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The limit theorems for general supercritical Markov branching processes presented in [1,2] are incomplete insofar, as the limit degenerates, unless an additional moment condition ("xlogx") is satisfied. Also, they provide no further information on the limit distribution except possibly existing moments. It is the purpose of this paper to overcome these deficiencies. To avoid technical conditions, we work in the setting of multigroup branching diffusions (Sec.1) already adopted in [16]. The reader will not find it difficult to extract a more abstract formulation in the style of [15]. It is not always pointed out, but the treatment covers also the case of a finite set of types, including to some extent non-embeddable discrete time processes.

Normalizing constants leading to a non-degenerate, finite limit without "xlogx" were first given for Bienaymé-Galton-Watson (BGW) processes by Seneta [28]. His convergence result was later strengthened by Heyde [17], who discovered the relevant martingale. For n-type processes proper normalizing constants were constructed by Hoppe [18,19]. Fusing elements of his approach with the machinery of [15,16] and a sufficiently sharp transience result (Sec.2), we obtain a solution of the normalizing problem in general (Sec.3).

The investigation of the limit distribution function itself has a longer history. Already the paper by Harris [12] on BGW processes with finite second moments gives proof of the existence of a continuous density on the positive reals, assuming the distribution is not concentrated at one point. It also contains some information on the behaviour near zero and infinity. The conditions for the existence of a continuous density were gradually relaxed by Levinson [25], Stigum [32], and Dubuc [6-9], who has continuity and positivity of the density on the positive reals without assumptions beyond first moments. A different existence proof for the density has been given by Athreya [3]. Another positivity proof can be found in Athreya and Ney [4]. For n-type processes, whose limit distribution is not concentrated at one point, Kesten and Stigum [23] have shown existence and continuity of the density, assuming "xlogx". Hoppe [18] has existence without "xlogx". We obtain existence and positivity in general (Sec.5). Continuity is guaranteed under an additional assumption.

The behaviour of the limit distribution near zero has been studied for BGW processes by Dubuc [6-9], See also Karlin and McGregor [22], Athreya and Ney [4]. There seem to be no results on the multitype case in the literature. In the continuous time setting some of the problems encountered by Dubuc do not occur. On the other hand we have to go through additional analytical preparations, due to the non-trivial set of types. Our statement arises as a simple consequence of a result on the rate at which the generating functional converges to the extinction probability ("Q-limit", Sec.4). No serious new problem occurs in connection with the behaviour near infinity. Here the argument of Seneta [30,31] goes through in general.

Finally we turn to theproof of strong convergence for arbitrary non-negative averaging functionals. It was already indicated in [16] that it does not suffice to simply replace the geometric normalization in the proofs of [1] with a generalized normalizing function. Instead we look at the Kurtz ratio of the process, working with a random cut-off and a different type of estimate for the remainder term (Sec.6). We conclude with a strong convergence result for processes with immigration (Sec.7). For BGW processes with stationary immigration strong convergence with generalized normalization was obtained by Seneta [29]. The necessity of his logarithmic moment condition on the immigration was shown by Cohn [5]. Hoppe [18] has convergence in probability for the n-type case. He also considers Markovian immigration schemes. We proceed in the spirit of [2], admitting not only a general set of types, but also a completely general immigration process. As in [2] the theory becomes strikingly simple when restricted to processes with a finite set of types.

1. Multigroup branching diffusions

Let Ω be the union of K connected open sets Ω_ν, $\nu = 1,\ldots,K$, in an N-dimensional, orientable manifold of class C^∞, let the closures $\overline{\Omega}_\nu$ be compact and pairwise disjoint, and let the boundary $\partial\Omega$ consist of a finite number of simply connected (N-1)-dimensional hypersurfaces of class C^3. Let X be the union of K Borel sets X_ν such that

$$\Omega_\nu \subset X_\nu \subset \overline{\Omega}_\nu, \quad \nu = 1,\ldots,K ,$$

in a way to be determined, and suppose to be given a uniformly elliptic differential operator $\underset{\sim}{A}|D(\underset{\sim}{A})$, represented in local coordinates on X by

$$\underset{\sim}{A}: = \sum_{i,j=1}^{N} \frac{1}{\sqrt{a(x)}} \frac{\partial}{\partial x^i} a^{ij}(x)\sqrt{a(x)} \frac{\partial}{\partial x^j} + \sum_{i=1}^{N} b^i(x) \frac{\partial}{\partial x^i}$$

$$D(\underset{\sim}{A}): = \{u|_X : u \in C^2(\overline{\Omega}) \wedge (\alpha u + \beta\frac{\partial u}{\partial n})|_{\partial\Omega} = 0\} ,$$

where (a^{ij}) and (b^i) are the restrictions to X of a symmetric, second-order, contravariant tensor of class $C^{2,\lambda}(\overline{\Omega})$ and a first-order, contravariant tensor of class $C^{1,\lambda}(\overline{\Omega})$,

$$a: = \det(a^{ij})^{-1} ,$$
$$0 \leq \alpha,\beta \in C^{2,\lambda}(\partial\Omega), \quad \alpha + \beta \equiv 1 ,$$
$$\overline{\Omega} \setminus X: = \{\beta = 0\} .$$

By $\frac{\partial}{\partial n}$ we denote the exterior normal derivative according to (a^{ij}) at $\partial\Omega$.

Define B as the Banach algebra of all complex-valued, bounded, Borel-measurable functions on X with supremum-norm $\|\cdot\|$, B_+ as the cone of all non-negative functions in B, further

$$C^\ell: = \{u|_X : u \in C^\ell(\overline{\Omega})\} ,$$
$$C_0^\ell: = \{u|_X : u \in C^\ell(\overline{\Omega}) \wedge u|_{\overline{\Omega}\setminus X} \equiv 0\} .$$

the closure of $\underset{\sim}{A}|\{\xi \in D(\underset{\sim}{A}): \underset{\sim}{A}\xi \in C_0^0\}$ in B is the C_0^0-generator of a contraction semigroup $\{T_t\}_{t\in\mathbb{R}_+}$ in B. This semigroup is non-negative respective B_+, stochastically continuous in $t \geq 0$ on B, and strongly continuous in $t \geq 0$ on C_0^0, with $T_t B \subseteq C_0^2$ for $t > 0$. It can be represented in the form

$$T_t\xi(x) = \int_X p_t(x,y)\xi(y)dy,$$

where $p_t(x,y)$ is the fundamental solution of $\partial p_t/\partial t = \underset{\sim}{A}p_t$. That is, $p_t(x,y)$ is given as a continuous function on $\{t>0\}\otimes\overline{\Omega}\otimes\overline{\Omega}$, continuously differentiable in x and y for $t>0$, such that

(T.1)
$$p_t(x,y) > 0, \quad (x,y) \in X_\nu \otimes X_\nu, \quad \nu = 1,\ldots,K,$$
$$p_t(x,y) \equiv 0, \quad (x,y) \in X_\nu \otimes X_\mu, \quad \nu \neq \mu,$$

(T.2)
$$p_t(x,\cdot) = p_t(\cdot,x) \equiv 0, \quad x \in \overline{\Omega}\setminus X,$$

(T.3)
$$\frac{\partial p_t}{\partial n_x}(x,y) < 0, \quad (x,y) \in (\overline{\Omega}_\nu \setminus X_\nu)\otimes X_\nu,$$
$$\frac{\partial p_t}{\partial n_y}(x,y) < 0, \quad (x,y) \in X_\nu \otimes (\overline{\Omega}_\nu \setminus X_\nu), \quad \nu = 1,\ldots,K,$$

and for $0 < t \leq t_0$, t_0 arbitrary but fixed,

(T.4)
$$\sup_{x,y\in X}\{|\frac{\partial p_t}{\partial x^i}(x,y)| + |\frac{\partial p_t}{\partial y^i}(x,y)|\} = O(t^{-(N+1)/2}), \quad i = 1,\ldots,N,$$

(T.5)
$$\sup_{x\in X}\int_X\{|\frac{\partial p_t}{\partial x^i}(x,y)| + |\frac{\partial p_t}{\partial x^i}(y,x)|\}dy = O(t^{-1/2}), \quad i = 1,\ldots,N,$$

cf. [21], [26].

The semigroup $\{T_t\}$ determines a conservative, continuous, strong Markov process $\{x_t, P^x\}$ on $X \cup \{\partial\}$, where ∂ is a trap. Now suppose to be given a $k \in B_+$, and define $k_0(x) := k(x)$ for $x \in X$, $k_0(\partial) := 0$, and

$$\eta_t: = \exp\{-\int_0^t k_0(x_s)ds\} .$$

Let $\{x_t^0, P_0^x\}$ be the η_t - subprocess of $\{x_t, P^x\}$, defined as a conservative process on $X \cup \{\partial\} \cup \{\Delta\}$, where Δ is a trap corresponding to the stopping by η_t. For $\xi \in B$ define $\xi_0(x): = \xi(x)$, if $x \in X$, and $\xi_0(\partial): = \xi_0(\Delta): = 0$. Then

$$T_t^0\xi(x): = E_0^x\xi_0(x_t^0), \quad x \in X, \quad t \geq 0 ,$$

defines a non-negative contraction semigroup $\{T_t^0\}_{t\in\mathbb{R}_+}$ on B. It is the unique solution of

$$T_t^0 = T_t - \int_0^t T_s k T_{t-s}^0 ds, \quad t \geq 0 , \tag{1.1}$$

and it is stochastically continuous in $t \geq 0$ on B and strongly continuous in $t \geq 0$ on C_0^0, with $T_t^0 B \subseteq C_0^1$ for $t \geq 0$.

Let $X^{(n)}$, $n \geq 1$, be the symmetrization of the direct product of n disjoint copies of X, $X^{(0)}: = \{\theta\}$ with some extra point θ. Define

$$\hat{X}: = \bigcup_{n=0}^{\infty} X^{(n)} ,$$

and let $\hat{\underline{A}}$ be the σ-algebra on $\hat{X}$ induced by the Borel algebra on X. Define

$$\hat{x}[\xi]: = 0, \qquad \hat{x} = \theta ,$$

$$: = \sum_{\nu=1}^{n} \xi(x_\nu), \quad \hat{x} = \langle x_1, \ldots, x_n \rangle \in X^{(n)}, \quad n > 0$$

for every finite-valued Borel-measurable ξ on X. Suppose to be given a stochastic kernel $\pi | X \otimes \hat{\underline{A}}$ such that

$$m\xi(x): = \int_{\hat{X}} \hat{x}[\xi]\pi(x, d\hat{x}), \quad \xi \in B, \quad x \in X,$$

defines a bounded operator m on B.

The pair (x_t^0, π) determines our (multigroup) branching diffusion, a conservative, right-continuous strong Markov process $\{\hat{x}_t, P^{\hat{x}}\}$ on $(\hat{X}, \underline{\hat{A}})$, constructed according to the following intuitive rules: All particles at a time move independently of each other, each according to $\{x_t^0, P_0^x\}$. A particle hitting ∂ disappears, a particle hitting Δ is replaced by a population of new particles according to $\pi(x_{t_\Delta -}, \cdot)$, where $x_{t_\Delta -}$ is the left limit of the path at the hitting time of Δ, cf. [20], [27].

In terms of the generating functional

$$F_t(\hat{x}, \eta) := E^{\hat{x}} \hat{\eta}(\hat{x}_t) ,$$

$$\hat{\eta}(\hat{x}) := 1, \qquad \hat{x} = \theta ,$$

$$:= \prod_{\nu=1}^{n} \hat{\eta}(x_\nu), \qquad \hat{x} = \langle x_1, \ldots, x_n \rangle ,$$

$$t \geq 0, \qquad \hat{x} \in \hat{X}, \qquad \eta \in \overline{S} := \{\xi \in B: \|\xi\| \leq 1\},$$

the assumption of independent motion and branching takes the form

$$\text{(F.1)} \qquad F_t(\hat{x}, \eta) = 1, \qquad \hat{x} = \theta ,$$

$$= \prod_{\nu=1}^{n} F_t(\langle x_\nu \rangle, \eta), \quad \hat{x} = \langle x_1, \ldots, x_n \rangle, \quad n > 0 .$$

Defining $F_t : \overline{S} \to \overline{S}$ by $F_t[\cdot](x) := F_t(\langle x \rangle, \cdot)$, $x \in X$, (F.1) combined with the Chapman-Kolmogorov equation yields

$$\text{(F.2)} \qquad F_{t+s}[\cdot] = F_t[F_s[\cdot]], \qquad t, s \geq 0 .$$

For every $t > 0$ define $\hat{x}_{t-}$ on $\hat{Y}$ with $Y := X \cup \{\partial\}$, and let $\underline{A}_0$ be the set of open spheres intersected with X. Define

$$\tau := \inf\{t > 0: \exists\, U \in \underline{A}_0: \hat{x}_{t-}[1_U] \neq \hat{x}_t[1_U]\} .$$

It follows from the strong Markov property of $\{\hat{x}_t, P^{\hat{x}}\}$ that for every $\eta \in \bar{S}$ the function $F_t[\eta](x)$, $t > 0$, $x \in X$, solves

$$u_t(x) = E^{\langle x \rangle} \hat{\eta}(\hat{x}_t) 1_{\{t < \tau\}} + E^{\langle x \rangle}(E^{\hat{x}_\tau} \hat{\eta}(\hat{x}_{t-s}))|_{s=\tau} 1_{\{\tau \leq s\}}$$

$$= T_t^0 \, \eta(x) + P_0^x(x_\tau^0 = \partial, \ \tau \leq t)$$

$$+ \int_{0}^{t}\int_{X} P_0^x(x_\tau^0 = \Delta, \ x_{\tau-}^0 \in dy, \ \tau \in ds) \int_X \pi(y, d\hat{x}) F_{t-s}(\hat{x}, \eta)$$

$$= T_t^0 \eta(x) + H_t(x) + \int_0^t T_s^0 \{k f[u_{t-s}]\}(x) ds,$$

$$H_t(x) := 1 - T_t^0(x) - \int_0^t T_s^0 k(x) ds ,$$

$$f[\eta](x) := \int_X \pi(x, d\hat{x}) \hat{\eta}(\hat{x}).$$

The uniqueness of the solution is easily verified by means of

$$\| f[\eta] - f[\xi] \| \leq \|m\| \ \|\eta - \xi\| .$$

We shall use the equation in the more convenient form of

$$\text{(IF)} \qquad 1 - F_t[\eta](x) = T_t^0(1-\eta)(x) + \int_0^t T_s^0\{k(1 - f[F_{t-s}[\eta]])\}(x) ds .$$

The assumptions guarantee that for every $t \geq 0$

$$M_t \xi(x) := E^{\langle x \rangle} \hat{x}_t[\xi], \quad \xi \in B, \ x \in X,$$

defines a non-negative, linear-bounded operator M_t on B. It follows from (F.1) that

$$\text{(1.2)} \qquad E^{\hat{x}} \xi(x_t) = \hat{x}[M_t \xi], \ \hat{x} \in \hat{X}, \ \xi \in B, \ t \geq 0,$$

and from (F.2) that $\{M_t\}_{t \in \mathbb{R}_+}$ is a semigroup: Simply set $\eta = \zeta + \lambda\xi$, differentiate with respect to λ at $\lambda = 0$ and let $\zeta \to 1$, using dominated convergence. Similarly, (IF) implies that for every $\xi \in B$

$$\text{(IM)} \qquad M_t\xi(x) = T_t^0\xi(x) + \int_0^t T_s^0\{kmM_{t-s}\xi\}(x)ds \ .$$

Again, the solution is unique.

We assume that the K×K-matrix with elements

$$m_{\nu\mu} := \int_{X_\nu} k(x)m1_{X_\mu}(x)dx, \quad \nu,\mu = 1,\ldots,K.$$

is irreducible. This entails primitivity of the moment semigroup $\{M_t\}$. To obtain a satisfactory limit theory we need a positivity result for $\{M_t\}$ which is stronger than what can be inferred from the Kreĭn-Rutman theorem. To this end we assume that m has a bounded extension to L^2 such that

$$\sup_{\xi\in B:\,||\xi||_1=1} ||km^*\xi||_\infty < \infty$$

or, if the X_ν are congruent,

$$km\xi(x) = \sum_{\nu=1}^{K} \mu_\nu(x)\xi(\varkappa_\nu x) + m_0\xi(x) \ ,$$

$$m_0 \geq 0 \quad [B_+],$$

$$\sup_{\xi\in B:\,||\xi||_1=1} ||m_0^*\xi||_\infty < \infty \ ,$$

where m^* and m_0^* are the adjoints of m and m_0, $\mu_\nu \in B_+$, $||\cdot||_p$ denotes the norm in L^p, $1 \leq p \leq \infty$, and $\varkappa_\nu x$ is the picture of x produced in X_ν by the given congruence.

A simple example for the first kind of branching law is the following model: A branching event at x results with probability $p_{n_1..n_K}(x)$ in $n_1+..+n_K$ new particles, n_ν of them in X_ν, $\nu=1,..,K$. The places of birth are distributed independently, a location in X_ν with the distribution density $f_\nu(x,\cdot)$, $\nu = 1,\ldots,K$. That is

$$m\xi(x) = \int_X m(x,y)\xi(y)\,dy ,$$

$$m(x,y) = \sum_{\nu=1}^{K} 1_{X_\nu}(y) f_\nu(x,y) \sum_{n_1,\dots,n_K \geq 0} n_\nu p_{n_1,\dots,n_K}(x) .$$

The idea behind the second type of branching law, $m_0 = 0$, is this: There are K different kinds of particles moving on the same physical domain. To the kind ν we assign X_ν as abstract domain of diffusion, $\nu = 1,..,K$. In the physical domain new particles are always born at the termination point (left limit) of their immediate ancestor. That is,

$$\pi(x,\hat{A}) = p_{0..0}(x) 1_{\hat{A}}(\theta) + \sum_{\substack{n_1\geq 0,..,n_K\geq 0 \\ n_1+..+n_K>0}} p_{n_1..n_K}(x)$$
$$\times 1_{\hat{A}}(\overbrace{\varkappa_1 x,..,\varkappa_1 x}^{n_1},...,\overbrace{\varkappa_K x,..,\varkappa_K x}^{n_K}), \quad x \in X, \quad \hat{A} \in \underline{\hat{A}} ,$$

where $1_{\hat{A}}$ is the indicator function of $\hat{A}$, $\{p_{n_1..n_K}(x)\}$ a probability distribution on $\mathbb{Z}_+^K$ for every $x \in X$. Here

$$m\xi(x) = \sum_{\nu=1}^{K} m_\nu(x)\xi(\varkappa_\nu x), \quad \xi \in B, \quad x \in X,$$

$$m_\nu := \sum_{n_1\geq 0,..,n_K\geq 0} n_\nu p_{n_1..n_K}, \quad \nu = 1,..,K .$$

Define

$$D_0^+ := \{u|_X : u \in C^1(\overline{\Omega}),\ u>0 \text{ on } X, u=0 \wedge \frac{\partial u}{\partial n} < 0 \text{ on } \overline{\Omega}\setminus X\}.$$

(1.3) PROPOSITION ([14], [16]). <u>The moment semigroup</u> $\{M_t\}_{t\geq 0}$ <u>is stochastically continuous in</u> $t \geq 0$ <u>on</u> B, <u>strongly continuous in</u> $t \geq 0$ <u>on</u> C_0^0 <u>with</u> $M_t B \subseteq C_0^1$ <u>for</u> $t > 0$. <u>It can be represented in the form</u>

(M) $$M_t = \rho^t \varphi\Phi^* + \Delta_t, \quad t > 0 ,$$

$$\Phi^*[\xi] = \int_X \varphi^*(x)\xi(x)dx, \quad \xi \in B,$$

<u>where</u> $0 < \rho \in \mathbb{R}_+$, $\varphi \in D_0^+$, $\varphi^* \in D_0^+$, $\Phi^*[\varphi] = 1$, <u>and</u> $\Delta_t: B \to B$ <u>such that for all</u> $t > 0$

$$\varphi\Phi^*\Delta_t = \Delta_t\varphi\Phi^* = 0 ,$$

$$-\alpha_t\varphi\Phi^* \leq \Delta_t \leq \alpha_t\varphi\Phi^* \qquad [B_+],$$

<u>with</u> $\alpha.: \mathbb{R}_+ \to \mathbb{R}_+$ <u>satisfying</u>

$$\rho^{-t}\alpha_t \downarrow 0 \quad \text{as} \quad t \uparrow \infty .$$

By first-order Taylor expansion

(FM) $$1-F_t[\xi] = M_t[1-\xi] - R_t(\xi)[1-\xi], \quad \xi \in \overline{S} ,$$

$$R_t(\eta)\zeta(x) := E^{\langle x\rangle}\omega(\eta,\zeta,\hat{x}_t) ,$$

$$\omega(\eta,\zeta,\hat{x}) := 0, \qquad \hat{x}[1] \leq 1 ,$$

$$:= \sum_{\nu=1}^{n} \zeta(x_\nu)\Big(1 - \int_0^1 \prod_{\mu\neq\nu}[1-\lambda(1-\eta(x_\mu))]d\lambda\Big) ,$$

$$\hat{x} = \langle x_1,\ldots,x_n\rangle, \quad n > 1 .$$

The mapping $R_t(\cdot)[\cdot]:\overline{S}\otimes B \to B$ is sequentially continuous respective the product topology on bounded regions, non-increasing in the first and linear-bounded in the second variable, and it satisfies

(RM) $$0 = R_t(1)\zeta \leq R_t(\eta)\zeta \leq M_t\eta, \quad (\eta,\zeta) \in \overline{S}_+\otimes B_+ ,$$

where $\overline{S}_+ := \overline{S} \cap B_+$.

To obtain a sufficiently sharp estimate for R_t in terms of M_t, we assume that for some and thus every $\xi \in D_0^+$ there exist constants c, c^* such that

(C) $$k m \xi \leq c \xi$$

(C*) $$\int_X \xi(x) k(x) m \eta(x)\, dx \leq c^* \int_X \xi(x)\, \eta(x)\, dx, \qquad \eta \in B_+ .$$

In case of the first type of branching law with

$$m\xi(x) = \int_X m(x,y)\, \xi(y)\, dy, \qquad \xi \in B,$$

it suffices for (C), (C^*), to hold that

$$m\xi(x) = \int_X m(x,y)\, \xi(y)\, dy, \quad \xi \in B, \quad x \in X ,$$

$$m(x,y) \leq \overline{m}(x,y), \quad (x,y) \in X \otimes X ,$$

$$\overline{m} \in C^1(\overline{\Omega} \otimes \overline{\Omega}), \ \overline{m}(\cdot, x) = \overline{m}(x, \cdot) \equiv 0, \quad x \in \overline{\Omega} \setminus X ,$$

$$dy := \sqrt{a(y)}\, dy^1 \ldots dy^N ,$$

where $y^1, \ldots, y^N$ are local coordinates of y. In case of the second type of branching law, $m_0 = 0$, the conditions are always satisfied.

(1.4) PROPOSITION ([15], [16]). <u>For every</u> $t > 0$ <u>there exists a mapping</u> $g_t : \overline{S}_+ \to B_+$ <u>such that</u>

(R)
$$R_t(\xi)[1-\xi] = g_t[\xi] \rho^t \Phi^*[1-\xi] \varphi, \quad \xi \in \overline{S}_+ ,$$
$$\lim_{\|1-\xi\| \to 0} \|g_t[\xi]\| = 0 ,$$

<u>where the convergence is uniform in</u> t <u>on any closed bounded interval</u> $[a,b]$ <u>with</u> $a > 0$.

We assume throughout that $\{\hat{x}_t, P^{\hat{x}}\}$ is supercritical, i.e., that $\rho > 1$. By c_ν, $\nu = 1,2,\ldots$, we denote suitable positive real constants.

2. Extinction probability and transience

Note that $P^{\hat{x}}(\hat{x}_t = \theta) = F_t(\hat{x}, 0)$. It follows from (F.1-2) that $F_t[0]$ is nondecreasing in t. Hence, the limit

$$q(x) := \lim_{t\to\infty} P^{\langle x\rangle}(\hat{x}_t = \theta), \quad x \in X,$$

exists. For the moment fix $t > 0$. By (FM), (M) with $\rho > 1$, and (R) we can find an $\epsilon > 0$ such that $\Phi^*[1-F_t[1-\xi]] > \Phi^*[\xi]$ whenever $\|\xi\| < \epsilon$. Suppose $\Phi^*[1-q] = 0$. Then $\Phi^*[1-F_s[0]] \to 0$, as $s \to \infty$. By (F.2), (FM), (RM), and (M) there must then exist an $s > 0$ such that $\|1-F_s[0]\| < \epsilon$ and consequently $\Phi^*[1-F_{t+s}[0]] > \Phi^*[1-F_s[0]]$. But this contradicts the fact that $F_s[0]$ is non-decreasing. Hence, $q < 1$ on a set of positive measure. From (IF) and $q = F_t[q]$, $t > 0$,

$$(2.1) \qquad 1-q = T_t^0(1-q) + \int_0^t T_s^0\{k(1-f[q])\}ds \ .$$

By (T.1), the boundedness of k, and the irreducibility of $(m_{\nu\mu})$, iteration of this equation yields $q < 1$ on X. Using $T_s^0 B \subseteq C_0^1$, $s > 0$, and (1.1) with (T.3-5) we get

$$1 - q \in D_0^+ \ .$$

A prerequisite of the limit theory we are aiming at is a sufficiently strong transience result. We shall need that $F_t[\xi] \to q$, as $t \to \infty$, for a rather large class of $\xi \in \overline{S}_+$. Particulary if $\sup q = 1$, as is the case if $\{\beta = 0\}$ is non-empty, our task is facilitated by a transformation first used for the Bienaymé-Galton-Watson process by Harris [12]: The functional $\overline{F}_t(x,\cdot)|\overline{S}$ given by

$$\bar{F}_t[\xi]: = \frac{F_t[q+(1-q)\xi]-q}{1-q},$$

$$\bar{F}_t(\hat{x},\xi): = 1; \qquad x = \theta,$$

$$: = \prod_{\nu=1}^{n} \bar{F}_t[\xi](x_\nu); \hat{x} = \langle x_1,\ldots,x_n\rangle ,$$

generates the transition function of a Markov branching process on $(\hat{X},\hat{\underline{A}})$. In fact, from (IF)

$$(\overline{\mathrm{IF}}) \qquad 1-\bar{F}_t[\xi] = \bar{T}_t^0(1-\xi) + \int_0^t \bar{T}_s^0\bar{k}(1-\bar{f}[\bar{F}_{t-s}[\xi]])ds,$$

$$\bar{T}_t^0\xi: = \frac{T_t^0[(1-q)\xi]}{1-q}, \quad \xi \in B,$$

$$\bar{k}: = \frac{1-f[q]}{1-q}\,k ,$$

$$\bar{f}[\xi]: = \frac{f[q+(1-q)\xi]-f[q]}{1-q}, \quad \xi \in \bar{S} .$$

Clearly, $\bar{T}_t^0$ is a non-negative contraction semigroup on B. It is stochastically continuous in $t \geq 0$ on B, and using the continuous differentiability of $p_t(x,y)$ with (T.5) and $1-q \in D_0^+$, we have $\bar{T}_t^0 B \subset C^0$ for $t > 0$. Hence, $\bar{T}_t^0$ has a restriction to C^0 which is strongly continuous in $t \geq 0$, cf. [11]. Expanding

$$1 - f[q] = m(1-q) - r(q)(1-q)$$

in analogy to (FM), it follows from (C) that $\bar{k}$ is bounded. From (2.1)

$$1 = \bar{T}_t^0 1 + \int_0^t \bar{T}_s^0\bar{k}\, ds .$$

That is, the process determined (up to equivalence) by $\overline{T}_t^0$ is the subprocess, corresponding to stopping with density $\overline{k}$, of a conservative process, whose transition semigroup $\{T_t\}$ is simply the (unique) solution of

$$\overline{T}_t = \overline{T}_t^0 + \int_0^t \overline{T}_s^0 \overline{k}\, \overline{T}_{t-s}\, ds \;.$$

REMARK. Assuming $1-q \in C_0^2$, we can formally calculate the differential generator of $\overline{T}_t$ as

$$\underset{\sim}{L}\xi = \frac{\underset{\sim}{A}[(1-q)\xi]}{1-q} - k\xi + k\,\frac{1-f[q]}{1-q}\xi \;.$$

Using

$$0 = \frac{\partial q}{\partial t} = \frac{\partial F_t[q]}{\partial t} = \underset{\sim}{A}F_t[q] + k(f[F_t[q]] - F_t[q])$$

$$= \underset{\sim}{A}q + k(f[q] - q)$$

this becomes

$$\underset{\sim}{L}\xi = \frac{\underset{\sim}{A}[(1-q)\xi]}{1-q} - \frac{\underset{\sim}{A}(1-q)}{1-q} \cdot \xi \;,$$

or explicitly,

$$\underset{\sim}{L} = \underset{\sim}{A} - 2 \sum_{i,j} \frac{a_{ij}}{1-q}\left(\frac{\partial}{\partial x_i} q\right)\frac{\partial}{\partial x_j} \;,$$

$$D(\underset{\sim}{L}) = \{\xi|_X \in C^2(\overline{\Omega}) : (1_{\{\beta=0\}}\xi + 1_{\{\beta>0\}} \frac{\partial \xi}{\partial n})|_{\partial\Omega} = 0\}.$$

That is, recalling $1-q \in D_0^+$, the transformation preserves reflecting barriers, turns elastic barriers into reflecting ones, and makes absorbing barriers inaccessible.

Let $\overline{\pi}$ be the stochastic kernel generated by $\overline{f}$ and $(\hat{X}_t, \overline{P}^{\hat{X}})$ the Markov branching process determined by $(\overline{T}_t^0, \overline{\pi})$. By definition, the extinction probability of $(\hat{X}_t, \overline{P}^{\hat{X}})$ is zero,

$$\overline{F}_t[0] \equiv 0 .$$

Its moment semigroup is given by

$$\overline{M}_t\xi(x) = \overline{E}^{\langle x\rangle}\hat{x}_t[\xi] = \lim_{\epsilon\to 1} \frac{\partial}{\partial\lambda}\overline{F}[\epsilon 1 + \lambda\xi](x)\big|_{\lambda=0}$$

$$= (1-q)^{-1}M_t[(1-q)\xi](x) .$$

Defining

$$\overline{\varphi} := (1-q)^{-1}\varphi, \quad \overline{\varphi}^* := (1-q)\varphi^*,$$

$$\overline{\Delta}_t\xi := (1-q)^{-1}\Delta_t[(1-q)\xi] ,$$

the following statement is an immediate consequence of (M) and $1-q \in D_0^+$:

The semigroup $\{\overline{M}_t\}$ is stochastically continuous on B and strongly continuous on C^0 in $t \geq 0$, and it can be represented in the form

$$(\overline{M}) \qquad \overline{M}_t = \rho^t\overline{\varphi}\,\overline{\Phi}^* + \overline{\Delta}_t , \quad t > 0 ,$$

$$\overline{\Phi}^*[\xi] := \int_X \overline{\varphi}^*(x)\xi(x)dx, \quad \xi \in B ,$$

with $\overline{\varphi}, \overline{\varphi}^* \in C^0$, $\inf \overline{\varphi} > 0$, $\inf \overline{\varphi}^* > 0$, and $\overline{\Delta}_t : B \to B$ such that

$$\overline{\varphi}\,\overline{\Phi}^*\Delta_t = \overline{\Delta}_t\overline{\varphi}\,\overline{\Phi}^* = 0 ,$$

$$-\alpha_t\overline{\varphi}\,\overline{\Phi}^* \leq \overline{\Delta}_t \leq \alpha_t\overline{\varphi}\,\overline{\Phi}^*, \quad t > 0,$$

where ρ and α_t are the same as in (M). Recall that $\rho^{-t}\alpha_t \downarrow 0$, as $t \uparrow \infty$.

Similarly, we can expand

$$1-\overline{F}_t[\xi] = \overline{M}_t[1-\xi] - \overline{R}_t(\xi)[1-\xi],$$

$$\overline{R}_t(\xi)[1-\xi]: = (1-q)^{-1}\overline{R}_t(q+(1-q)\xi)[(1-q)(1-\xi)],$$

and obtain the following analog of (R):

For every $t > 0$ there exists a map $\overline{g}_t : \overline{S}_+ \to B_+$, namely

$$\overline{g}_t[\xi]: = g_t[q+(1-q)\xi]$$

such that

$$(\overline{R}) \qquad \overline{R}_t(\xi)[1-\xi] = \overline{g}_t[\xi]\overline{\Phi}^*[1-\xi]\overline{\varphi},$$

$$\lim_{||1-\xi||\to 0} ||\overline{g}_t[\xi]|| = 0,$$

uniformly in $t \in [a,b]$, $0 < a < b < \infty$.

Thus, we can switch freely between $\{\hat{x}_t, P^{\hat{x}}\}$ and $\{\hat{x}_t, \overline{P}^{\hat{x}}\}$, according to convenience. The advantage of the second process is its monotonicity, which follows from the fact that $\overline{T}_t$ is conservative and $\overline{f}[0] = 0$, i.e., $\overline{\pi}(x,\{\theta\}) \equiv 0$.

(2.2) PROPOSITION. <u>For every</u> $n > 0$ <u>and</u> $\hat{x} \neq \theta$

$$\lim_{t\to\infty} \overline{P}^{\hat{x}}(\hat{x}_t[1] \leq n) = 0.$$

<u>Proof.</u> Irreducibility of $(m_{\nu\mu})$ implies irreducibility of $(\overline{m}_{\nu\mu})$,

$$\overline{m}_{\nu\mu}: = \int_{X_\mu} \overline{k}\,\overline{m}\,1_{X_\nu}.$$

Hence, $\{\overline{k} > 0\} \cap X_\nu$ has positive Lebesgue measure for every ν. Moreover, since $\rho > 1$, there exist a μ and a $\delta > 0$ such that even $\{\bar{k} > 0\} \cap \{x:\bar{\pi}(x,\hat{x}[1] > 1) \geq \delta\} \cap X_\mu$ has positive Lebesgue measure. Define

$$A_\nu: = \{\bar{k} > 0\} \cap \{\bar{\pi}(\cdot,\hat{x}[1] > 1) \geq \delta\} \cap X_\nu, \quad \nu = \mu,$$

$$: = \{\bar{k} > 0\} \cap X_\nu, \qquad \nu \neq \mu,$$

$$\bar{\tau}: = \inf\{t > 0: \hat{x}_{t-}[1] \neq \hat{x}_t[1]\}.$$

Then, using (T.1-5) and $1-q \in D_0^+$,

$$\bar{P}^{\langle x\rangle}\{\bar{\tau} \leq 1, \hat{x}_{\bar{\tau}-}[1_{A_\nu}] = 1\} = \int_0^1 \bar{T}_s^0 \bar{k} 1_{A_\nu}(x)ds$$

$$\geq e^{-\|\bar{k}\|} \int_0^1 (1-q)^{-1} T_s[(1-q)\bar{k}1_{A_\nu}](x) \geq \epsilon_\nu > 0, \quad x \in X_\nu,$$

ϵ_ν independent of x. Hence, by irreducibility and monotonicity, there exists an integer n such that

$$\bar{P}^{\langle x\rangle}(\hat{x}_n[1] > 1) \geq \delta\epsilon^n, \quad x \in X,$$

$$\epsilon: = \min_\nu \epsilon_\nu .$$

From this, by homogeneity,

$$\bar{P}^{\langle x\rangle}(\hat{x}_{nm}[1] = 1) \leq (1-\delta\epsilon^n)^m, \quad m = 1,2,\ldots .$$

Using the branching independence and again the monotonicity, this proves (2.2). □

(2.3) COROLLARY. <u>For all</u> $\xi \in S:=\{\eta \in B: \|\eta\| < 1\}$

$$\lim_{t\to\infty} \|\bar{F}_t[\xi]\| = 0.$$

<u>Proof</u>. Pointwise convergence follows from (2.2) by

$$|\bar{F}_t[\xi](x)| \leq \sum_{n=1}^{\ell} \bar{P}^{\langle x\rangle}(\hat{x}_t[1]=n)\|\xi\|^n + (1-\|\xi\|)^{-1}\|\xi\|^{\ell+1},$$

convergence in norm from pointwise convergence by

$$\| \overline{F}_t[\xi] \| = \| \overline{F}_s[\overline{F}_{t-s}[\xi]] - \overline{F}_s[\overline{F}_{t-s}[0]] \|$$

$$\leq \| \overline{M}_s[|\overline{F}_{t-s}[\xi] - \overline{F}_{t-s}[0]|]\| = \| \overline{M}_s[|\overline{F}_{t-s}[\xi]|] \|$$

and $(\overline{M})$. □

(2.4) COROLLARY. <u>If</u> $\xi = q+(1-q)\zeta$, $\zeta \in S$, <u>then</u>

$$\lim_{t\to\infty} \| F_t[\xi]-q \| = 0.$$

(2.5) PROPOSITION. <u>For</u> $\xi \in \overline{S}_+$ <u>with</u> $\xi < 1$ <u>on a set of positive Lebesgue measure</u>

$$\lim_{t\to\infty} \| F_t[\xi]-q \| = 0 .$$

Proof. We have $F_t[0](x) \to q(x)$ for every x. As in the proof of (2.3) $\| q-F_t[0] \| \to 0$. Now fix ξ as assumed. Clearly,

$$F_t[0] \leq F_t[\xi] \leq F_t[\xi \vee q],$$

so that we may consider $\xi \vee q$ instead of ξ. By (F.2) and (2.4) it suffices to show for some $t > 0$ that $(1-q)^{-1}(1-F_t[\xi \vee q])$ is bounded from below by a positive constant. For $\eta \in \overline{S}_+$ define

$$T_t^{(0)}\eta: = T_t^0\eta, \quad T_t^{(n+1)}\eta: = \int_0^t T_s^0 k(1-f[1-T_{t-s}^{(n)}(1-\eta)])ds .$$

The irreducibility of $(m_{\nu\mu})$ and (T.1) imply the existence of an n such that $\{k(1-f[1-T_s^{(n)}(1-\xi\vee q)])> 0\} \cap X_\nu$ has positive Lebesgue measure for all $s > 0$ and ν. Hence, using (IF) and (T.1-5),

$$\frac{1-F_t[\xi\vee q]}{1-q} \geq \frac{e^{-\| k \|}}{1-q} \int_0^t T_s k(1-f[1-T_{t-s}^{(n)}(1-\xi\vee q)])ds \geq c_1 > 0,$$

which completes the proof. □

3. Proper normalizing functions

As in case of a finite set of types proper normalizing functions can be obtained via solving the backward iterate problem for F_t. We call $(\xi_t)_{t\in\mathbb{R}_+} \subset \overline{S}_+$ a sequence of backward iterates, if

$$\xi_t = F_s[\xi_{t+s}], \quad t,s \geq 0 .$$

Such a sequence is non-trivial, if for some $t \geq 0$ neither $\xi_t = 1$ a.e., nor $\xi_t = q$ a.e.

(3.1) PROPOSITION. There exists a non-trivial sequence of backward iterates.

Proof. Except for the use of Arzela's theorem the argument coincides with the proof given for processes with a finite set of types by Hoppe [18,19].

Step 1. Let $Q := \{\xi \in \overline{S}_+ : \xi \geq q\}$. Since q and 1 are fixed points, $F_t[Q]$ is decreasing in t, by (F.2) and the monotonicity of $F_t[\xi]$ in ξ. The continuity of $F_t[\xi]$ in ξ implies connectedness and compactness of $F_t[Q]$ in the topology of pointwise convergence. Hence, $Q_\infty := \bigcap_{n\in\mathbb{N}} F_n[Q]$ is connected, and as $q, 1 \in Q_\infty$ and $q < 1$, there exists a $\xi_0 \in Q_\infty$ such that $q < \xi_0 < 1$ on a set of positive measure. By definition of Q_∞, there exists for every $n \in \mathbb{N}$ a finite sequence $(\xi_{n,j})_{j=0,1,\ldots,n-1} \subset Q$ such that $\xi_{n,j} = F_1[\xi_{n,j+1}]$.

Step 2. It follows from (IF) and the continuity properties of T_t^0 that the family $\{\xi_{n,j} : j < n, n \in \mathbb{N}\}$ is equicontinuous and thus, by Arzela's theorem, relatively compact in the topology of uniform convergence on $Q \cap C_0^0$. Hence, there exists for every $j \in \mathbb{N}$ a sequence $(\xi_{n_\ell,j})_{\ell\in\mathbb{N}}$, $n_\ell \to \infty$, as $\ell \to \infty$, converging in norm to some $\xi_j \in Q$, and by continuity of $F_1[\xi]$ in ξ, $\xi_j = F_1[\xi_{j+1}]$. Finally, define $\xi_t := F_{[t+1]-t}[\xi_{[t+1]}]$ for $t \in \mathbb{R}_+$ and recall (F.2). $\square$

If $\xi < 1$ on a set of positive measure, it follows from (T.1) and the irreducibility of $(m_{\nu\mu})$, by iterating (IF), that $F_t[\xi] < 1$ on X for all $t > 0$. Similarly, if $\xi > q$ on a set of positive measure, it follows by iterating the equation obtained by subtracting (IF) from (2.1) that $F_t[\xi] > q$ on X for all $t > 0$. In particular, let (ξ_t) be a non-trivial sequence of backward iterates. By definition, $\xi_s < 1$ on a set of positive measure for some $s > 0$. Consequently, $\xi_t < 1$ on X for all $t < s$, but also $\xi_t < 1$ on a set of positive measure, depending on t, for $t > s$. Hence, $\xi_t < 1$ on X for all t. On the other hand, $\xi_t \geq F_n[0] \to q$, $n \to \infty$. That is, $\xi_t \geq q$ on X for all t. By definition $\xi_s > q$ on a set of positive measure. Hence, by the same argument as before, $\xi_t > q$ on X for all t.

(3.2) PROPOSITION. <u>If (ξ_t) is a non-trivial sequence of backward iterates, then</u>

$$\lim_{t\to\infty} \| 1-\xi_t \| = 0.$$

<u>Proof.</u> In view of

$$1-\xi_t = F_s[1] - F_s[\xi_{t+s}] \leq M_s[1-\xi_{t+s}]$$

and (M), it suffices to show $\Phi^*[1-\xi_t] \to 0$. Suppose $\Phi^*[1-\xi_t] \not\to 0$. Then there exist a ν, a sequence $t_j \uparrow \infty$, and a constant $c_2 > 0$ such that

$$\Phi^*[1_{X_\nu}(1-\xi_{t_j})] \geq c_2, \quad j \in \mathbb{N}.$$

Restricted to functions on X_ν, the semigroup $\{T_t^0\}$ has a representation analogous to (M): The role of $k(m-1)$ is simply taken by $-k$. Thus for s fixed sufficiently large, there exists a $c_3 > 0$ such that

$$T_s^0 \xi \geq c_3 \Phi^*[1_{X_\nu} \xi] 1_{X_\nu} \varphi, \quad \xi \in B_+ .$$

Recalling (F.2) and (IF),

$$\xi_0 = F_{t_j - s}[F_s[\xi_{t_j}]] \leq F_{t_j - s}[1 - T_s^0(1 - \xi_{t_j})]$$

$$\leq F_{t_j - s}[1 - c_2 c_3 1_{X_\nu} \varphi], \quad j \in \mathbb{N}.$$

By (2.5) the righthand side converges to q, as $j \to \infty$. This is a contradiction, and (3.2) is proved. □

(3.3) PROPOSITION. Let (ξ_t) be a non-trivial sequence of backward iterates, and define

$$\zeta_t := -\log \xi_t, \quad \gamma_t := \Phi^*[\zeta_t].$$

Then there exists, for every $t > 0$, a sequence (ε_t) in B such that

$$\zeta_t = (1 + \varepsilon_t) \gamma_t \varphi, \quad t > 0,$$

$$\lim_{t \to \infty} \| \varepsilon_t \| = 0.$$

Furthermore,

$$\gamma_t = \rho^{-t} L(\rho^{-t}), \quad t > 0,$$

where $L(s)$ is slowly varying, as $s \to 0$.

The statement follows from the next two lemmata, which will be needed again later.

(3.4) LEMMA. <u>For every non-trivial sequence of backward iterates</u> (ξ_t) <u>there exists a sequence</u> (h_t) <u>in</u> B <u>such that</u>

$$1-\xi_t = (1+h_t)\Phi^*[1-\xi_t]\varphi, \quad t > 0,$$

$$\lim_{t\to\infty} \| h_t \| = 0 .$$

<u>Proof</u>. The proof is similar to that of Lemma 1 of [15], or Lemma 4 of [16]. Using (F.2), (FM), (M), and (R),

$$\rho^s(1-\rho^{-s}\alpha_s - \| g_s[\xi_{t+s}] \|)\Phi^*[1-\xi_{t+s}]\varphi$$

$$\leq 1-F_s[\xi_{t+s}] \leq \rho^s(1+\rho^{-s}\alpha_s)\Phi^*[1-\xi_{t+s}]\varphi .$$

Combining these inequalities with those obtained by applying Φ^* to them yields

$$-\frac{2\rho^{-s}\alpha_s+\| g_s[\xi_{t+s}] \|}{1+\rho^{-s}\alpha_s}\varphi \leq \frac{1-F_s[\xi_{t+s}]}{\Phi^*[1-F_s[\xi_{t+s}]]} - \varphi$$

$$\leq \frac{2\rho^{-s}\alpha_s+\| g_s[\xi_{t+s}] \|}{1-\rho^{-s}\alpha_s-\| g_s[\xi_{t+s}] \|}\varphi .$$

Replace $F_s[\xi_{t+s}]$ by ξ_t, and let $t \to \infty$, using (3.2) and (R). Then let $s \to \infty$. □

(3.5) LEMMA. <u>If</u> (ξ_t) <u>is a non-trivial sequence of backward iterates, then</u>

$$\lim_{t\to\infty} \frac{\Phi^*[1-\xi_t]}{\Phi^*[1-\xi_{t+s}]} = \rho^s, \quad s > 0 .$$

<u>Proof</u>. Note that, by the definition of (ξ_t), (FM), (M), and (R),

$$\Phi^*[1-\xi_t] = \rho^s\Phi^*[1-\xi_{t+s}]\Phi^*[(1-g_s[\xi_{t+s}])\varphi]. \quad \square$$

Proof of (3.3). Note that $1-\xi_t = \zeta_t(1+\underset{\sim}{o}(\zeta_t))$, where $\|\underset{\sim}{o}(\xi)\| = o(\|\xi\|)$, recall (3.2), and apply (3.4-5). □

(3.6) PROPOSITION. Let (ξ_t) be a non-trivial sequence of backward iterates. Then there exists a random variable W such that

$$\lim_{t\to\infty} \hat{x}_t[\zeta_t] = \lim_{t\to\infty} \gamma_t \hat{x}_t[\varphi] = W \quad \text{a.s. } [P^{\hat{x}}],$$

$$P^{\hat{x}}(W=0) = \hat{q}(\hat{x}), \quad P^{\hat{x}}(W<\infty) = 1, \quad \hat{x} \in \hat{X},$$

with $\Phi(s)(x) := E^{\langle x\rangle}\exp\{-sW\}$, $s \in \mathbb{R}_+$, $x \in X$, satisfying

$$(\Phi) \qquad \Phi(s\rho^t) = F_t[\Phi(s)], \quad s,t \geq 0 .$$

Proof. Let $\underline{F}_s = \sigma\{\hat{x}_s; s \leq t\}$. Then for $t, s \geq 0$

$$E^{\hat{x}}(\hat{\xi}_{t+s}(\hat{x}_{t+s})|\underline{F}_t) = F_{t+s}(\hat{x}_t, \xi_{t+s}) = \hat{\xi}_t(\hat{x}_t) \quad \text{a.s. } [P^{\hat{x}}].$$

By the martingale convergence theorem and (3.5), this implies the convergence statements. Using (F.2), (3.5), and dominated convergence,

$$\Phi(s\rho^t) = \lim_{u\to\infty} F_{u+t}[\exp\{-s\rho^t\gamma_{t+u}\varphi\}]$$

$$= F_t[\lim_{u\to\infty} F_u[\exp\{-s(\rho^t\gamma_u^{-1}\gamma_{t+u})\gamma_u\varphi\}] = F_t[\Phi(s)].$$

It follows that $\Phi(0+)$ and $\Phi(\infty-)$ are fixed points of F_t in $C^0 \cap \overline{S}_+$. By (2.5) the only fixed points in $C^0 \cap \overline{S}_+$ are q and 1. Clearly, $\Phi(\infty-) \leq \Phi(1) \leq \Phi(0+)$. On the other hand,

$$\Phi(1) = \lim_{t\to\infty} F_t[e^{-\zeta_t}] = \xi_0 \in C^0 \cap \overline{S}_+ ,$$

and $q < \xi_0 < 1$ on X. Hence, $\Phi(\infty-) = q$ and $\Phi(0+) = 1$. □

(3.7) PROPOSITION. <u>For every non-trivial sequence of backward iterates</u>

$$0 < \lim_{t\to\infty} \rho^t \gamma_t = \gamma \leq \infty ,$$

$$E^{\hat{x}} W = \gamma \hat{x}[\varphi] , \quad \hat{x} \neq \theta .$$

<u>Here</u> $\gamma < \infty$ <u>if and only if for some and thus all</u> $t > 0$

(XLOG X) $$\Phi^*[E^{\langle\cdot\rangle} \hat{x}_t[\varphi] \log \hat{x}_t[\varphi]] < \infty .$$

<u>Proof</u>. Using (F.2) and (FM),

$$\rho^t \Phi^*[1-\xi_t] = \rho^t \Phi^*[1-F_s[\xi_{t+s}]] \leq \rho^{t+s} \Phi^*[1-\xi_{t+s}] .$$

Hence,

$$\lim_{t\to\infty} \rho^t \gamma_t = \lim_{t\to\infty} \rho^t \Phi^*[1-\xi_t] = \gamma$$

exists and is positive, possibly infinite.

It was shown in [1] that $\rho^{-t} \hat{x}_t[\varphi]$ has a finite, non-degenerate strong limit if and only if (XLOGX) is satisfied. That is, $(\gamma < \infty)$ and (XLOGX) are equivalent. □

REMARK. It is known that (XLOGX) is equivalent to

$$\Phi^*[\int_{\hat{X}} \pi(\cdot, d\hat{x}) \hat{x}[\varphi] \log \hat{x}[\varphi]] < \infty,$$

cf. [1], [16].

(3.8) PROPOSITION. <u>For any</u> $t > 0$ <u>the solutions of</u>

$$\Psi(\rho^t s) = F_t[\Psi(s)], \quad s \geq 0,$$

<u>in</u> $\overline{S}_+$ <u>are</u>

$$\Psi_0(s) \equiv 1, \quad \Psi_\alpha(s) = \Phi(\alpha s), \quad 0 < \alpha < \infty, \quad \Psi_\infty(s) \equiv q.$$

Proof. We have to show that, given a solution $\Psi(s)$ not identical 1 or q, there exists a positive real α such that $\Psi(s) = \Phi(\alpha s)$, $s > 0$. Consider the family of backward iterate sequences

$$\xi_t := \Phi(\rho^{-t}), \quad \xi_t^{(s)} := \Psi(s\rho^{-t}), \quad s > 0 .$$

Since by (3.6) the corresponding normalizing functions $\gamma_t, \gamma_t^{(s)}$, lead to non-degenerate, finite, strong limits $W, W^{(s)}$, we must have

$$\lim_{t\to\infty} \gamma_t^{-1}\gamma_t^{(s)} = \alpha(s) > 0, \quad s > 0.$$

Using (3.5),

$$\lim_{t\to\infty} \frac{\gamma_t^{(a)}}{\gamma_t^{(b)}} = \lim_{t\to\infty} \frac{\Phi^*[1-\xi_t^{(a)}]}{\Phi^*[1-\xi_t^{(b)}]} = \lim_{t\to\infty} \frac{\Phi^*[1-\xi_t^{(a)}]}{\Phi^*[1-\xi^{(a)}_{t+\log_\rho(a/b)}]} = \frac{a}{b} .$$

That is, $\alpha(s) = \alpha s$, α a positive real constant. Accordingly,

$$(1-\delta_t)\alpha s\zeta_t \leq \zeta_t^{(s)} \leq (1+\delta_t)\alpha s\zeta_t ,$$

$$\lim_{t\to\infty} \delta_t = 0 .$$

Taking the exponential function and applying F_t,

$$\Phi_t((1-\delta_t)\alpha s) \leq F_t[\Psi(s\rho^{-t})] \leq \Phi_t((1+\delta_t)\alpha s).$$

Note that the middle term is equal to $\Psi(s)$, and let $t \to \infty$. □

(3.9) COROLLARY. For every non-trivial sequence of backward iterates (ξ_t) there exists an $a \in \mathbb{R}$ such that

$$\xi_t = \Phi(\rho^{a-t}), \quad t > 0 .$$

4. Extinction probability and transience continued

With an additional indecomposibility assumption we prove a rate of convergence result corresponding to (2.3). It will be used to obtain the behaviour of the distribution function of W near zero.

Let $\delta F_t[\eta;\xi]$ be the Fréchet derivative of F_t at $\eta \in S$ in the direction of ξ. Since M_t is bounded,

$$\delta F_t(q)\xi := \lim_{\epsilon\uparrow 1} \delta F_t[\epsilon q;\xi], \qquad \xi \in B$$

defines a linear-bounded operator on B. Since q is a fixed point of F_t for all t, $\{\delta F_t(q)\}$ is a semigroup. The same is true for $\{\delta\overline{F}_t(0)\}$. The two are connected through

(4.1) $$\delta\overline{F}_t(0)\xi = (1-q)^{-1}\delta F_t(q)[(1-q)\xi], \quad \xi \in B, \ t > 0.$$

(4.2) LEMMA. *Suppose the matrix* $(\overline{f}_{\nu\mu})$,

$$\overline{f}_{\nu,\mu} := \int_{X_\mu} \overline{k}\,\delta\overline{f}(0)1_{X_\nu}, \quad \nu,\mu = 1,\dots,K,$$

is irreducible. Then $\{\delta\overline{F}_t(0)\}$ *can be represented in the form*

$$\delta\overline{F}_t(0) = \sigma^t\overline{\psi}\overline{\Psi}^* + \overline{\Gamma}_t, \quad t > 0,$$

$$\overline{\Psi}^*[\xi] = \int_X \overline{\psi}^*(x)\xi(x)dx, \qquad \xi \in B,$$

where $\sigma \in (0,1)$, $\overline{\psi},\overline{\psi}^* \in C^0$, $\inf\overline{\psi} > 0$, $\inf\overline{\psi}^* > 0$, *and* $\overline{\Gamma}_t : B \to B$ *such that*

$$\overline{\psi}\overline{\Psi}^*\overline{\Gamma}_t = \overline{\Gamma}_t\overline{\psi}\overline{\Psi}^* = 0,$$

$$-\beta_t\overline{\psi}\overline{\Psi}^* \le \overline{\Gamma}_t \le \beta_t\overline{\psi}\overline{\Psi}^*, \quad t > 0,$$

$$\sigma^{-t}\beta_t \downarrow 0, \quad t\uparrow\infty.$$

Proof. It follows from $\overline{k}\delta\overline{f}(0)\xi = k(1-q)^{-1}\delta f(q)[(1-q)\xi]$ that irreducibility of $k\delta f(q)$ and $\overline{k}\delta f(0)$ are equivalent. From (IF)

$$\delta F_t(q)\xi = T_t^0\xi + \int_0^t T_s^0 k\delta f(q)\,\delta F_{t-s}(q)\xi ds .$$

Also,

$$\delta f(q) \leq m \qquad [B_+] .$$

Hence, the proof of (M) also applies to $\{\delta F_t(q)\}$. Application of (4.1) to the resulting representation leads to the proposed representation for $\{\delta\overline{F}_t(0)\}$. To see that $\sigma < 1$, note that

$$\delta\overline{F}_t(0)1_X(x) = \overline{P}^{\langle x\rangle}\{\hat{x}_t[1] = 1\} \to 0, \quad x \in X ,$$

by (2.2), further

$$\delta\overline{F}_{t+s}(0)1_X(x) \leq (\sigma^s + \beta_s)\overline{\psi}\overline{\Psi}^*[\delta\overline{F}_t(0)1_X] ,$$

so that $\|\delta\overline{F}_t(0)1_X\| \to 0, \quad t \to \infty$. □

(4.3) PROPOSITION. <u>There exists a functional</u> $\overline{Q}$ <u>on</u> $\overline{S}_+$ <u>such that for all</u> $\xi \in \overline{S}_+$

$$\sigma^{-t}\overline{F}_t[\xi] = (1 + \delta_t[\xi])\overline{Q}[\xi]\overline{\psi} ,$$

$$\lim_{t\to\infty} \|\delta_t[\xi]\| = 0 .$$

<u>We have</u> $\overline{Q}[\xi] = 0$ <u>if and only if</u> $\xi = 0$ a.e., <u>while</u> $\overline{Q}[\xi] = \infty$ <u>if and only if</u> $\xi = 1$ a.e.

The proof will be based on three lemmata. Note that

$$\overline{F}_t[\xi] = \delta\overline{F}_t(0)\xi + \overline{G}_t[\xi] ,$$

(4.4)

$$\overline{G}_t[\xi] := \sum_{\nu=2}^{\infty} \frac{1}{\nu!}\, \delta^\nu \overline{F}_t[0;\xi,\ldots,\xi] .$$

(4.5) LEMMA. *For every* $t > 0$ *there exists a mapping* $a_t : \overline{S}_t \to B$ *such that*

$$\overline{G}_t[\xi] = a_t[\xi]\sigma^t \overline{\psi}\,\overline{\Psi}^* ,$$

$$\lim_{\|\xi\| \to 0} \| a_t[\xi] \| = 0$$

uniformly in $t \in [a,b]$, $0 < a < b < \infty$.

Proof. The proof resembles the proof of (R). We have

$$\overline{f}[\xi] = \delta\overline{f}(0)\xi + \overline{g}[\xi] ,$$

$$\overline{F}_t[\xi] = \overline{T}^0_t \xi + \int_0^t \overline{T}^0_s \, \overline{k}\overline{f}[\overline{F}_{t-s}[\xi]]ds ,$$

$$\delta\overline{F}_t(0)\xi = \overline{T}^0_t \xi + \int_0^t \overline{T}^0_s \, \overline{k}\delta\overline{f}(0)\,\delta\overline{F}_{t-s}(0)\xi ds .$$

For every $\epsilon > 0$ and $\xi \in \overline{S}_+$, $\overline{G}_t[\xi]$ is the only bounded solution in $[\epsilon, \epsilon+\lambda]$, $\lambda > 0$, of

$$v_t = A_t + B^\epsilon_t + \int_0^{t-\epsilon} \overline{T}^0_s \overline{k}\delta\overline{f}(0) v_{t-s} ds ,$$

$$A_t := \int_0^t \overline{T}^0_s \overline{k}\overline{g}[\overline{F}_{t-s}[\xi]]ds,$$

$$B^\epsilon_t := \int_0^\epsilon \overline{T}^0_{t-s} \overline{k}\delta\overline{f}(0)\overline{G}_t[\xi] \; ds .$$

By the inherent positivity, this solution equals the limit of the iteration sequence $(v^{(\nu)}_t(x))_{\nu \in \mathbb{N}}$, $v^{(0)}_t \equiv 0$, which we now estimate.

By the mean-value theorem we have for every $\xi \in \overline{S}_+$

$$\overline{g}[\xi] \leq \overline{f}[\xi] \leq \delta\overline{f}(\;\xi)\xi \leq \overline{m}\xi ,$$

$$\overline{G}_t[\xi] \leq \overline{F}_t[\xi] \leq \delta\overline{F}_t(\;\xi)\xi \leq \overline{M}_t \xi .$$

Fixing λ, let $0 < \delta < \epsilon/2$ and $\epsilon \le t \le \epsilon+\lambda$. Then using $(\overline{M})$,

$$A_t \le \int_0^\delta + \int_{t-\delta}^{\delta} \overline{M}_s \overline{kmM}_{t-s} \xi \, ds + \int_\delta^{t-\delta} \overline{M}_s \overline{kg}[\overline{M}_{t-s}\xi] ds$$

$$\le \delta C_1(\epsilon) \rho^t \overline{\varphi}\overline{\Phi}^*[\xi]$$

$$+ C_2(\delta,\epsilon)\overline{\Phi}^*[\overline{g}[C_3(\delta)\overline{\Phi}^*[\xi]\varphi]] C_3(\delta)\rho^t \overline{\varphi}\overline{\Phi}^*[\xi] ,$$

where C_1, C_2, C_3 are constants depending on the choice of ϵ and δ, as indicated. Hence, there exists for every ϵ a functional Θ_ϵ on $\overline{S}_+$ such that

$$A_t \le \Theta_\epsilon[\xi] \rho^t \overline{\Phi}^*[\xi]\overline{\varphi} ,$$

(4.6)

$$\lim_{||\xi|| \to 0} \Theta_\epsilon[\xi] = 0 .$$

Secondly,

$$B_t^\epsilon \le \overline{M}_{t-\epsilon} \int_0^\epsilon \overline{M}_{\epsilon-s} \overline{kmM}_s \xi ds =: \tilde{B}_t^\epsilon ,$$

(4.7)

$$\int_0^{t-\epsilon} \overline{T}_s^0 \overline{km\tilde{B}}_{t-s}^\epsilon \, ds \le \tilde{B}_t^\epsilon .$$

Using (4.6), (4.7), $(\overline{M})$, and the fact that

$$\overline{\varphi} \le C\overline{\psi} , \quad \overline{\Phi}^* \le C^* \overline{\Psi}^* \; [B_+] ,$$

we get

$$\lim_{t\to\infty} v_t^{(\nu)} \le CC^* (\tfrac{\rho}{\sigma})^t (e^{ct} \Theta_\epsilon[\xi] + \epsilon c_4 (1+\rho^{-t+\epsilon}\alpha_{t+\epsilon}) \sigma^t \overline{\Psi}^*[\xi]\overline{\psi} .$$

Since ϵ was arbitrary, this proves (4.5). □

(4.8) LEMMA. *If* $(\overline{f}_{\nu\mu})$ *is irreducible, there exists for every* $t > 0$ *a mapping* $b_t : \overline{S}_+ \to B$ *such that*

$$\overline{F}_t[\xi] = (1 + b_t[\xi])\overline{\Psi}^*[\overline{F}_t[\xi]]\overline{\psi} ,$$

$$\lim_{t\to\infty} \|b_t[\xi]\| = 0$$

for every $\xi \in \overline{S}_+$ *with* $\xi < 1$ *on a set of positive measure.*

Proof. The proof is similar to the proof of (3.4). From (F.2), (4.4),

$$\overline{F}_s[\overline{F}_{t-s}[\xi]] = \delta\overline{F}_s(0)\overline{F}_{t-s}[\xi] + \overline{G}_s[\overline{F}_{t-s}[\xi]] ,$$

and from this, by (4.2), (4.5),

$$(1-\sigma^{-s}\beta_s)\sigma^s\overline{\Psi}^*[\overline{F}_{t-s}[\xi]]\overline{\psi} \leq \overline{F}_t[\xi]$$

$$\leq (1+\sigma^{-s}\beta_s + \|a_s[\overline{F}_{t-s}[\xi]]\|)\sigma^s\overline{\Psi}^*[\overline{F}_{t-s}[\xi]]\overline{\psi} .$$

To estimate $(\overline{\Psi}^*[\overline{F}_t[\xi]])^{-1}\overline{F}_t[\xi] - \overline{\psi}$, combine these two inequalities with those obtained by applying $\overline{\Psi}^*$ to them. First let $t \to \infty$, then $s \to \infty$, recalling that $\|\overline{F}_t[0]\| \to 0$, as $t \to \infty$, by (2.5). □

(4.9) LEMMA. *For* $t > 0$ *and* $\eta \in S_+$ *let* $\sigma_t(\eta)$ *be the spectral radius of* $\delta\overline{F}_t(\eta)$. *Then*

$$\lim_{\|\eta\|\to 0} \sigma_t(\eta) = \sigma^t .$$

Proof. For $t > 0$ and $\eta \in S_+$

$$\delta\overline{F}_t(\eta)1 = \delta\overline{F}_t(0)1 + \sum_{\nu=1}^{\infty} \frac{1}{\nu!} \delta^{\nu+1}\overline{F}_t[0,\eta,\ldots,\eta,1]$$

$$\leq \delta\overline{F}_t(0)1 + 2\|\eta\|(1-\|\eta\|)^{-2}1 .$$

Hence, for $\|\eta\| \leq \frac{1}{2}$,

$$\|\delta\overline{F}_t(\eta)^n 1\| \leq \|\sum_{\nu=0}^{n}\binom{n}{\nu}\|8\eta\|^{n-\nu}\delta\overline{F}_t(0)^{\nu}1\|$$

$$\leq c_5 \sum_{\nu=0}^{n} \binom{n}{\nu}\|8\eta\|^{n-\nu}\sigma^{t\nu}$$

$$= c_5(8\|\eta\|+\sigma^t)^n \quad .$$

Since we are dealing with positive operators, this proves (4.9). □

Proof of (4.3). For $\xi \in \overline{S}_+$, using (4.4),

$$\sigma^{-t-s}\overline{\Psi}^*[\overline{F}_{t-s}[\xi]]$$

$$= \sigma^{-t-s}\overline{\Psi}^*[\delta\overline{F}_s(0)\overline{F}_t[\xi]] + \sigma^{-t-s}\overline{\Psi}^*[\overline{G}_s[\overline{F}_t[\xi]]]$$

$$\geq \sigma^{-t}\overline{\Psi}^*[\overline{F}_t[\xi]] \quad .$$

That is, $\sigma^{-t}\overline{\Psi}^*[\overline{F}_t[\xi]]$ converges to some functional $\overline{Q}[\xi]$. Combined with (4.8), this implies $\sigma^{-t}\overline{F}_t[\xi] \to \overline{Q}[\xi]\overline{\psi}$ in the way proposed.

If $\xi > 0$ on a set of positive measure, then by ($\overline{\text{IF}}$) and (T.1)

$$\overline{F}_t[\xi] \geq \overline{T}_t^0\xi \geq e^{-\|k\|t}(1-q)^{-1}T_t[(1-q)\xi] > 0 \quad .$$

on a set of positive measure, thus $\overline{Q}[\xi] > 0$.

Finally, suppose $\xi < 1$ on a set of positive measure, $\xi \neq 0$. Using (4.4),

$$\sigma^{-n}\overline{\Psi}^*[F_n[\xi]] = \sigma^{-1}\overline{\Psi}^*[\overline{F}_1[\xi]]\prod_{\nu=1}^{n-1}\{1+\sigma^{-1}\frac{\overline{\Psi}^*[\overline{G}_1[\overline{F}_\nu[\xi]]]}{\overline{\Psi}^*[\overline{F}_\nu[\xi]]}\} ,$$

$$\sum_{\nu=1}^{n}\frac{\overline{\Psi}^*[\overline{G}_1[\overline{F}_\nu[\xi]]]}{\overline{\Psi}^*[\overline{F}_\nu[\xi]]} \leq \sum_{\nu=1}^{n}(1-\|\overline{F}_\nu[\xi]\|)^{-1}\frac{\|\overline{F}_\nu[\xi]\|}{\overline{\Psi}^*[\overline{F}_\nu[\xi]]}\|\overline{F}_\nu[\xi]\| \quad .$$

In view of (4.8), it suffices to show that $\|\overline{F}_n[\xi]\| = O(\lambda^n)$ with some $\lambda < 1$, in order to secure that the limit of $\sigma^{-n}\overline{\Psi}^*[\overline{F}_n[\xi]]$ is finite.

Since $\overline{F}_t[0] \equiv 0$, (F.2) and the mean-value theorem yield

$$\overline{F}_{t+s}[\xi] \leq \delta\overline{F}_t(\overline{F}_s[\xi])\overline{F}_s[\xi] .$$

Iterating this inequality, we get

$$\overline{F}_{n+j}[\xi] \leq \delta\overline{F}_1(\overline{F}_{j+n-1}[\xi])\,\delta\overline{F}_1(\overline{F}_{j+n-2}[\xi])\cdots\delta\overline{F}_1(\overline{F}_j[\xi])\overline{F}_j[\xi]$$

$$\leq \delta\overline{F}_1(\sup_{\nu\geq j} \|\overline{F}_\nu[\xi]\|)^n 1, \qquad n, j \in \mathbb{N} .$$

Recall that $\|\overline{F}_\nu[\xi]\| \to 0$, as $\nu \to \infty$, and apply (4.9). $\square$

REMARK. The specialization of (4.3) to Bienaymé-Galton-Watson processes is well known, cf. [12], [13]. In fact, it first occured in a paper by Koenigs [24] on the classical problem of fractional iteration. In view of the useful role this result has played as a tool, a theory of the general case with decomposable $(\overline{f}_{\nu\mu})$ seems desirable.

5. Properties of the limit distribution.

We prove that the distribution function of W has a positive density and look at its behaviour near zero and infinity. As before let σ be the spectral radius of $\delta F_1(q)$, and define

$$\epsilon_0 := -\log_\rho \sigma .$$

(5.1) PROPOSITION. For every $\hat{x} \neq \theta$ there exists a measurable function $w_{\hat{x}} | \mathbb{R}_+$ such that

$$P^{\hat{x}}(W \leq \lambda) = \hat{q}(\hat{x}) + \int_0^\lambda w_{\hat{x}}(u)\,du, \qquad \lambda \geq 0.$$

If we have $\hat{x}[1]\epsilon_0 > 1$, or if (XLOGX) is satisfied, then $w_{\hat{x}}(u)$ is bounded and continuous in u.

It suffices to prove that the distribution function of the limit $\overline{W}$ for the transformed process $\{\hat{x}_t, \overline{P}^{\hat{x}}\}$ has a Lebesgue density $\overline{w}_{\hat{x}}$ and that in case $\hat{x}[1]\epsilon_0 > 1$, or (XLOGX) is satisfied, this density is bounded and continuous. Recall that $\delta F_1(q)$ and $\delta\overline{F}_1(0)$ have the same spectral radius, and note that (XLOGX) is equivalent to its analog for $\{\hat{x}_t, \overline{P}^{\hat{x}}\}$. Let $\overline{\Phi}(s)(x)$ denote the Laplace transform of $\overline{W}$ respective $\overline{P}^{\langle x \rangle}$.

(5.2) LEMMA. For every bounded, closed interval $I \subset \mathbb{R}$ not containing zero,

$$\sup_{t \in I} \| \overline{\Phi}(it) \| > 1 .$$

Proof. Suppose $|\overline{\Phi}(it)| \equiv 1$ for all t. Then

$$\overline{\Phi}(it) = e^{it\xi},$$

ξ measurable, finite. Inserting this into

(5.3) $$\overline{\Phi}(i\rho^s t) = \overline{F}_s[\overline{\Phi}(it)]$$

yields

$$\overline{E}^{\hat{x}}\exp\{it(\hat{x}_s[\xi]-\rho^s\hat{x}[\xi])\} = 0, \quad t \geq 0.$$

From this

$$\hat{x}_s[\xi] = \rho^s\hat{x}_0[\xi] \quad \text{a.s.},$$

$$\overline{M}_s\xi(x) = \overline{E}^{\langle x\rangle}\hat{x}_s[\xi] = \rho^s\xi(x), \quad x \in X .$$

That is, $\xi = \alpha\overline{\varphi}$, where α is a constant. Since $\overline{W}$ is non-degenerate, $\alpha \neq 0$. However, since our process is super-critical and $\overline{\varphi}$ continuous, we cannot have $\hat{x}_s[\overline{\varphi}] = \rho^s\hat{x}_0[\overline{\varphi}]$ a.s. for all s. Hence, for some $t \neq 0$, $x \in X$,

$$\overline{\Phi}(it)(x) < 1.$$

Since $\overline{F}_s[\xi](x)$ is continuous in x, by $(\overline{IF})$ and the continuity and boundedness of $\overline{T}^0_s$, so is $\overline{\Phi}(it)(x)$, by (5.3). We can therefore find for every $x \in X$ a $\delta(x) > 0$, and for every t with $0 < |t| \leq \delta(x)$ a $U(x,t) \subset X$ of positive measure such that

$$|\overline{\Phi}(it)(y)| < 1, \quad 0 \leq |t| \leq \delta(x), \quad y \in U(x,t).$$

If $\eta \in \overline{S}_+$ is positive on a set of positive measure in X_ν, then $\overline{T}^0_s\eta \geq e^{-\overline{k}s}\overline{T}_s\eta$, which is uniformly positive in X_ν, by (T.1-5) and $1-q \in D^+_0$. Iterating

$$1-|\overline{\Phi}(\rho^s it)| \geq 1-\overline{F}_s[|\overline{\Phi}(it)|]$$

$$= \overline{T}^0_s(1-|\overline{\Phi}(it)|) + \int_0^s \overline{T}^0_u\overline{k}(1-\overline{f}[\overline{F}_{s-u}[|\overline{\Phi}(it)|]])ds$$

and recalling the irreducibility of $(\overline{m}_{\nu\mu})$, we get

$$\| \overline{\Phi}(it) \| < t, \ t \neq 0 .$$

Since $\overline{\Phi}(it)$ is pointwise continuous in t and bounded, it follows by

$$\| \overline{\Phi}(it)-\overline{\Phi}(i(t+\varepsilon)) \| \leq \| \overline{M}_s[|\overline{\Phi}(it\rho^{-s})-\overline{\Phi}(i(t+\varepsilon)\rho^{-s})|] \|$$

and $(\overline{M})$ that $\overline{\Phi}(it)$ is strongly continuous in t. Hence, $\| \overline{\Phi}(it) \|$ is continuous, and the proof is complete. □

(5.4) LEMMA. <u>For every positive</u> $\varepsilon < \varepsilon_0$

$$\| \overline{\Phi}(it) \| = O(|t|^{-\varepsilon}), \ t \neq 0 .$$

<u>Proof</u>. As in the proof of (4.3)

$$|\overline{F}_{n+j}[\overline{\Phi}(it)]| \leq \overline{F}_n[\overline{F}_s[|\overline{\Phi}(it)|]]$$

$$\leq \delta\overline{F}_1(\sup_{\nu\geq j}\| \overline{F}_\nu[|\overline{\Phi}(it)|] \|)^n\overline{F}_j[|\overline{\Phi}(it)|] .$$

By (5.2), $\| \overline{\Phi}(it) \| \leq c_6 < 1$ on $[1,\rho]$, so that $\| \overline{F}_\nu[|\overline{\Phi}(it)|] \| \to 0$, as $\nu \to \infty$, uniformly in $t \in [1,\rho]$. According to (4.9), there then exists for every positive $\varepsilon < \varepsilon_0$ a j such that

$$\sup_{1\leq t\leq\rho} \| \overline{F}_{j+n}[\Phi(it)] \| = O(\rho^{-n\varepsilon}),$$

or equivalently,

$$\sup_{1\leq t\leq\rho} \| \overline{\Phi}(it\rho^{j+n}) \| = O(\rho^{-n\varepsilon}) .$$

Hence,

$$\sup_{\rho^{n+j}\leq t\leq\rho^{n+j+1}} \| \overline{\Phi}(it) \| \leq c_7|t|^{-\varepsilon}$$

with c_7 independent of n. □

(5.5) COROLLARY. *If* (XLOGX) *is satisfied, then*

$$\left\| \frac{\partial}{\partial t}\overline{\Phi}(it) \right\| = O(|t|^{-1-\epsilon}), \quad t \neq 0.$$

Proof. Denoting $\partial\overline{\Phi}(s)/\partial s$ by $\overline{\Phi}'(s)$

$$\rho^{n+j}|\overline{\Phi}'(it\,\rho^{n+j})| \leq \delta\overline{F}_n(\overline{F}_j[|\overline{\Phi}(it)|])\,\delta\overline{F}_j(|\overline{\Phi}(it)|)|\overline{\Phi}'(it)|.$$

Also,

$$\delta\overline{F}_j(|\overline{\Phi}(it)|) \leq \overline{M}_j \quad [B_+],$$

and by assumption of (XLOGX),

$$\overline{E}^{\langle x\rangle}\overline{W} = \overline{\varphi}(x), \quad x \in X,$$

cf. [1]. That is, $|\overline{\Phi}'(it)| \leq \overline{\varphi}$. Continue as in the proof of (5.4).

□

Proof of (5.1). Given $\hat{x}_0[1]\epsilon_0 > 1$, or (XLOGX), the characteristic function of $\overline{W}$ is absolutely integrable, by (5.4-5), and this implies that the distribution function of $\overline{W}$ has a bounded continuous density. Given $\hat{x}_0[1]\epsilon_0 < 1$, the argument of [3] can be adapted: For every $\hat{x} \neq \theta$ the probability space carrying $\{\hat{x}_t\}$ can be enlarged in such a way that it also carries a set of independent random variables $\overline{W}_x$, $x \in X$, which are independent of $\hat{x}_t$ and satisfy

$$\overline{P}^{\hat{x}}(\overline{W}_x \in I) = \overline{P}^{\langle x\rangle}(\overline{W} \in I), \quad x \in X,$$

for every Borel set $I \subset \mathbb{R}_+$. Since the characteristic functions of $\rho^{-n}\hat{x}_n[\overline{W}_x]$ and $\overline{W}$ coincide, so do their distributions. That is, for every I of Lebesgue measure zero,

$$\overline{P}^{\hat{x}}(\overline{W} \in I) = \int_{\hat{X}} \overline{P}^{\hat{x}}(\hat{x}_n \in d\hat{y})\overline{P}^{\hat{x}}(\hat{y}[W_\cdot] \in \rho^n I)$$

$$\leq \overline{P}^{\hat{x}}(\hat{x}_n[1]\epsilon_0 \leq 1) + \int_{\{\hat{y}[1]\epsilon_0 > 1\}} \overline{P}^{\hat{x}}(\hat{x}_n \in d\hat{y})\overline{P}^{\hat{x}}(\hat{y}[W_\cdot] \in \rho^n I),$$

the first term on the right vanishes, as $n \to \infty$, by (2.2), and the second term is identical zero for all n. □

(5.6) PROPOSITION. <u>For every</u> $\hat{x} \neq \theta$ <u>the density</u> $\overline{w}_{\hat{x}}$ <u>is positive on the positive reals.</u>

<u>Proof.</u> Let $\lambda^{(n)}$ be the measure induced on $A^{(n)} := \{\hat{A} \cap X^{(n)} : \hat{A} \in \underline{\hat{A}}\}$ by the Lebesgue measure on $\underline{A}$, and let $\underline{P}^{(n)}$ be the class of elements of $A^{(n)}$ which have positive $\lambda^{(n)}$-measure. Defining

$$\langle x_1, \ldots, x_n \rangle + \langle y_1, \ldots, y_j \rangle := \langle x_1, \ldots, x_n, y_1, \ldots, y_j \rangle,$$

we have

$$\overline{w}_{\hat{x}+\hat{y}}(s) = \int_{0+}^{s} \overline{w}_{\hat{x}}(s-u)\overline{w}_{\hat{y}}(u)\,du .$$

Let $\overline{P}_t(x,\hat{A})$ be the transition function of $\{\hat{x}_t, \overline{P}^{\hat{x}}\}$. From (5.3)

$$\overline{w}_{\hat{x}}(s) = \int_{\hat{X}} \overline{P}_t(\hat{x}_t, d\hat{y})\rho^t \overline{w}_{\hat{y}}(\rho^t s). \tag{5.7}$$

<u>Step 1.</u> For $\hat{x} \in X^{(n)}$, $n\epsilon_0 > 1$, $\overline{w}_{\hat{x}}(s)$ is continuous, and since $\overline{W}$ is non-degenerate, it is also positive somewhere. That is, $\overline{w}_{\hat{x}} > 0$ on some open interval $I_{\hat{x}} \in \mathbb{R}_+$. It follows by (5.7) that for each $u \in \rho^t I_{\hat{x}}$ there exist a $j = j(\hat{x},u) \in \mathbb{N}$ and an $\hat{A}_{\hat{x},k} \in \underline{P}^{(j)}$ such that $\overline{P}_t(\hat{x}, \hat{A}_{\hat{x},u}) > 0$ and $\overline{w}_{\hat{y}}(u) > 0$ for $\hat{y} \in \hat{A}_{\hat{x},u}$. Because of (T.1) and $k \in B$, there then exists a neighbourhood $\hat{U}_{s,\hat{x}} \in \underline{P}^{(n)}$ of $\hat{x}$ on which $\overline{P}_t(\cdot, \hat{A}_{\hat{x},u}) > 0$, and thus

$$\overline{w}_{\hat{y}}(s) > 0, \quad s \in I_{\hat{x}}, \ \hat{y} \in \hat{U}_{s,\hat{x}}. \tag{5.8}$$

Step 2. In view of (T.1), the irreducibility of $(m_{\nu\mu})$, and $\rho > 1$, there exists an integer d such that for $\hat{x} \neq \theta$

$$(5.9) \qquad \bar{P}_t(\hat{x}, \hat{A}) > 0, \ t > 0, \ \hat{A} \in \underline{P}^{(\varkappa d)}, \ \varkappa \in \mathbb{N}.$$

If $\bar{w}_{\hat{x}} > 0$ on I and $\bar{w}_{\hat{y}} > 0$ on J, then $\bar{w}_{\hat{x}+\hat{y}} > 0$ on $I+J := \{z = x+y : x \in I, y \in J\}$. That is, given any $s_0 > 0$, we can choose $\hat{y} \in X^{(\varkappa d)}$, $\varkappa d \epsilon_0 > 1$, such that $\bar{w}_{\hat{y}} > 0$ on some interval (a,b), $b > s_0$, and thus, by (5.7-9), $\bar{w}_{\hat{x}} > 0$ on $(0,b)$. Now let $s_0 \to \infty$. □

(5.10) PROPOSITION. <u>There exists a function</u> $L(s)$ <u>on</u> $\mathbb{R}_+$, <u>slowly varying as</u> $s \to 0$, <u>such that for</u> $x \in X$

$$P^{\langle x \rangle}(W > \lambda) = o(\lambda^{-1} L(\lambda^{-1}))\varphi(x), \quad \lambda \to \infty.$$

<u>Proof</u>. By (3.4) and (3.5) with $\xi_t = \Phi(\rho^{-t})$,

$$1 - \Phi(s) = (1 + h_s^*) s L(s) \varphi,$$

$$\lim_{s \to 0} \| h_s^* \| = 0 .$$

From this, by Karamata's Tauberian theorem, cf. [33],

$$\int_0^\lambda P^{\langle x \rangle}(W > u)du \sim L(\lambda^{-1})\varphi(x), \quad \lambda \to \infty .$$

Now apply Seneta's version of the Tauberian theorem of Landau and Feller, [30]. □

(5.11) PROPOSITION. <u>If</u> $(f_{\nu\mu})$ <u>is irreducible, then for</u> $x \in X$

$$P^{\langle x \rangle}(W \leq \lambda) - q(x) \sim \lambda^{\epsilon_0} \frac{\bar{Q}[\bar{\Phi}(1)]}{\Gamma(\epsilon_0 + 1)} (1 - q(x))^{\epsilon_0} \psi(x), \quad \lambda \to 0,$$

where Γ *is the Gamma function and* $\psi := (1-q)\overline{\psi}$. *If in addition* $\epsilon_0 > 1$, *then*

$$\overline{w}_{\langle x\rangle}(\lambda) \sim \epsilon_0 \lambda^{\epsilon_0 - 1} \frac{\overline{Q}[\overline{\Phi}(1)]}{\Gamma(\epsilon_0+1)} \overline{\psi}(x), \quad \lambda \to 0 .$$

Proof. From (Φ) and (4.3)

$$\sigma^{-t}\overline{\Phi}(\rho^t)(x) = \sigma^{-t}\overline{F}_t[\overline{\Phi}(1)](x) \sim \overline{Q}[\overline{\Phi}(1)]\overline{\psi}(x), \quad t \to \infty .$$

That is,

$$\overline{\Phi}(s)(x) \sim s^{-\epsilon_0}\overline{Q}[\overline{\Phi}(1)]\overline{\psi}(x), \quad s \to \infty.$$

Now apply Karamata's Tauberian theorem, to obtain the first statement, and the argument on p.250 of [4] together with Corollary 4.4a on p.194 of [33], to obtain the density version. $\square$

6. Strong convergence with general non-negative test functions

(6.1) PROPOSITION. For every a.e. continuous $\eta \in B$

$$\lim_{t\to\infty} \gamma_t \hat{x}_t[\eta] = \Phi^*[\eta]W \quad \text{a.s.} \quad [P^{\hat{x}}] .$$

On account of the following lemma, it suffices to prove convergence of discrete skeletons.

(6.2) LEMMA. If for all $\epsilon > 0$ and $\xi \in B_+$

$$\lim_{\mathbb{N}\ni n\to\infty} \frac{\hat{x}_{n\epsilon}[\xi]}{\hat{x}_{n\epsilon}[\varphi]} = \Phi^*[\xi] \text{ a.s. } [P^{\hat{x}}] \quad \text{on} \quad \{W > 0\} ,$$

then for all a.e. continuous $\eta \in B$

$$\lim_{t\to\infty} \frac{\hat{x}_t[\eta]}{\hat{x}_t[\varphi]} = \Phi^*[\eta] \text{ a.s. } [P^{\hat{x}}] \quad \text{on} \quad \{W > 0\} .$$

Proof. The lemma is a special case of Lemma 9 of [16].

Proof of (6.1). It suffices to prove skeleton convergence for $\epsilon = 1$. Due to (F.1) we can assume without loss of generality that

$$\hat{x}_{n+\ell}[\xi] = \sum_{j=1}^{\hat{x}_n[1]} \hat{x}^{n,j}_{n+\ell}[\xi] \text{ a.s. } [P^{\hat{x}}]$$

where the $\hat{x}^{n,j}_{n+\ell}$, $j = 1,\dots,\hat{x}_n[1]$, are $\underline{F}_n$-measurable and independent conditioned on $\underline{F}_n$, and for every $\hat{A} \in \underline{\hat{A}}$

$$P^{\hat{x}}(\hat{x}^{n,j}_{n+\ell} \in \hat{A} | \underline{F}_n) = P^{\langle x_j \rangle}(\hat{x}_\ell \in \hat{A}) \text{ a.s. } [P^{\hat{x}}]$$

with $\hat{x}^{n,j}_n = \langle x_j \rangle$. Define

$$\beta_n := \hat{x}_n[\varphi], \qquad 1_n := 1_{\{\beta_n > 0\}} ,$$

$$Z^j_{n,\ell} := \hat{x}^{n,j}_{n+\ell}[\xi] , \qquad Z^{j*}_{n,\ell} := Z^j_{n,\ell} 1_{\{Z^j_{n,\ell} \le \beta_{n-1}\}} ,$$

$$S_{n,\ell}: = 1_{n-1}\beta_{n-1}^{-1} \sum_{j=1}^{\hat{x}_n[1]} Z_{n,\ell}^{j}, \qquad S_{n,\ell}^{*}: = 1_{n-1}\beta_{n-1}^{-1} \sum_{j=1}^{\hat{x}_n[1]} Z_{n,\ell}^{j*},$$

$$\delta_{n,\ell}: = E^{\hat{x}}(S_{n,\ell} - S_{n,\ell}^{*} \mid \underline{F}_n) .$$

Then

$$1_{n+\ell}\beta_{n+\ell}^{-1}\hat{x}_{n+\ell}[1]$$

$$= 1_{n+\ell}\beta_{n+\ell}^{-1}\beta_{n-1}(S_{n,\ell} - E^{\hat{x}}(S_{n,\ell}^{*} \mid \underline{F}_n) - \delta_{n,\ell} + E^{\hat{x}}(S_{n,\ell} \mid \underline{F}_n) .$$

We estimate the terms on the right, proceeding as indicated in [16]. Define

$$H_{\langle\cdot\rangle}^{\ell}(\lambda): = P^{\langle\cdot\rangle}(\hat{x}_{\ell}[\xi] \leq \lambda), \qquad \lambda > 0 .$$

<u>Step 1</u>. Using (1.2), (M), and

$$\lim_{n\to\infty} \beta_n^{-1}\beta_{n+\ell} = \lim_{n\to\infty} \gamma_{n+\ell}^{-1}\gamma_n = \rho^{\ell} > 1 \text{ a.s. } [P^{\hat{x}}], \qquad \ell > 0 , \tag{6.3}$$

we get

$$\sum_{n=1}^{\infty} E^{\hat{x}}\{[S_{n,\ell} - E^{\hat{x}}(S_{n,\ell}^{*} \mid \underline{F}_n)]^2 \mid \underline{F}_{n-1}\}$$

$$\leq \sum_{n=1}^{\infty} E^{\hat{x}}\{1_{n-1}\beta_{n-1}^{-2} \sum_{j=1}^{\hat{x}_n[1]} E^{\hat{x}}([Z_{n,\ell}^{j*}]^2 \mid \underline{F}_n) \mid \underline{F}_{n-1}\}$$

$$\leq \sum_{n=1}^{\infty} E^{\hat{x}} 1_{n-1}\beta_{n-1}^{-2}\hat{x}_{n-1}[M_1[\int_0^{\beta_{n-1}} \lambda^2 dH_{\langle\cdot\rangle}^{\ell}(\lambda)]]$$

$$\leq C_4 \sum_{n=1}^{\infty} 1_{n-1}\beta_{n-1}^{-1} \int_0^{\beta_{n-1}} \lambda^2 d\Phi^*[H_{\langle\cdot\rangle}^{\ell}(\lambda)]$$

$$\leq C_5 \int_0^{\infty} \lambda d\Phi^*[H_{\langle\cdot\rangle}^{\ell}(\lambda)] + C_6 ,$$

$$\sum_{n=1}^{\infty} P^{\hat{x}}\{S_{n,\ell} \neq S_{n,\ell}^{*} \mid \underline{F}_{n-1}\}$$

$$\leq \sum_{n=1}^{\infty} E^{\hat{x}}\{1_{n-1} \sum_{j=1}^{\hat{x}_n[1]} P^{\hat{x}}(Z^j_{n,\ell} > \beta_{n-1} | \underline{F}_n) | \underline{F}_{n-1}\}$$

$$\leq \sum_{n=1}^{\infty} 1_{n-1}\hat{x}_{n-1}[M_1[\int_{\beta_{n-1}}^{\infty} dH^{\ell}_{\langle\cdot\rangle}(\lambda)]]$$

$$\leq C_7 \sum_{n=1}^{\infty} 1_{n-1}\beta_{n-1} \int_{\beta_{n-1}}^{\infty} d\Phi^*[H^{\ell}_{\langle\cdot\rangle}(\lambda)]$$

$$\leq C_8 \int_0^{\infty} \lambda \, d\Phi^*[H^{\ell}_{\langle\cdot\rangle}(\lambda)] + C_9 \quad .$$

The $C_4,\ldots,C_9$ are finite, but in general random. It follows by application of Chebychev's inequality and the conditional Borel-Cantelli lemma that

$$\lim_{n\to\infty} (S_{n,\ell} - E^{\hat{x}}(S^*_{n,\ell} | \underline{F}_n)) = 0 \quad \text{a.s.} \quad [P^{\hat{x}}] \ .$$

Step 2. Using (M),

$$\delta_{n,\ell} \leq 1_{n-1}\beta^{-1}_{n-1}\hat{x}_n[M_\ell[\xi]] \leq (\rho^{\ell}+\alpha_\ell)\beta^{-1}_{n-1}\beta_n \,\Phi^*[\xi].$$

Hence, in particular,

$$\limsup_{n\to\infty} \beta_n^{-1}\hat{x}_n[1] < \infty \qquad \text{a.s. on } \{W > 0\}.$$

Now

$$\delta_{n,\ell} \leq \|\xi\| 1_n\beta^{-1}_{n-1}\hat{x}_n[1] \sup_{x\in X} \int_{\beta_{n-1}}^{\infty} \lambda \, dP^{\langle x\rangle}(\hat{x}_\ell[1] \leq \lambda) \ .$$

From (IF)

$$\int_y^{\infty} \lambda \, dP^{\langle x\rangle}(\hat{x}_\ell[1] \leq \lambda) = \int_0^{\ell} T^0_s k N^y_{\ell-s}(x) ds, \quad y > 1 \ ,$$

$$N^y_t(x) := \int_{\hat{X}} \sum_{n\geq y} n\pi(x,d\hat{x}) P^{\hat{x}}(\hat{x}_t[1] = n) \ .$$

Notice that

$$\lim_{y\to\infty} N_s^y(x) = 0, \qquad s \geq 0, \quad x \in X ,$$

$$N_s^y \leq m[M_s[1]] \leq e^{||km||} m[1], \quad s \geq 0, \quad y \geq 0 .$$

Hence, by dominated convergence

$$\lim_{y\to\infty} \sup_{x\in X} \int_y^\infty \lambda \, dP^{\langle x\rangle}(\hat{x}_\ell[1] \leq \lambda) = 0 ,$$

using $T_s^0 \leq T_s$, $||T_s|| \leq 1$, and the boundedness of $p_s(x,y)$ on $[\alpha,t]\otimes X\otimes X$ for $\alpha > 0$. Thus, since $\beta_n \to \infty$ on $\{W > 0\}$,

$$\lim_{n\to\infty} \delta_{n,\ell} = 0 \quad \text{a.s. } [P^{\hat{x}}] \quad \text{on } \{W > 0\}, \quad \ell > 0.$$

<u>Step 3</u>. We have

$$1_{n+\ell}\beta_{n+\ell}^{-1}\beta_{n-1}E^{\hat{x}}(S_{n,\ell}|\underline{F}_n) = 1_{n+\ell}\beta_{n+\ell}^{-1}\hat{x}_n[M^\ell[\xi]].$$

Again by (M) and (6.3),

$$\limsup_{n\to\infty} 1_{n+\ell}\beta_{n+\ell}^{-1}\hat{x}_n[M^\ell[\xi]] \leq (1+\rho^{-\ell}\alpha_\ell)\,\Phi^*[\xi] ,$$

$$\liminf_{n\to\infty} 1_{n+\ell}\beta_{n+\ell}^{-1}\hat{x}_n[M^\ell[\xi]] \geq (1-\rho^{-\ell}\alpha_\ell]\,\Phi^*[\xi] .$$

Now let $\ell \to \infty$. □

REMARK. The difference to the method of [1] lies in the use of a random normalization, a random cut-off, and the argument of Step 2. Notice that the result of Step 2 is trivial in case of a finite X.

7. Processes with immigration

The aim is a theory in the spirit of [2], but with general normalizing functions, that is, a non-degenerating limit. As in [2] we adopt the superposition point of view.

In addition to our branching process we assume to be given an immigration process $\{\tau_\nu, \hat{y}_\nu, P\}$, where (τ_ν), $0 \leq \tau_\nu \uparrow \infty$ is a sequence of (not necessarily finite) random times and $(\hat{y}_\nu)_{\nu\in\mathbb{N}}$ a random sequence in $(\hat{X}, \hat{\underline{A}})$, both defined on the same probability space with measure P.

Let $\{\hat{x}_{\nu,t}; t \geq \tau_\nu\}$ be the branching process initiated at τ_ν by $\hat{y}_\nu$, and denote

$$N_t := \max\{\nu : \tau_\nu \leq t\} .$$

Define

$$\hat{x}+\hat{y} := \hat{x}, \qquad \hat{y} = \theta,$$
$$:= \langle x_1, \ldots, x_n, y_1, \ldots, y_\ell \rangle, \quad \hat{x} = \langle x_1, \ldots, x_n \rangle$$
$$\hat{y} = \langle y_1, \ldots, y_\ell \rangle .$$

The immigration-branching process $\{\hat{z}_t, \tilde{P}\}$ is then given by

$$\hat{z}_t = \sum_{\nu \leq N_t} \hat{x}_{\nu,t}$$

and the corresponding probability measure $\tilde{P}$ defined on the appropriate product space.

Let $\tilde{\underline{F}}_t$ be the σ-algebra generated by $\{\hat{z}_s; s \leq t\}$ and define

$$W_\nu := \lim_{t\to\infty} \gamma_{t-\tau_\nu} \hat{x}_{\nu,t}[\varphi],$$

$$\tilde{W}^* := \sum_{\nu=1}^{\infty} \rho^{-\tau_\nu} W_\nu .$$

(7.1) PROPOSITION. If

$$\sum_{\nu=1}^{\infty} \gamma_{\tau_\nu} \hat{y}_\nu[\varphi] < \infty,$$

then

$$\tilde{W} = \lim_{t\to\infty} \gamma_t \hat{z}_t[\varphi]$$

exists almost surely, and

$$\tilde{W} = \tilde{W}^* < \infty \quad \text{a.s. } [\tilde{P}].$$

Proof. We condition throughout on

$$\underline{I} := \sigma(\tau_\nu, \hat{y}_\nu; \nu \in \mathbb{N}).$$

Step 1. The limit $\tilde{W}^*$ always exists, but may be infinite. Since

$$\Big(\prod_{\nu\leq N_t} \hat{\xi}_t(\hat{x}_{\nu,t}), \tilde{\underline{F}}_t\Big)$$

is a positive supermartingale, $\tilde{W}$ also exists, but again may be infinite.

Step 2. We first show that $\tilde{W}$ and $\tilde{W}^*$ are finite almost everywhere. The Laplace transform of the distribution function of $\hat{z}_t[\zeta_t]$ is

$$\tilde{\Psi}_t(s) = \prod_{\nu\leq N_t} F_{t-\tau_\nu}(\hat{y}_\nu, \xi_t^s).$$

Let $\hat{y}_\nu = \langle y_\nu^1, \ldots, y_\nu^n\rangle$. For $0 < u < t < \infty$ and $0 < s \leq 1$

$$\sum_{\nu=N_u+1}^{N_t} \sum_{j=1}^{\hat{y}_\nu[1]} (1-F_{t-\tau_\nu}[\xi_t^s](y_\nu^j)) \leq \sum_{\nu=N_u+1}^{N_t} \sum_{j=1}^{\hat{y}_\nu[1]} (1-\xi_{\tau_\nu}(y_\nu^j))$$

$$\leq (1+\epsilon_u) \sum_{\nu=N_u+1}^{\infty} \gamma_{\tau_\nu} \hat{y}_\nu[\varphi],$$

where (ε_u) is a numerical null sequence. Hence, still for $0 < s \leq 1$,

$$\tilde{\Psi}(s) := \lim_{t\to\infty} \tilde{\Psi}_t(s) = \prod_{\nu=1}^{\infty} \lim_{t\to\infty} F_{t-\tau_\nu}(\hat{y}_\nu, \xi_t^s)$$

$$= \prod_{\nu=1}^{\infty} \hat{\Phi}(\rho^{-\tau_\nu} s)(\hat{y}_\nu) = \tilde{E}\exp\{-\tilde{W}^* s\} =: \tilde{\Psi}^*(s)$$

with

$$\tilde{\Psi}(0+) = \tilde{\Psi}^*(0+) = 1.$$

That is,

$$\tilde{P}(\tilde{W} < \infty) = \tilde{P}(\tilde{W}^* < \infty) = 1.$$

<u>Step 3</u>. We now show that $\tilde{W}$ and $\tilde{W}^*$ are equal almost surely. Let

$$W_{\nu,t} := \gamma_{t-\tau_\nu} \hat{x}_{\nu,t}[\varphi], \quad \tilde{W}_t := \gamma_t \hat{z}_t[\varphi].$$

Then

$$\tilde{W}-\tilde{W}^* = (\tilde{W}-\tilde{W}_t) + \sum_{\tau_\nu \leq s} (\gamma_{t-\tau_\nu}^{-1} \gamma_t W_{\nu,t} - \rho^{-\tau_\nu} W_\nu)$$

$$+ \sum_{s<\tau_\nu \leq t} \gamma_{t-\tau_\nu}^{-1} \gamma_t W_{\nu,t} - \sum_{\tau_\nu > s} \rho^{-\tau_\nu} W_\nu .$$

As $t \to \infty$, for fixed s, the first term on the right tends to zero and the third term to a finite limit $U_s \geq 0$, non-increasing in s. Next let $s \to \infty$. Then U_s tends to a finite limit $U \geq 0$, and the last term tends to zero. Thus, we have $\tilde{W} = \tilde{W}^*+U$, all three variables being finite and positive. Since the Laplace transforms $\tilde{\Psi}(s)$ and $\tilde{\Psi}^*(s)$ of $\tilde{W}$ and $\tilde{W}^*$ coincide for $s \in (0,1]$, this can only be true if $U = 0$ almost surely. □

Recall from (5.10) and its proof that there exists a sequence (θ_t) in $\bar{S}_+$ such that

$$t\,P^{\langle x\rangle}(W > t) = \theta_t(x)\int_0^t P^{\langle x\rangle}(W > u)\,du, \quad t > 0,$$

$$\lim_{t\to\infty} \theta_t(x) = 0, \quad x \in X.$$

Note further that always

$$\liminf_{t\to\infty} \tilde{W}_t \geq \tilde{W}^* \qquad \text{a.s. } [\tilde{P}].$$

(7.2) PROPOSITION. *If*

$$\limsup_{t\to\infty} \| \theta_t \| < 1 ,$$

then

$$\tilde{W}^* = \infty \quad \text{a.s.} \quad \underline{\text{on}} \quad \{ \sum_{\nu=1}^{\infty} \gamma_{\tau_\nu} \hat{y}_\nu[\varphi] = \infty \}.$$

Proof. Condition on $\underline{I}$. Then either $\tilde{W}^* < \infty$ a.s., or $\tilde{W}^* = \infty$ a.s., by Kolmogorov's zero-one law. Moreover, $\tilde{W}^* < \infty$ a.s. only if

$$\tilde{S} := \sum_{\nu=1}^{\infty} \rho^{-\tau_\nu} \hat{y}_\nu [E^{\langle\cdot\rangle} W 1_{\{\rho^{-\tau_\nu} W \leq 1\}}] < \infty ,$$

by Kolmogorov's three series criterion. Since $\gamma_t \sim \rho^{-t} L(\rho^{-t})$, cf. (3.3), and

$$\Phi^*[\int_0^t P^{\langle\cdot\rangle}(W > u)\,du] \sim L(t^{-1}), \quad t \to \infty ,$$

cf. the proof of (5.10), there exists for every $s > 0$ and $\mu \geq \mu_s$, μ_s sufficiently large, a $C_{s,\mu}$ such that

$$\tilde{S} \geq C_{s,\mu} \sum_{\nu \geq \mu} \gamma_{\tau_\nu} \frac{\hat{y}_\nu[\int_0^{\rho^{\tau_\nu}} uP^{\langle\cdot\rangle}(W \leq u)\,du]}{\Phi^*[\int_0^{\rho^{\tau_\nu}} P^{\langle\cdot\rangle}(W > \rho^s u)\,du]} .$$

Observing that

$$\int_0^t uP^{\langle x\rangle}(W \leq u)\,du = (1-\theta_t(x))\int_0^t P^{\langle x\rangle}(W > u)\,du$$

and

$$P^{\langle x\rangle}(W > u) = E^{\langle x\rangle}P^{\langle x\rangle}(W > u|F_s)$$
$$\geq E^{\langle x\rangle}\hat{x}_s[P^{\langle\cdot\rangle}(W > \rho^s u)]$$
$$\geq (\rho^s - \alpha_s)\varphi(x)\Phi^*[P^{\langle\cdot\rangle}(W > \rho^s u)],$$

we get

$$\tilde{S} \geq c_{s,\mu}(\rho^s - \alpha_s) \inf_{t \geq \rho^{\tau_\mu}} (1 - \|\theta_t\|) \sum_{\nu \geq \mu} \gamma_{\tau_\nu} \hat{y}_\nu[\varphi] .$$

By (M), we can choose s such that $\alpha_s < \rho^s$. □

REMARK. From (IF)

$$P^{\langle x\rangle}(W > t) = \int_0^t T_s^0 k \int_{\hat{X}} \pi(\cdot, d\hat{x}) P^{\hat{x}}(W > \rho^s t) ds .$$

Using this, it is easily verified that the existence of a $c_8 > 0$ such that

$$k(x) \int_{\hat{X}} \pi(x, d\hat{x}) h(\hat{x}) \leq c_8 \Phi^*[k \int_{\hat{X}} \pi(\cdot, d\hat{x}) h(\hat{x})], \quad x \in X,$$

for all bounded measurable h, is a sufficient condition for $\|\theta_t\| \to 0$.

(7.3) PROPOSITION. <u>If</u>

$$\sum_{\nu=1}^{\infty} \gamma_{\tau_\nu} \hat{y}_\nu[1] < \infty \quad \text{a.s. } [P],$$

<u>then for all a.e. continuous</u> $\eta \in B$

$$\lim_{t\to\infty} \gamma_t \hat{z}_t[\eta] = \Phi^*[\eta]\tilde{W} \quad \text{a.s. } [\tilde{P}].$$

We break the proposition up into two lemmata. Define

$$U_+ := \{\xi \in B_+ : \lim_{t\to\infty} \gamma_t \hat{x}_t[\xi] = \Phi^*[\xi]W\}.$$

(7.4) LEMMA. <u>Given a</u> $\vartheta \in \bar{S}_+$ <u>such that</u>

$$\lim_{t\to\infty} \gamma_t \hat{z}_t[\vartheta] = \Phi^*[\vartheta]\tilde{W} \qquad \text{a.s. } [\tilde{P}],$$

<u>we have for every</u> $\xi \in U_+$

$$\lim_{t\to\infty} \gamma_t \hat{z}_t[\vartheta\xi] = \Phi^*[\vartheta\xi]\tilde{W} \qquad \text{a.s. } [\tilde{P}].$$

<u>Proof</u>. The argument is essentially the same as in [2]: From the definition of $\hat{z}_t$ as a superposition

$$\liminf_{t\to\infty} \gamma_t \hat{z}_t[\eta] \geq \Phi^*[\eta]\tilde{W}$$

for $\eta \in B_+$, in particular, for $\eta = \vartheta\xi$. Clearly $\xi \in U_+$ and $\xi(1-\vartheta) \in U_+$. Thus

$$\limsup_{t\to\infty} \gamma_t \hat{z}_t[\vartheta\xi] = \lim_{t\to\infty} \gamma_t \hat{z}_t[\vartheta] - \liminf_{t\to\infty} \gamma_t \hat{z}_t[\vartheta(1-\xi)]$$

$$\leq \Phi^*[\vartheta]\tilde{W} - \Phi^*[\vartheta(1-\xi)]\tilde{W} = \Phi^*[\vartheta\xi]\tilde{W}. \qquad \square$$

This already proves (7.3) if $\inf \varphi > 0$, as is the case if and only if $\{\beta > 0\} = \partial\Omega$. In that case we simply take $\vartheta := \varphi$.

(7.5) LEMMA. <u>If</u>

$$\sum_{\nu=1}^{\infty} \gamma_{\tau_\nu} \hat{y}_\nu[1] < \infty \quad \text{a.s. } [P],$$

<u>then</u>

$$\limsup_{t\to\infty} \gamma_t \hat{z}_t[1] \leq \Phi^*[1]\tilde{W} \qquad \text{a.s. } [\tilde{P}].$$

Proof. Similarly as in [2] we first consider discrete skeletons, assuming τ_ν to take its values on the time skeleton, and then reduce the continuous time case to the treatment of skeletons. As in the preceding section we work with a random cutoff.

Part I. First let τ_ν be integer valued. Define

$$\tilde{\beta}_n := \hat{z}_n[\varphi], \quad \tilde{1}_n := \{\tilde{\beta}_n > 0\},$$

$$z_{n,\ell} := \sum_{\nu=1}^{N_n-1} \hat{x}_{\nu,n+\ell}[1] = \sum_{j=1}^{z_{n,0}} z^j_{n,\ell},$$

$$\tilde{S}_{n,\ell} := \tilde{1}_{n-1}\tilde{\beta}^{-1}_{n-1} z_{n,\ell}, \quad \tilde{S}^*_{n,\ell} := 1_{n-1}\tilde{\beta}^{-1}_{n-1} \sum_{j=1}^{z_{n,0}} z^j_{n,\ell} 1_{\{z^j_{n,\ell} \le \tilde{\beta}_{n-1}\}},$$

$$\Delta_{n,\ell} := \tilde{1}_{n+\ell}\tilde{\beta}^{-1}_{n+\ell} \sum_{n-1<\tau_\nu\le n+\ell} \hat{x}_{\nu,n+\ell}[1].$$

Then

$$\tilde{1}_{n+\ell}\tilde{\beta}^{-1}_{n+\ell}\hat{z}_{n+\ell}[1] \le \tilde{1}_{n+\ell}\tilde{\beta}^{-1}_{n+\ell}[(\tilde{S}_{n,\ell}-\tilde{E}(\tilde{S}^*_{n,\ell}|\tilde{F}_n)) + \tilde{E}(\tilde{S}_{n,\ell}|\tilde{F}_n)] + \Delta_{n,\ell}.$$

Step 1. As in the proof of (6.1), step 1,

$$\lim_{n\to\infty}(\tilde{S}_{n,\ell}-\tilde{E}(\tilde{S}^*_{n,\ell}|\tilde{F}_n)) = 0 \quad \text{a.s. } [\tilde{P}], \ \ell > 0.$$

Step 2. Define

$$\Delta^*_{n,\ell} := \gamma_{n+\ell}\tilde{\beta}_{n+\ell}\Delta_{n,\ell}.$$

Then

$$\sum_{n=1}^{\infty} \tilde{E}(\Delta^*_{n,\ell}|\underline{I}) \le \sum_{n=1}^{\infty} \gamma_{n+\ell} \sum_{n-1<\tau_\nu\le n+\ell} \hat{y}_\nu[M_{n+\ell-\tau_\nu}[1]]$$

$$\le c_9(\ell+1) \sum_{\nu=1}^{\infty} \gamma_{\tau_\nu} \hat{y}_\nu[1].$$

That is, $\Delta^*_{n,\ell} \to 0$ a.s., and thus

$$\lim_{n\to\infty} \Delta_{n,\ell} = 0 \quad \text{a.s. } [\tilde{P}], \ \ell > 0.$$

Step 3. Using (M), we have

$$\limsup_{n\to\infty} \tilde{1}_{n+\ell}\tilde{\beta}^{-1}_{n+\ell}\tilde{\beta}_{n-1}\tilde{E}(\tilde{S}_{n,\ell}|\tilde{F}_n)$$

$$= \limsup_{n\to\infty} \tilde{1}_{n+\ell}\tilde{\beta}^{-1}_{n+\ell}\hat{z}_n[M_\ell[1]]$$

$$\leq (1+\rho^{-\ell}\alpha_\ell)\Phi^*[1], \quad \ell > 0 .$$

Combining steps 1 to 3 yields

$$\limsup_{n\to\infty} \tilde{\beta}^{-1}_n \hat{z}_n[1] \leq \Phi^*[1] \qquad \text{a.s. } [\tilde{P}].$$

Part II. We now return to general τ_ν. For every $\epsilon > 0$ the process $\{\hat{z}_{n\epsilon}\}$ can be considered as discrete-time process with immigration

$$\tau^*_\nu := ([\tau_\nu/\epsilon]+1)\epsilon, \quad \hat{y}^*_\nu := \hat{x}_{\nu,\tau^*_\nu} .$$

Doing this from now on, define

$$Z^\epsilon := \hat{x}_\epsilon[1]+\#\{t:\hat{x}_{t-}[1] > \hat{x}_t[1];\ 0 < t \leq \epsilon\},$$

$$\tilde{Z}^{\epsilon,j}_n := \hat{z}^{n\epsilon,j}_{(n+1)\epsilon}[1]+\#\{t:\hat{z}^{n\epsilon,j}_{t-}[1] > \hat{z}^{n\epsilon,j}_t[1]; n\epsilon < t \leq (n+1)\epsilon\}$$

Step 1. Repeating step 1 of part I, with $z^j_{n,\ell}$ replaced by $\tilde{Z}^{\epsilon,j}_n$ and $\ell = 1$,

$$\limsup_{t\to\infty} \tilde{\beta}^{-1}_t \hat{z}_t[1] \leq \limsup_{t\to\infty} \tilde{\beta}^{-1}_t \sum_{j=1}^{\hat{z}_{n\epsilon}[1]} \tilde{Z}^{\epsilon,j}_n$$

$$\leq \limsup_{n\to\infty} \tilde{\beta}^{-1}_{n\epsilon}\hat{z}_{n\epsilon}[E^{\langle\cdot\rangle}Z^\epsilon] \quad \text{a.s. } [\tilde{P}] \ \text{ on } \ \{\tilde{W} > 0\}.$$

Step 2. We have

$$\tilde{E}(\hat{y}^*_\nu[1] \mid \underline{I}) \leq \| E^{\langle \cdot \rangle} Z^\epsilon \| \hat{y}_\nu[1],$$

$$\sum_{\nu=1}^{\infty} \gamma_{\tau^*_\nu} \hat{y}^*_\nu[1] < \infty .$$

Hence, according to part I,

$$\limsup_{n\to\infty} \tilde{\beta}^{-1}_{n\epsilon} \hat{z}_{n\epsilon}[1] \leq \Phi^*[1] \text{ a.s. } [\tilde{P}] \text{ on } \{\tilde{W} > 0\}.$$

Step 3. Notice that

$$\| E^{\langle \cdot \rangle} Z^\epsilon \| \leq e^{\| k \| (\| m \| + 1)\epsilon} \downarrow 1, \quad \epsilon \downarrow 0 .$$

Combine steps 1 to 3 to complete the proof. □

EXAMPLE. As a special case, suppose $\tau_1, \tau_2, \ldots$ are the epochs of a renewal process and $\hat{y}_1, \hat{y}_2, \ldots$ are independent, identically distributed, and independent of $\{\tau_\nu\}$. Then, for any $\eta \in B_+$, the condition

(7.6) $$E \log^+ \hat{y}_1[\eta] < \infty$$

is sufficient, and if the mean interarrival time λ is finite, also necessary for

$$\sum_{\nu=1}^{\infty} \gamma_{\tau_\nu} \hat{y}_\nu[\eta] < \infty \quad \text{a.s. } [P].$$

This follows from the fact (cf.[2]) that (7.6) is sufficient, and if $\lambda < \infty$, also necessary for

$$\sum_{\nu=1}^{\infty} \beta^{-\nu} \hat{y}_\nu[\eta] < \infty \quad \text{a.s. } [P]$$

with any $\beta > 1$: Since $\tau_\nu/\nu \to \lambda$, as $\nu \to \infty$, and $\gamma_t = \rho^{-t}L(\rho^{-t})$, we can find a $\beta_1 > 1$ such that $\gamma_{\tau_\nu} \leq \beta_1^{-\nu}$ for sufficiently large ν, and if $\lambda < \infty$, a $\beta_2 > 1$ such that $\gamma_{\tau_\nu} \leq \beta_2^{-\nu}$ for large enough ν.

REMARK. In this section we have made no explicit use of our branching diffusion setup. We needed (M), the existence of a non-trivial sequence of backward iterates, and - only in the case of continuous time with $\inf \varphi = 0$ - the existence of $Z^t \geq \hat{x}_s[1]$, $t \geq s > 0$, such that $\| E^{\langle \cdot \rangle} Z^t \| \downarrow 1$, as $t \downarrow 0$. As in [2] the special case of a finite X has emerged as almost a triviality: Finiteness of X implies $\inf \varphi > 0$ and $\| \vartheta_t \| \to 0$, $t \to \infty$.

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