta_n\left(\frac{M(\tau)}{\tau} - \mu\right), \Delta_n\left(\frac{R(\tau)}{\tau} - \frac{\mu}{\lambda}\right)\right)$$

$$= V\left(M-\mu T, U+\frac{\lambda-\mu}{\lambda}T\right) = \frac{\mu^3\sigma^2}{\lambda(\mu-\lambda)}$$

To find the limiting distribution of $(\sqrt{\tau}(\frac{M(\tau)}{\tau}-\mu),\ \sqrt{\tau}(\frac{M(\tau)}{R(\tau)}-\lambda))$ we write

$$\sqrt{\tau}(\frac{M(\tau)}{R(\tau)}-\lambda) = \frac{\tau}{R(\tau)}(\sqrt{\tau}(\frac{M(\tau)}{\tau}-\mu)-\lambda\sqrt{\tau}(\frac{R(\tau)}{\tau}-\frac{\mu}{\lambda})) .$$

By lemma 4

$$(\sqrt{\tau}(\frac{M(\tau)}{\tau}-\mu), \frac{\lambda}{\mu}(\sqrt{\tau}(\frac{M(\tau)}{\tau}-\mu)-\lambda\sqrt{\tau}(\frac{R(\tau)}{\tau}-\frac{\mu}{\lambda}))$$
$$\Rightarrow N(0, \begin{pmatrix} \mu^3\sigma^2 & 0 \\ 0 & \frac{\lambda^2}{\mu} \end{pmatrix}) \text{ for } \tau \to \infty$$

Moreover $\frac{\tau}{R(\tau)} \xrightarrow{P} \frac{\lambda}{\mu}$ for $\tau \to \infty$. Hence by Slutskys theorem

$$(\sqrt{\tau}(\frac{M(\tau)}{\tau}-\mu), \sqrt{\tau}(\frac{N(\tau)}{R(\tau)}-\lambda)) \Rightarrow N(0, \begin{pmatrix} \mu^3\sigma^2 & 0 \\ 0 & \frac{\lambda^2}{\mu} \end{pmatrix}) \text{ for } \tau \to \infty.$$

This completes the proof.

For the traffic intensity $\rho = \frac{\lambda}{\mu}$ we have

Corollary 2: $\hat{\rho} = \frac{\hat{\lambda}}{\hat{\mu}} = \frac{N(\tau)}{R(\tau)} \cdot \frac{\tau}{M(\tau)}$ and

$$\sqrt{\tau}\ (\hat{\rho} - \frac{\lambda}{\mu}) \Rightarrow N(0, \frac{\lambda^2}{\mu^2} + \lambda^2\sigma^2) \text{ for } \tau \to \infty.$$

Proof: We have

$$\sqrt{\tau}(\frac{N(\tau)}{R(\tau)} \cdot \frac{\tau}{M(\tau)} - \frac{\lambda}{\mu}) = \sqrt{\tau}\ \frac{\tau}{M(\tau)}((\frac{N(\tau)}{R(\tau)}-\lambda) - \frac{\lambda}{\mu}(\frac{M(\tau)}{\tau}-\mu)).$$

Since $\frac{\tau}{M(\tau)} \xrightarrow{P} \frac{1}{\mu}$, the result follows from theorem 2.

We have restricted ourselves to the case $\lambda<\mu$. It is therefore natural to require $\hat{\lambda}<\hat{\mu}$. That this is the case asymptotically follows from corollary 2, since $\frac{\hat{\lambda}}{\hat{\mu}} \xrightarrow{P} \frac{\lambda}{\mu}<1$.

This result enables us to find asymptotic distributions for various quantities connected with the steady state, since most of them only depends on ρ, se e.g. Lilliefors [5].

5. Hypothesis testing.

In this section we specialize to the case M/M/1, i.e. we assume exponential service times with departure intensity μ.

In this case we have

$$L_\tau(\lambda,\mu) = \lambda^{N(\tau)} e^{-R(\tau)} \mu^{M(\tau)} e^{-\mu\tau} h(U,V)$$

$$\text{and} \quad \hat{\lambda}_\tau = \frac{N(\tau)}{R(\tau)}, \quad \hat{\mu}_\tau = \frac{M(\tau)}{\tau} \qquad (18)$$

We want to test the hypothesis

$$H_0: \ \rho = \rho_0 < 1$$

Under H_0 we have

$$L_\tau(\rho_0\mu,\mu) = (\rho_0\mu)^{N(\tau)} e^{-\rho_0\mu R(\tau)} \mu^{M(\tau)} e^{-\mu\tau}$$

$$\text{and} \quad \hat{\hat{\mu}}_\tau = \frac{N(\tau)+M(\tau)}{\tau+\rho_0 R(\tau)} \ . \qquad (19)$$

The distribution of $\hat{\hat{\mu}}$ under H_0 is given by

Theorem 3. For $\tau \to \infty$

$$\sqrt{\tau}\left(\frac{N(\tau)+M(\tau)}{\tau+\rho_0 R(\tau)} - \mu\right) \Rightarrow N(0,\tfrac{\mu}{2}) \ .$$

Proof: We have

$$\sqrt{\tau}\left(\frac{N(\tau)+M(\tau)}{\tau+\rho_0 R(\tau)} - \mu\right) = \frac{R(\tau)}{\tau+\rho_0 R(\tau)} \sqrt{\tau}\left(\frac{N(\tau)}{R(\tau)} - \rho_0\mu\right) +$$

$$\frac{\tau}{\tau+\rho_0 R(\tau)} \sqrt{\tau}\left(\frac{M(\tau)}{\tau} - \mu\right) .$$

Now
$$\frac{R(\tau)}{\tau+\rho_0 R(\tau)} \xrightarrow{P} \frac{1}{2\rho_0} \quad \text{for } \tau \to \infty$$

$$\frac{\tau}{\tau+\rho_0 R(\tau)} \xrightarrow{P} \frac{1}{2} \quad \text{for } \tau \to \infty,$$

and by theorem 2

$$\frac{1}{2\rho_0}\sqrt{\tau}\left(\frac{N(\tau)}{R(\tau)} - \rho_0\mu\right)+\frac{1}{2}\sqrt{\tau}\left(\frac{M(\tau)}{\tau} -\mu\right) \Rightarrow N\left(0, \frac{\mu}{2}\right) \text{ for } \tau \to \infty$$

An application of Slutskys theorem completes the proof.

Note that (18) and (19) are valid also for $\rho \geqq 1$ (and $\rho_0 \geqq 1$).

The likelihood ratio test for H_0 is

$$Q_\tau = \frac{L_\tau(\rho_0\hat{\hat{\mu}}_\tau, \hat{\hat{\mu}}_\tau)}{L_\tau(\hat{\lambda}_\tau, \hat{\mu}_\tau)},$$

or

$$- 2 \log Q = -2\left(N(\tau)\log\left(1+\frac{\rho_0 R(\tau)M(\tau)-N(\tau)\tau}{N(\tau)(\tau+\rho_0 R(\tau))}\right)\right.$$
$$\left.+M(\tau)\log\left(1+\frac{\tau N(\tau)-\rho_0 R(\tau)M(\tau)}{M(\tau)(\tau+\rho_0 R(\tau)}\right)\right)$$

By a Taylor expansion we obtain

$$- 2 \log Q = \frac{N(\tau)+M(\tau)}{\tau+\rho_0 R(\tau)} \frac{\tau\left(\rho_0\frac{M(\tau)}{\tau} - \frac{N(\tau)}{R(\tau)}\right)^2}{\frac{N(\tau)}{R(\tau)} \cdot \frac{M(\tau)}{\tau} \cdot \frac{\tau+\rho_0 R(\tau)}{R(\tau)}} + S(\tau).$$

Since

$$\frac{N(\tau)+M(\tau)}{\tau+\rho_0 R(\tau)} \cdot \left(\frac{N(\tau)}{R(\tau)} \cdot \frac{M(\tau)}{\tau} \cdot \frac{\tau+\rho_0 R(\tau)}{R(\tau)}\right)^{-1} \xrightarrow{P} \frac{1}{2\mu\rho_0^{\,2}}, \text{ for } \tau \to \infty$$

and

$$\sqrt{\tau}\left(\rho_0\frac{M(\tau)}{\tau} - \frac{N(\tau)}{R(\tau)}\right) \Rightarrow N(0,2\rho_0^{\,2}\mu)$$

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we have

$$- 2 \log Q \Rightarrow \chi^2 \text{ with } f=1 \quad \text{for } \tau \to \infty$$

if we can prove

$$|S(\tau)| \xrightarrow{P} 0 \quad \text{for } \tau \to \infty$$

Let

$$Y(t) = \frac{\tau N(\tau)-\rho_0 R(\tau) M(\tau)}{\tau+\rho_0 R(\tau)}$$

Be theorem 2 we have

$$\frac{1}{\sqrt{\tau}} Y(\tau) \Rightarrow N(0,\tfrac{\mu}{2}) \ , \tag{20}$$

hence

$$\frac{Y(\tau)}{M(\tau)} \xrightarrow{P} 0, \ \frac{Y(\tau)}{N(\tau)} \xrightarrow{P} 0 \quad \text{for } \tau \to \infty \tag{21}$$

We have

$$|S(\tau)| \leqq \frac{1}{2}|g''(\theta_{1\tau})-g''(\tfrac{Y(\tau)}{N(\tau)})| N(\tau) (\tfrac{Y(\tau)}{N(\tau)})^2$$

$$+\frac{1}{2}|g''(\theta_{2\tau})-g''(\tfrac{Y(\tau)}{M(\tau)})| M(\tau) (\tfrac{Y(\tau)}{M(\tau)})^2 \ ,$$

where

$$\begin{aligned} &0<\theta_{1\tau}<\frac{Y(\tau)}{N(\tau)} \\ &0<\theta_{2\tau}<\frac{Y(\tau)}{M(\tau)} \end{aligned} \tag{22}$$

and

$$g''(x) = \frac{-1}{1+x} \ .$$

(20) and (21) imply for $\tau \to \infty$

$$\frac{1}{2}|g''(\theta_{1\tau})-g''(\tfrac{Y(\tau)}{N(\tau)})| \xrightarrow{P} 0$$

$$\frac{1}{2}|g''(\theta_{2\tau})-g''(\tfrac{Y(\tau)}{M(\tau)})| \xrightarrow{P} 0 \ .$$

Moreover by (20) we have for $\tau \to \infty$

$$N(\tau)\left(\frac{Y(\tau)}{N(\tau)}\right)^2 \Rightarrow \chi^2, f=1$$

$$M(\tau)\left(\frac{Y(\tau)}{M(\tau)}\right)^2 \Rightarrow \chi^2, f=1.$$

Hence $|S(\tau)| \xrightarrow{P} 0.$

This completes the proof.

Manuskriptark. A4 = A5

References

[1] Clarke, A.B. (1957). "Maximum likelihood estimates in a simple queue",Ann. Math. Stat. Vol. 28, pp. 1036-1040.

[2] Cox, D.R. (1965). "Some problems of statistical analysis connected with congestion", Proceedings of the symposium on congestion theory, North Carolina.

[3] Keiding, N. and Lauritzen, S.L. (1977). "Maximum likelihood estimation of the offspring mean in a simple branching process".Preprint No. 4, Inst. Math. Statist., Univ. Copenhagen.

[4] Kendall, D.G. (1953). "Stochastic processes occuring in the theory of queues and their analysis by the method of the imbedded Markov chain." Ann. Math. Stat. Vol. 24, pp. 338-354.

[5] Lilliefors, H.W. (1966). "Some confidence intervals for queues", Op. Res. Vol. 14, pp. 723-727.

[6] Prabhu, N.U. (1965). "Queues and Inventeries", Wiley, New York.

[7] Smith, W.L. (1955). "Regenerative stochastic processes", Proc. Roy. Soc. A,Vol. 232, pp. 6-31.

[8] Smith, W.L. (1958). "Renewal Theory and its Ramifications", Jour. Roy. Stat. Soc. B, Vol. 20, pp. 243-284.

[9] Syski, R. (1965). "Markovian Queues", Proceedings of the symposium on congestion theory, North Carolina.

[10] Wolff, R.W. (1965). "Problems of statistical inference for birth and death queuing models", Op. Res. Vol. 13, pp. 343-357.

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