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# On the Statistical Theory of Sampling Inspection by Attributes

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# GENERALIZATION OF
# THE CLASSICAL ATTRIBUTE SAMPLING INSPECTION PLANS *)

by

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Summary.

A survey is given of the three most important models for a producer's inspection system, the LTPD, the AOQL and the IQL system. The engineering background for the models is explained and it is pointed out that the restrictions are introduced to get protection against the effects of outliers. The classical theory is generalized by introducing a continuous distribution of the process average. Asymptotic expansions for the acceptance number and the sample size as functions of the lot size, the cost parameters and the parameters of the prior distribution are given and the efficiency of the classical solution in relation to the generalized solution is discussed.

Key words. Inspection by attributes. Producer's inspection system. LTPD. AOQL. IQL. Asymptotic expansions. Prior distribution. Efficiency.

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The problem.

We shall consider the problem of lot-by-lot sampling inspection for a producer who manufactures items classified as defective or non-defective so that the quality of a lot of items is given by its fraction defective. From each lot a sample is taken without replacement and it is decided to accept or reject the lot based on the number of defectives in the sample.

We shall not discuss the problems involved in designing a system of sampling inspection for a consumer, neither shall we discuss the relations between the producer's and the consumer's inspection systems.

It seems reasonable to assume that the producer from past data knows the distribution of lot quality under normal manufacturing conditions, the prior distribution. Ideally the manufacturing process is in binomial control with a known process average, p say, which means that the probability that a lot of size N contains X defectives equals the binomial probability $b(X,N,p)$. Suppose that the process average varies at random from lot to lot. The lot quality distribution then becomes a mixed binomial

$$b_w(X,N) = \int_0^1 b(X,N,p)\,dW(p).$$

According to my experience most empirical lot quality distributions may be described rather well by a beta-binomial distribution. However, there is one important reservation to this statement, namely the occurrence of outliers. Sooner or later the process goes out of control and the product becomes of poorer quality until the change is detected and the process adjusted. Lots produced during such an out-of-control period will be called outliers and these lots are not incorporated into the description of the lot quality distribution above.

We shall also assume that the producer knows the costs of inspection and the costs of rejection per item. The term rejection is used for all the possible actions taken on lots which are not accepted. Hence, rejection may mean rectifying inspection of rejected lots so that defective items are corrected or replaced by good items.

If the producer also knows the costs of accepting a defective item we are able to find the break-even quality, $p_r$ say, defined as the fraction defective for which the costs of acceptance and the costs of rejection are equal.

We shall give a survey of the three most important models and study the consequences of generalizing the prior distribution. For simplicity we shall keep to single sampling. Some results for double and sequential sampling exist but the theory is far from complete. A single sampling plan is characterized by the sample size, n, and the acceptance number, c, so that a lot is accepted if the number of defectives in the sample is at most c, otherwise the lot is rejected.

For an infinite series of lots from a process in binomial control the average probability of acceptance then becomes $P(p) = B(c,n,p)$, where B denotes the binomial distribution function. We shall set $Q(p) = 1 - P(p)$. If the lot quality distribution is a mixed binomial distribution the average probability of acceptance becomes

$$B_W(c,n) = \int_0^1 B(c,n,p)dW(p).$$

Under the assumptions stated we may find the average costs as function of $(c,n)$ and determine the optimum plan as the one minimizing the average costs. However, we also have to take the outliers into regard and to get protection against the effects of outliers we introduce a restriction on the system and minimize under this restriction. Instead of minimizing average costs we may just as well minimize average regret defined as costs minus unavoidable costs due to the defectives produced under normal manufacturing conditions.

The regret functions we are going to consider may be written as

$$R(c,n,N) = n + (N-n)d(c,n,N)$$

for $c = -1,0,\ldots,n$, $n = 0,1,\ldots,N$, $N = 0,1,\ldots,$ where $d(c,n,N)$ represents the average decision loss per item.

We shall denote the optimum plan by $(\hat{c},\hat{n})$ and the corresponding regret as $\hat{R}(N)$ so that the efficiency of a given plan $(c,n)$ is defined as $e = \hat{R}(N)/R(c,n,N)$, $0 \leq e \leq 1$.

The restrictions considered lead to a relation between c,n and N, $n = n_{c,N}$ say, which may be used to eliminate n from the regret function. To find the optimum relation between c and N we ask the question: For what values of N is a given value of c better than a neighbouring value, c + 1 say? The answer is obtained by solving the inequality $R(c,n_{c,N},N) \leqq R(c+1,n_{c+1,N},N)$ with respect to N. The solution $N \leqq N_c$ has to be found numerically by iteration. If $N_c$ is an increasing function of c then c will be the optimum acceptance number for $N_{c-1} < N \leqq N_c$.

Since explicit solutions for $\hat{c}$ and $\hat{n}$ in terms of the lot size, the cost parameters and the parameters of the prior distribution do not exist we shall study the asymptotic solution for $N \to \infty$ to find out how the optimum plan and the average decision loss depend on the parameters. In particular we shall study the following three problems: (1) how $\hat{c}/\hat{n}$ tends to the critical quality level defined by the restriction, (2) how $\hat{n}$ increases with N, and (3) how the average decision loss per item $d(\hat{c},\hat{n},N)$ decreases with N. We shall only give the main term of the asymptotic expansions.

More details and proofs may be found in a recent book by Hald [3] . Tables of the systems of sampling plans discussed in the following have been provided by Hald and Møller [4] .

A popular procedure for determining a sampling plan is to use the methods of hypothesis testing, i.e. to specify two quality levels, $p_1$ and $p_2$ say, $p_1 < p_2$, representing satisfactory and unsatisfactory quality, respectively, and to specify $Q(p_1) = \alpha$, the producer's risk, and $P(p_2) = \beta$, the consumer's risk, $0 < \beta < 1-\alpha < 1$. This is of course not an acceptable solution of the problem unless $\alpha$ and $\beta$ are given as functions of N and the other parameters.

## The LTPD system with minimum average costs of inspection and rejection.

We shall first present the Dodge-Romig [1] model, which is based on a binomial prior distribution, and afterwards study the consequences of using a mixed binomial distribution as prior.

Consider a producer's final inspection of an infinite series of lots under circumstances where each lot retains its identity. We shall assume that the manufacturing process normally is in binomial control with process average equal to $p_1$ and that the producer to get protection against marketing outliers chooses a Lot Tolerance Percent Defectice (LTPD), 100 $p_2$ say, $p_1 < p_2$, and imposes the condition $P_H(p_2) = \beta$, where $\beta$ is small, on the system. The choice of $p_2$ is based on technological considerations whereas the choice of $\beta$ to a large extent is arbitrary. $P_H(p_2)$ denotes the probability of acceptance based on the hypergeometric distribution.

Using the costs of sampling inspection as economic unit and denoting the costs of rejection per item as $\gamma$ the average regret equals

$$R(c,n,N) = n + (N-n)\gamma Q(p_1), \quad Q(p_1) = 1 - B(c,n,p_1). \qquad (1)$$

If there were no outliers all lots should be accepted without inspection. The restriction $P_H(p_2) = \beta$ is introduced to get protection against outliers and the price to be paid for this protection is sampling inspection of all lots plus the costs of rejecting some lots of normal quality. For $\gamma = 1$ and $\beta = 0.10$ we get the Dodge-Romig LTPD system.

The OC functions for some plans satisfying the restriction $P_H(p_2) = 0.1$ have been shown on Fig. 1.

Asymptotically, i.e. for $N \to \infty$, $n \to \infty$ and $n/N \to 0$, the restriction leads to the relation

$$c = np_2 - a_1 n^{\frac{1}{2}} + a_2 + 0(n^{-\frac{1}{2}}),\ a_1 > 0 \quad \text{for } \beta < \tfrac{1}{2} \tag{2}$$

so that c/n tends to $p_2$.

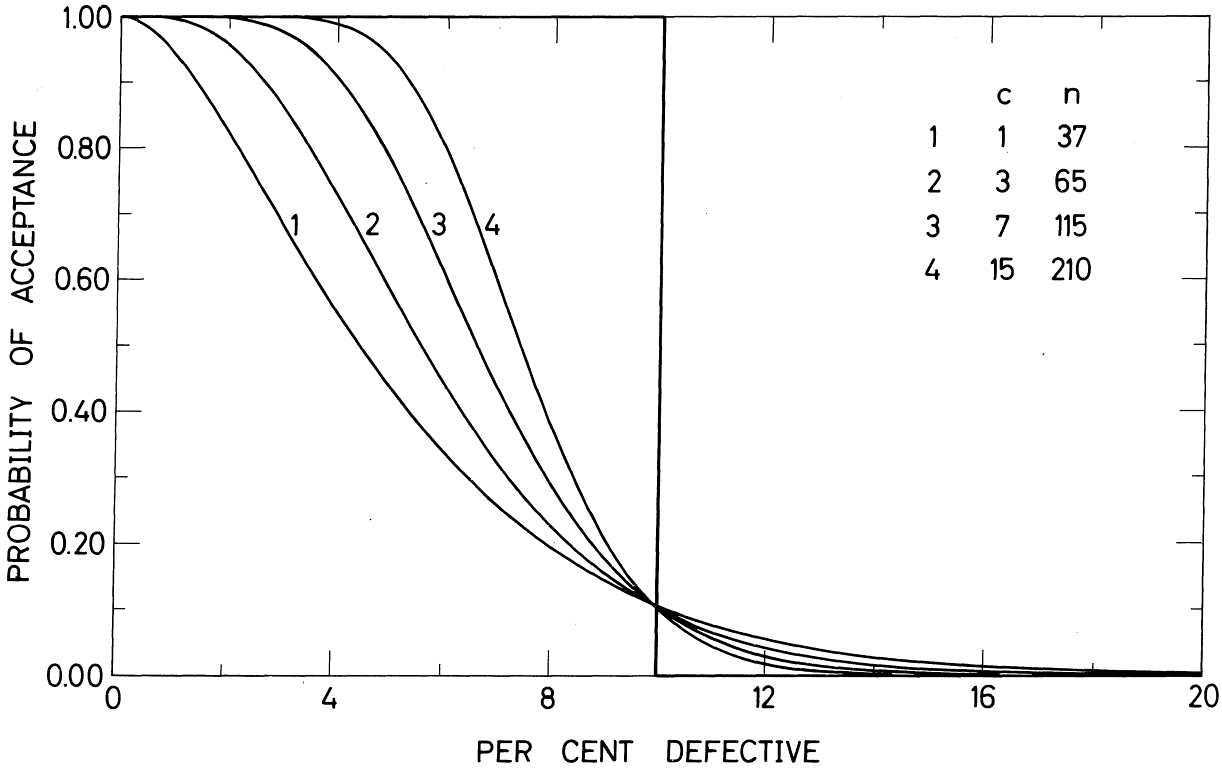


Fig. 1. Examples of OC curves from an LTPD system.

Hence, under the restriction $Q(p_1)$ tends exponentially to zero and the main term of the regret becomes

$$R \sim n + (N-n)\lambda n^{-\frac{1}{2}} e^{-n\Psi}, \tag{3}$$

where $\Psi = p_2 \ln(p_2/p_1) + q_2 \ln(q_2/q_1)$. The optimum value of n may thus be found by setting the derivative of R equal to zero and solving for n which leads to

$$\hat{n} \sim \Psi^{-1} \ln N \quad \text{and} \quad \hat{R} \sim \hat{n} + \Psi^{-1} \tag{4}$$

so that $(N-\hat{n})\,\gamma\, Q(p_1) \sim \Psi^{-1}$.

The interpretation of this result in terms of hypothesis testing is that for fixed power the size of the test should tend to zero inversely proportional to $N\gamma$.

We shall now assume that the prior distribution is a mixed binomial with a continuous distribution of p so that the average costs of inspection and rejection are obtained by integration of (1) with respect to p which gives $K = n + (N-n)\gamma(1 - B_W(c,n))$. Furthermore, we shall assume that $p_2$ belongs to the support of $W(p)$ so that for $N \to \infty$ the fraction $1 - W(p_2) = w_2$, say, of the lots should be rejected. This is a fundamental change of the model because the purpose of inspection now is twofold: (1) to reject outliers, and (2) to sort out lots of unsatisfactory quality from normal production. Deducting the costs of the unsatisfactory lots, $N\gamma w_2$, from the average costs and dividing by the constant $1 - \gamma w_2$ we get the regret in the standard form

$$R(c,n,N) = n + (N-n)\,\gamma_2\,(1 - B_W(c,n) - w_2), \quad \gamma_2 = \gamma/(1-\gamma w_2). \tag{5}$$

Under the restriction $P_H(p_2) = \beta$ it may be shown that

$$1 - B_W(c,n) - w_2 = a_1\, w(p_2)\, n^{-\frac{1}{2}} + O(n^{-1}), \tag{6}$$

where $w(\cdot)$ denotes the density of the prior distribution of p.

Hence, for $\beta < \frac{1}{2}$ the average decision loss per item is a decreasing function of n converging to zero as $n^{-\frac{1}{2}}$. Inserting into R and minimizing with respect to n we find

$$\hat{n} \sim \{\tfrac{1}{2}\, a_1\, w(p_2)\, \gamma_2 N\}^{2/3} \quad \text{and} \quad \hat{R} \sim 3\,\hat{n},$$

so that the average decision loss per lot asymptotically equals twice the regret due to sampling inspection.

The properties of the solution for a continuous prior are thus fundamentally different from the properties for a degenerate prior.

## The AOQL system with minimum average costs of inspection and rejection.

In the present section we shall assume that inspection is rectifying, i.e. all defectives found are replaced by good items, and that rejected lots are 100 per cent inspected. For an infinite series of lots from a process in binomial control with process average equal to p the Average Outgoing Quality (AOQ) then becomes

$$p_A = p\ P(p)(N-n)/N, \qquad 0 \leqq p \leqq 1.$$

Consider a producer's internal inspection where the size of the inspection lots is chosen mainly for convenience in handling and economy in inspection, for example as one hour's output of a production line, and where the inspection lots lose their identity in a common storeroom from which larger lots are delivered. Under these assumptions the producer is obviously interested in controlling the average quality of the lots in the storeroom and to that end he may specify a maximum average fraction defective to be permitted in the product without serious consequences for the consumer. Hence, the sampling plan has to satisfy the condition

$$\max_p \{p\ P(p)\,(N-n)/N\} = p_L ,$$

where $p_L$ denotes the specified Average Outgoing Quality Limit, AOQL. Under this condition the regret function (1) should be minimized. It is assumed that $p_1 < p_L$ so that the regret also here gives the price to be paid to get protection against outliers. For $\gamma = 1$, this is the AOQL model due to Dodge and Romig [2].

In the Dodge-Romig AOQL system engineering, economical and statistical concepts have been fully integrated in a natural way for the first time. (In the LTPD system the consumer's risk is arbitrary.) The two parameters $p_1$ and $p_L$ have clear engineering interpretations, the optimality criterion is chosen from economical considerations, and statistical concepts such as random sampling, statistical control, probability of acceptance, average quality and average costs enter the model in a natural way.

Fig. 2 gives the OC curves for some plans with AOQL equal to 0.03. It will be seen that c/n converges to $p_L$ from above and it may be proved that

$$c = np_L + (np_L q_L)^{\frac{1}{2}}\ (\ln\ n)^{\frac{1}{2}}\ (1 + 0(1)).$$

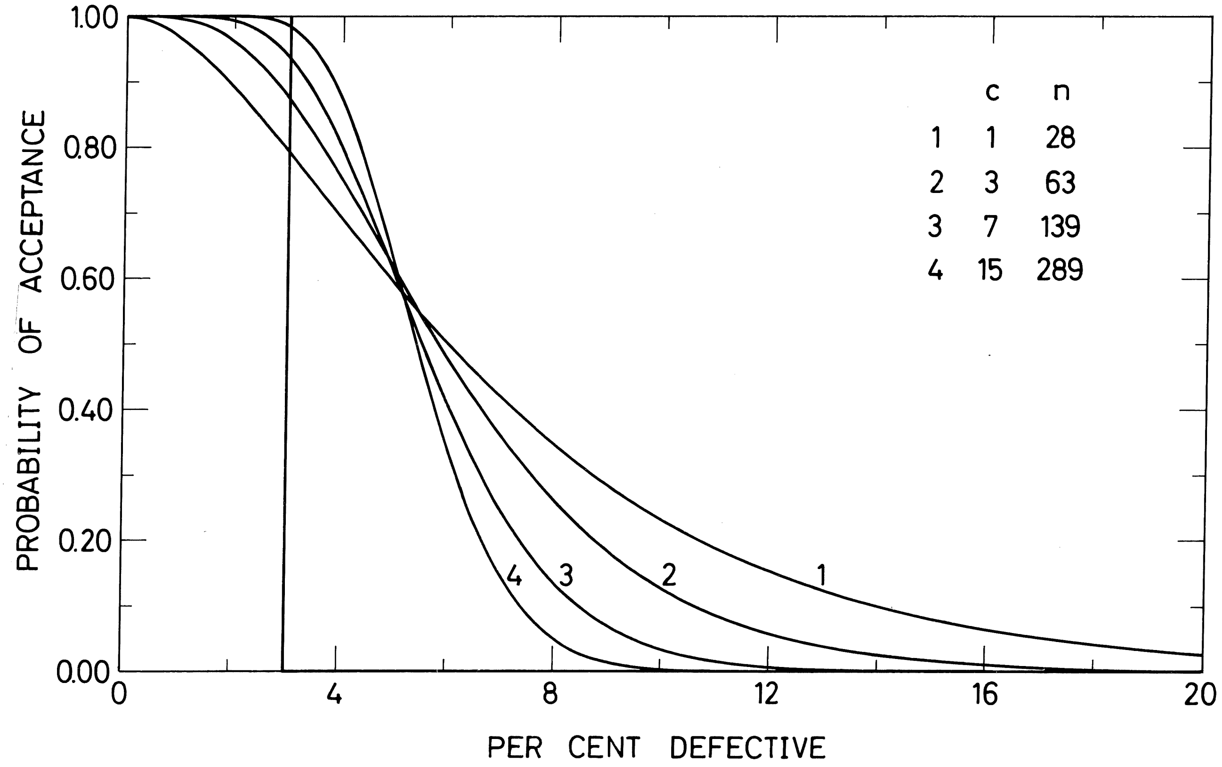


Fig. 2. Examples of OC curves from an AOQL system.

For $n \to \infty$ and $p_1 < p_L$ it follows that $Q(p_1)$ tends exponentially to zero and the regret function therefore equals (3) with $p_2$ replaced by $p_L$. Hence, the main terms of the asymptotic expansions for $\hat{n}$ and $\hat{R}$ are of the same form as for the LTPD system, see (4).

Generalizing to a continuous distribution of p with average product quality $\bar{p}$ and setting $w_L = 1 - W(p_L)$ the standardized average costs becomes equal to (5) with $w_2$ replaced by $w_L$. However, under the AOQL restriction we get

$$1 - B_W(c,n) - w_L = -w(p_L)\,(p_L q_L \ln n)^{\frac{1}{2}}\, n^{-\frac{1}{2}}\,(1 + 0(1))$$

which is an <u>increasing</u> function of n. Hence, for $N \to \infty$ the average cost attains its minimum for a <u>finite</u> value of n and $\hat{n}$ therefore becomes an increasing function of N with a finite upper limit.

To understand this (surprising) result it should be noticed that the AOQL is not a critical quality level for individual lots as the LTPD, but the AOQL relates only to average quality. If $\bar{p} < p_L$ then the prior distribution is acceptable and regardless of how many lots of poor quality there exist, i.e. regardless of the size of $w_L$, the whole output under normal manufacturing conditions should be accepted. Only the existence of outliers motivates inspection. However, to get the desired protection against outliers we do not have to reject all product with $p > p_L$ and therefore it is not necessary to let $\hat{n}$ tend to infinity.

It also follows from these considerations that (5) with $w_2$ replaced by $w_L$ is not a proper regret function because the costs of the unsatisfactory lots are not equal to $N\gamma w_L$.

## The IQL system with minimum average costs.

Consider a producer's internal inspection between two departments, the first one producing items to be used in the second, for example as pieces in assemblies. In such cases the producer will normally know the costs of accepting a defective item so that the break-even quality may be found. In the simplest case the break-even quality equals the costs of rejecting an item divided by the costs of accepting a defective item. We shall assume that the decision loss for product of quality p is proportional to $|p - p_r|$. Since the loss is symmetric about $p_r$ it seems reasonable to introduce the restriction $P(p_r) = \frac{1}{2}$, i.e. to use $p_r$ as Indifference Quality Level (IQL) and to minimize the regret function (1) with $p_1 < p_r$ under this restriction. This proposal is due to Weibull [5] .

Mathematically it may be considered as a special case of the Dodge-Romig LTPD system with $\beta = \frac{1}{2}$ and the asymptotic solution may therefore be obtained from the results in Section 2 by setting $a_1 = 0$ and replacing $p_2$ by $p_r$.

Fig. 3 shows some OC curves for an IQL system with $p_r = 0.05$.

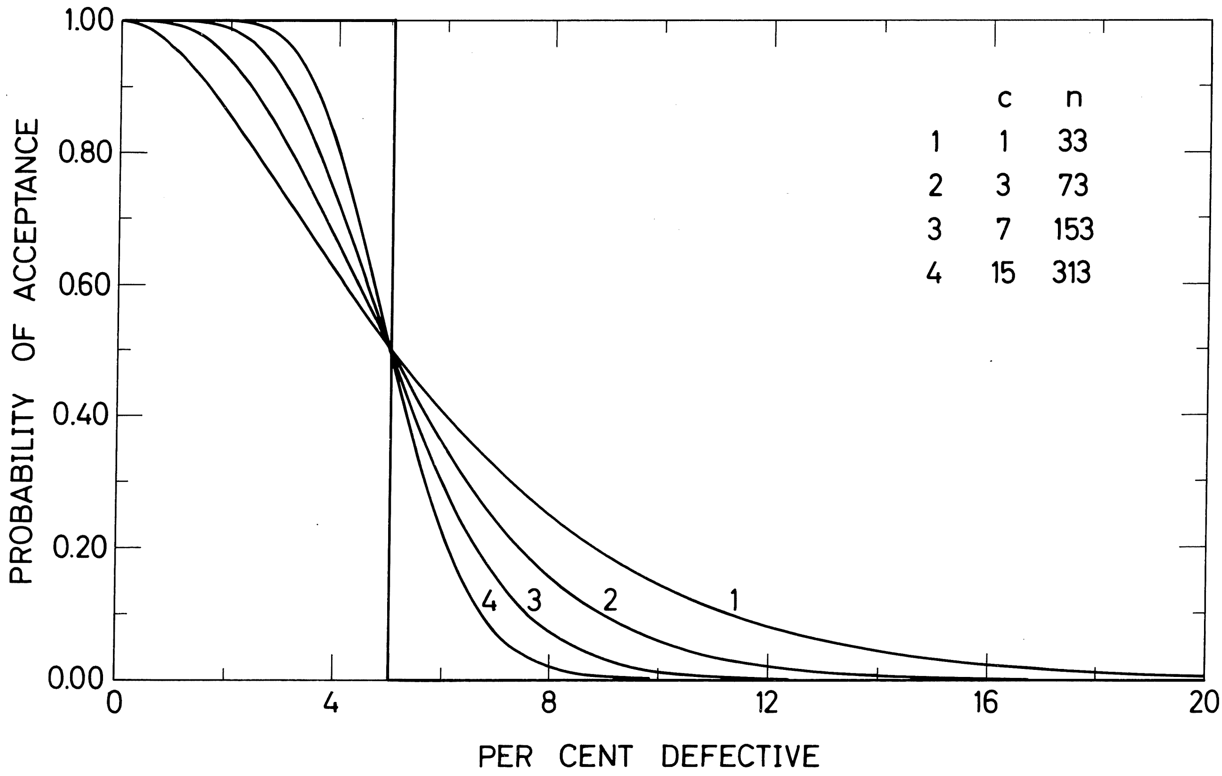


Fig. 3. Examples of OC curves from an IQL system.

For a continuous prior the regret function becomes

$$R(c,n,N) = n + (N-n)\gamma\{\int_0^{p_r} (p_r - p)Q(p)\,dW(p) + \int_{p_r}^{1} (p - p_r)P(p)\,dW(p)\}.$$

It may be proved that

$$R \sim n + (N-n)\gamma(\tfrac{1}{2}\, p_r q_r w(p_r) n^{-1})$$

which leads to

$$\hat{n} \sim (\tfrac{1}{2}\, p_r q_r w(p_r)\gamma N)^{\frac{1}{2}} \quad \text{and} \quad \hat{R} \sim 2\,\hat{n}.$$

The average decision loss per lot is thus asymptotically equal to the regret due to sampling inspection.

Minimizing the regret function with respect to both c and n without any restriction on the OC function we get the Bayesian single sampling plan for which c asymptotically is a linear function of n and the optimum values of n and R therefore have the same properties as for the IQL system above. However, for small values of N the Bayesian solution may be acceptance without inspection whereas the IQL system always leads to inspection.

## Efficiency.

We have seen that $\hat{n}$ asymptotically is proportional to ln N if the prior distribution of p is a one-point distribution and that this relationship (naturally) undergoes a drastic change if the prior instead is continuous with a support containing the critical quality level.

According to my experience the assumption of a continuous prior will usually be more realistic than the classical assumption. How can it then be that the Dodge-Romig plans have functioned rather well in practice? The answer depends on the following facts:
(1) Normally only a small portion of the prior will be above the critical quality level. (2) Inspection lot sizes are usually not very large. (3) The regret is rather insensitive to variations in sample size.

Let us discuss this in more detail for the LTPD system. Assuming that we only consider sampling plans satisfying the restriction $P_H(p_2) = \beta$ the efficiency of the plan (c,n) becomes a function of n only and asymptotically we have

$$e = 3 \Big/ \left\{ \frac{n}{\hat{n}} + 2\left(\frac{\hat{n}}{n}\right)^{\frac{1}{2}} \right\},$$

which clearly demonstrates the insensitivity mentioned. For $\frac{1}{2} < n/\hat{n} < 2$, say, the efficiency is larger than 0.88. To investigate the efficiency also for small samples we shall consider an example with a beta distribution of p defined by $\bar{p}/p_2 = 0.35$ and $w_2 = 0.05$. Fig. 4 shows $\hat{c}$ and consequently also $\hat{n}$ for $\gamma = 1$ and $\beta = 0.10$ as function of $Np_2$. (The gamma-Poisson approximation has been used.)

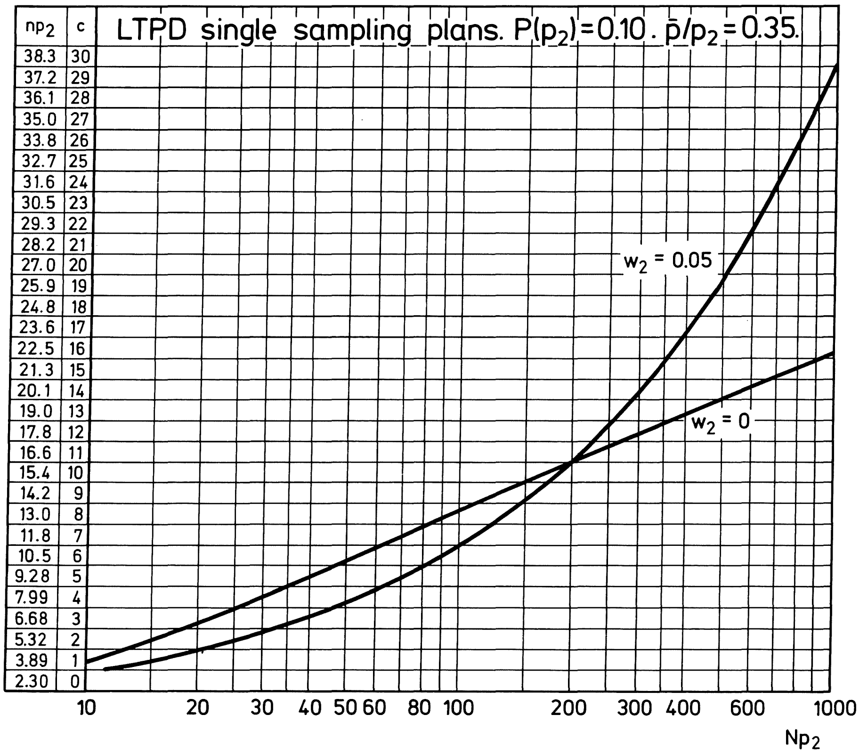


Fig. 4. The optimum LTPD single sampling plan (c,n) as function of $Np_2$ for a continuous prior with $w_2 = 0.05$ and for a one-point prior, $w_2 = 0$.

It will be seen that the two curves intersect one another for $Np_2 = 200$, $\hat{n}p_2 = 15.4$ and $\hat{c} = 10$. The efficiency of the Dodge-Romig solution is between 0.94 and 1.00 for $10 < Np_2 < 200$. For $Np_2 > 200$ the efficiency decreases towards zero being 0.90 and 0.81 for $Np_2 = 1000$ and 2000, respectively. Hence, for lots of small or medium size the efficiency of the Dodge-Romig solution will be rather large.

## Conclusions.

Besides the more specific results obtained above the following conclusions are made.

(1) It is necessary to introduce a restriction on the system to get protection against outliers and to make sure that the solution always leads to inspection.

(2) The three inspection problems described lead to different models and to systems of sampling plans with essentially different properties. In practice all three problems may occur within the same plant. Hence, it is not advisable to use a general-purpose system of sampling plans.

(3) Given that the correct model has been chosen we have shown that the regret function is rather insensitive to variations in the sample size. It is more important to choose the correct model than to have precise knowledge of the parameters within the model.

(4) The hypothesis-testing approach should only be used as a last resort and the two risks should be made dependent on the lot size and the cost parameters.

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