Søren Asmussen Thomas Kurtz

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Abstract

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Relationships between the growth of a sequence N_k and conditions on the tail of the distribution of a sequence X_ℓ of i.i.d. mean zero random variables are given that are necessary and sufficient for

$$\sum_{k=1}^{\infty} P\{|\frac{1}{N_k}\sum_{\ell=1}^{N_k} X_\ell|>\varepsilon\}<\infty.$$

The results are significant for distributions satisfying $E(|X_\ell|)<\infty$ but $E(|X_\ell|^\beta)=\infty$ for some $\beta>1$.

Necessary and sufficient conditions for the finiteness of sums of the form

$$\sum_{n=1}^{\infty}\gamma(n)P\{|\frac{1}{n}\sum_{\ell=1}^{n} X_\ell|>\varepsilon\}$$

are obtained as a corollary.

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Necessary and sufficient conditions for complete convergence in the law of large numbers

Søren Asmussen
Institute of Mathematical Statistics
University of Copenhagen
Copenhagen, Denmark

Thomas G. Kurtz
Department of Mathematics
University of Wisconsin-Madison
Madison, Wisconsin 53706

1. Introduction.

Let $X_1, X_2, X_3, \ldots$ be a sequence of independent identically distributed random variables with

(1.1) $$E|X_n|<\infty,\ EX_n=0$$

and define $F(t)=P(|X_n|\geq t)$, $p(n,\varepsilon)=P(|X_1+\ldots+X_n|/n>\varepsilon)$. We are interested in relationships between the growth of a sequence N_k and conditions on the tail F of the X_ℓ that imply

(1.2) $$\sum_{k=1}^{\infty} p(N_k,\varepsilon)<\infty \ \underline{\text{for}}\ \underline{\text{all}}\ \varepsilon>0.$$

Finiteness of the sum in (1.2) is an indication of the rate of convergence (termed "complete convergence" by Hsu and Robbins (1947)) of the summands to zero and is the necessary and sufficient condition (by Borel-Cantelli) for the almost sure convergence of a triangular array. The sum can also be interpreted as the expected number of times $|\Sigma_1^{N_k} X_\ell|>\varepsilon N_k$. Questions of this type arise in the study of sums of independent random variables indexed by lattices and more general partially ordered sets, for example Smythe (1974), as well as in the study of supercritical branching processes, for example Asmussen (1978) and Athreya and Kaplan (1976). In that setting, the sequence is replaced by a triangular array and $N_1, N_2, \ldots$ are the generation sizes of the branching process.

In Theorems A and B $\psi(x)$ will be a strictly increasing function on $[0,\infty)$ satisfying $\sup_{x\geq 1} \psi(x+1)/\psi(x)<\infty$ such that $\psi^{-1}(x)$ is absolutely continuous with derivative $\gamma(x)$ and for some θ and some $C>0$ $\gamma(x)$ satisfies $\gamma(yx)\leq Cy^{\theta}\gamma(x)$ for $x\geq 0$ and $y\geq 1$. Note that for some C_1, C_2 and some $\alpha,\beta>0$, $C_1 x^{\alpha}\leq\psi(x)\leq C_2 e^{\beta x}$. Also the requirement on γ is automatic (with $\theta=0$) if ψ is convex and also if $\psi(x)=x^{\alpha}$, $\alpha>0$.

<u>Theorem</u> A. <u>Let</u> t_k, N_k <u>satisfy</u>

(1.3) $$\lim_{k\to\infty} N_k/\psi(t_k) = 1 \quad .$$

<u>If</u> $\inf_k t_k - t_{k-\ell} \equiv a>0$ <u>for some</u> $\ell>0$ <u>and</u>

(1.4) $$\int_0^{\infty} t\gamma(t)F(t)dt = Eg(|X_\ell|)<\infty$$

<u>where</u> $g(t)=\int_0^t s\gamma(s)ds$, <u>then</u> (1.2) <u>holds</u>. <u>Conversely</u>, <u>if</u> $\sup_k t_k - t_{k-1} \equiv A<\infty$ <u>and</u> (1.2) <u>holds</u>, then (1.4) <u>holds</u>

<u>Corollary</u> I. (1.4) <u>holds if and only if</u> (1.2) <u>holds for some</u> [and then every] <u>sequence</u> N_k <u>satisfying</u> $\lim_{k\to\infty} N_k/\psi(hk)=1$ <u>for some</u> $h>0$. <u>Also</u>, (1.4) <u>is equivalent to the existence of sequences</u> t_k, N_k <u>satisfying</u> (1.2), (1.3) <u>and</u> $\lim_{k\to\infty} t_{k+1}-t_k=0$.

Corollary II. (1.4) *holds if and only if*

(1.5) $$\sum_{n=0}^{\infty} (\psi^{-1}(n+1)-\psi^{-1}(n))p(n,\varepsilon)<\infty \quad \textit{for all } \varepsilon>0.$$

Theorem B. *Define* $H(t)=\sup_{s\geq t} sF(s)$. *Then* (1.2) *holds for every sequence* N_k *satisfying*

(1.6) $$\inf_k N_k/\psi(hk)>0 \quad \textit{for some } h>0$$

if and only if

(1.7) $$\int_0^{\infty} \gamma(t)H(t)dt<\infty.$$

Remark. One might expect that if (1.2) holds and $M_k>N_k$, then $\Sigma p(M_k,\varepsilon)<\infty$ for all $\varepsilon>0$. A comparison of Theorems A and B shows that this is not in general the case. However, roughly speaking, moment assumptions only slightly stronger than (1.1) will ensure that (1.4) and (1.7) are the same condition and in fact both equivalent to

(1.8) $$\int_0^{\infty} \psi^{-1}(t)F(t)dt<\infty.$$

More precisely, always (1.8)$\Rightarrow$(1.7)$\Rightarrow$(1.4) and (1.4)$\Leftrightarrow$(1.8) for any concave ψ and also e.g. for $\psi(x)=x^{\beta}$, $\beta>0$. We verify these implications below. For an example where (1.4) holds but not (1.7), let $\gamma(t)=\frac{1}{t}$ and let the X_ℓ be concentrated on $\{\pm 2,\pm 2^4,\ldots,\pm 2^{j^2},\ldots\}$ with $P(X_\ell=2^{j^2})=P(X_\ell=-2^{j^2})=c/j^2 2^{j^2}$.

Example. In connection with the branching process example above, we note the following particular cases. If $N_k/wm^k\to 1$ with $m>1$, $0<w<\infty$, then (1.2) holds without further assumptions that (1.1) (Corollary I). This may fail if it is only known that $N_k\geq w_1 m_1^k$ with $0<w_1<\infty$, $m_1>1$, but here at least the condition $E|X_\ell|\log^+|X_\ell|<\infty$ is sufficient for (1.2) (Theorem B and the remark). However, if $\underline{\lim}\, N_{k+1}/N_k\geq m_2>1$, then again (1.1) is sufficient for (1.2)

(Theorem A with $\psi(x)=e^x$, $t_k=\log N_k$).

There are many results in the literature related to those given here (see the review paper by Hanson (1970) and Heathcote (1967)). In particular, Katz (1963) proved that for $\lambda \geq 1$

$$(1.9) \qquad \sum_{n=1}^{\infty} n^{\lambda-2} p(n,\varepsilon) < \infty \qquad \underline{\text{for every}}\ \varepsilon>0$$

if and only if $E|X_\ell|^\lambda<\infty$, thereby generalizing earlier results of Spitzer (1956), Erdös (1949) and Hsu and Robbins (1947). This result is the special case of Corollary II with $\psi(x)=x^{1/(\lambda-1)}$ $\lambda>1$ and $\psi(x)=e^x$ for $\lambda=1$. More recently Smythe (1974) has given a result similar to the first part of Theorem A but with more restrictions on ψ and on the relationship between ψ and N_k. In connection with the idea of replacing $tF(t)$ by $H(t)$ in Theorem B, see also Franck and Hanson (1966), and in connection with the last part of Corollary I, Dvoretzky (1949).

2. Proofs. The results are based on the following theorems from Kurtz (1972).

Theorem C. Let $X_1, X_2, \ldots$ be independent random variables with mean zero and let $S_m=\sum_{k=1}^{m} a_k X_k$ for some sequence $a_k \geq 0$. Let $F(t)=\sup_k P\{|X_k|\geq t\}$ and define $\eta=\sup_k a_k \int_{1/a_k}^{\infty} F(t)dt$ and $e=\sum_k a_k \int_{1/a_k}^{\infty} F(t)dt$.

(a) If $\eta<2\delta$ and $0\leq\alpha\leq 1$, then

$$P\{\sup_m |S_m|>\delta+e\} \leq (\alpha+1)\left[\frac{4}{(2\delta-\eta)^{\alpha+1}}+1\right]\sum_k \int_0^1 u^\alpha F(u/a_k)du.$$

(b) Let $\varepsilon>0$ and $L>1$. If

$$\alpha\leq 1+\varepsilon(L-1), \quad \eta<\delta/2^L, \quad \sum_k \int_0^1 u^{1-\varepsilon}F(u/a_k)du \leq M < \infty,$$

then

$$P\{\sup_m |S_m| > \delta + e\}$$
$$\leq \left(\sum_k \int_0^1 u^\alpha F(u/a_k)\,du \right) (\alpha+1) \left[1 + \sum_{\ell=1}^{L} \frac{(\alpha(\alpha+1)M)^{\ell-1}}{\prod_{m=0}^{\ell-1} (\delta/2^{m+1}-\eta)^{\alpha+1}} + \frac{\alpha(\alpha(\alpha+1)M)^{L-1}}{\prod_{m=0}^{L-1} (\delta/2^{m+1}-\eta)^{\alpha+1}} \right].$$

Note that

$$\sum_k \int_0^1 u^{1-\varepsilon} F(u/a_k)\,du = \sum_k a_k \int_0^{1/a_k} (a_k v)^{1-\varepsilon} F(v)\,dv \leq \sum_k a_k \int_0^\infty F(v)\,dv.$$

Consequently when $\sum_k a_k=1$, for example, we may take $\varepsilon=1$ and replace M by $\int_0^\infty F(v)\,dv$.

<u>Corollary</u> III. <u>Let</u> $a_k=1/N$, $k=1,\dots,N$ <u>and</u> $a_k=0$ <u>for</u> $k>N$. <u>Let</u> $\eta=\frac{1}{N}\int_N^\infty F(t)\,dt$, $e=\int_N^\infty F(t)\,dt$ <u>and</u> $M=\int_0^\infty F(t)\,dt$.

(a) If $\eta<2\delta$ <u>and</u> $\alpha\leq 1$, <u>then</u>

$$P\{\sup_m |S_m| > \delta+e\} \leq CN \int_0^1 u^\alpha F(uN)\,du$$

<u>where</u> C <u>depends</u> <u>on</u> α,δ <u>and</u> η, <u>and</u> <u>is</u> <u>increasing</u> <u>in</u> η.

(b) <u>Let</u> $L>1$, $\alpha\leq 1+(L-1)$ <u>and</u> $\eta<\delta/2^L$. <u>Then</u>

$$P\{\sup_m |S_m| > \delta+e\} \leq CN \int_0^1 u^\alpha F(uN)\,du$$

<u>where</u> C <u>depends</u> <u>on</u> α, δ, M <u>and</u> η <u>and</u> <u>is</u> <u>increasing</u> <u>in</u> η.

We start by proving the first part of Theorem A. Taking $\alpha=2+\theta$ in Corollary III, (1.2) is implied by

$$\sum N_k \int_0^1 u^{2+\theta} F(N_k u)\,du < \infty. \tag{2.1}$$

Let $M=\sup N_k/\psi(t_k)$, $m=\inf N_k/\psi(t_k)$. Since we can break the sum in (1.2) into ℓ parts, without loss of generality we can assume $\ell=1$.

Furthermore we can assume $t_1=2$ and hence

$$\sup_k \sup_{t_k-1\le t\le t_k} \frac{\psi(t_k)}{\psi(t)} \equiv K<\infty.$$

Then (2.1) is bounded by

$$M\sum_k \psi(t_k)\int_0^1 u^{2+\theta}F(mu\psi(t_k))\,du = M\int_0^1 u^{2+\theta}\sum_k \psi(t_k)F(mu\psi(t_k))\,du$$

(2.2)
$$\le M\int_0^1 u^{2+\theta}\frac{1}{a}\int_{t_{k-1}}^{t_k} \frac{\psi(t_k)F(mu\psi(t))}{\nu(t_k-1)}\,dt\,du$$

$$\le \frac{KM}{a}\int_0^1 u^{2+\theta}\sum \int_{t_{k-1}}^{t_k} \frac{\psi(t)F(mu\psi(t))}{\nu(t_k-1)}\,dt\,du$$

$$\le \frac{KM}{a}\int_0^1 u^{2+\theta}\int_0^\infty \psi(t)F(mu\psi(t))\,dt\,du.$$

Substituting $x=mu\psi(t)$ the inner integral becomes

(2.3) $$\int_0^\infty \frac{x}{m^2u^2}\,\gamma\left(\frac{x}{mu}\right)F(x)\,dx \le C\int_0^\infty \frac{x}{m^{2+\theta}u^{2+\theta}}\,\gamma(x)F(x)\,dx$$

and finiteness of (2.1) follows.

To prove the sufficiency of (1.7) in Theorem B we bound (2.1) by

(2.4) $$\sum_k \int_0^1 u^{\theta+1}H(N_k u)\,du$$

and then using the monotonicity of H approximate (2.4) by

(2.5)
$$\sum_k \int_0^1 u^{\theta+1}H(\psi(k)u)\,du \approx \int_0^1 u^{\theta+1}\int_0^\infty H(\psi(x)u)\,dx\,du$$

$$= \int_0^1 u^{\theta+1}\int_0^\infty \frac{1}{u}\gamma\left(\frac{t}{u}\right)H(t)\,dt\,du \le C\int_0^1\int_0^\infty \gamma(t)H(t)\,dt\,du < \infty.$$

The proof of necessity is based upon

<u>Lemma</u>. (i) <u>Let</u> $Y_1,\ldots,Y_N$ <u>be</u> <u>independent</u> <u>and</u> <u>symmetric</u>, $T_n=Y_1+\ldots+Y_n$. <u>Then</u>

(2.6) $$P(\max_{n=1,\ldots,N}|Y_n|>\varepsilon)\le 4P(|T_N|>\varepsilon).$$

(ii) (1.2) <u>implies that</u>

(2.7) $$\sum_{k=1}^{\infty} N_k F(N_k \varepsilon) < \infty \quad \underline{\text{for all}}\ \varepsilon > 0.$$

<u>Proof</u>. (i) Let $A_n^+ = \{Y_n > \varepsilon, |Y_\ell| \le \varepsilon\ \ \ell = 1,\dots,n-1\}$, $A_n^- = \{Y_n < \varepsilon, |Y_\ell| \le \varepsilon\ \ \ell=1,\dots,n-1\}$, $A_n = A_n^+ + A_n^-$. Since the Y_ℓ, $\ell \neq n$, are independent and symmetric conditioned upon A_n^+, A_n^-,

$$P(|T_N| > \varepsilon) \ge$$

$$\sum_{n=1}^{N} \{P(A_n^+, T_{n-1} \ge 0, T_N - T_n \ge 0) + P(A_n^-, T_{n-1} \le 0, T_N - T_n \le 0)\} \ge$$

$$\sum_{n=1}^{N} \frac{1}{4}\{PA_n^+ + PA_n^-\} = \frac{1}{4} \sum_{n=1}^{N} PA_n = \frac{1}{4} P(\max_{n=1,\dots,N} |Y_n| > \varepsilon)$$

(ii) Let $F^s, p^s(n,\varepsilon)$ be obtained from $F, p(n,\varepsilon)$ by symmetrization. Since

$$F(N_k(\varepsilon + \frac{|\text{med } X_\ell|}{N_k})) \le 2F^s(N_k \varepsilon), \quad p^s(n,\varepsilon) \le 2p(n,\varepsilon/2),$$

we can without loss of generality assume the X_ℓ to be symmetric. From $nF(n\varepsilon) \to 0$ we have

$$\lim_{n\to\infty} P(\max_{\ell=1,\dots,n} |X_\ell|/n > \varepsilon)/nF(n\varepsilon) = 1.$$

Thus (2.7) follows from (1.2) and (2.6) since

$$\sum_{k=1}^{\infty} P(\max_{\ell=1,\dots,N_k} |X_\ell|/N_k > \varepsilon) \le 4 \sum_{k=1}^{\infty} p(N_k, \varepsilon) < \infty .$$

To obtain the second part of Theorem A let $\kappa = \sup_{t \ge t_1} \sup_{0 \le S \le A} \frac{\psi(t+S)}{\psi(t)}$. Then

$$\sum_k N_k F(N_k/M) \ge m \sum_k \psi(t_k) F(\psi(t_k)) \ge \frac{m}{A} \sum_k \int_{t_k}^{t_{k+1}} \psi(t_k) F(\psi(t))\,dt$$

$$\ge \frac{m}{A\kappa} \int_{t_1}^{\infty} \psi(t) F(\psi(t))\,dt = \frac{m}{A\kappa} \int_{\psi^{-1}(t_1)} x\gamma(x) F(x)\,dx.$$

Clearly (1.4) is necessary in Theorem B as well. Therefore to verify the necessity of (1.7) we may assume (1.4) holds but that

$$\int_0^\infty \gamma(t)H(t)dt = \infty.$$

Let $T_1=\{t:H(t)\neq tF(t)\}$ and $T_2=\{t:H(t)=tF(t)\}$. Then

$$\int_0^\infty \gamma(t)H(t)dt = \int_{T_1} \gamma(t)H(t)dt+\int_{T_2} \gamma(t)tF(t)dt = \infty.$$

Since the second term on the right is finite the first is infinite.

For every $t\in T_1$ there is a $s_t>t$ such that $H(u)=s_tF(s_t)$ for $t\leq u\leq s_t$. Since $H(t)$ is left continuous T_1 is a union of intervals $(t_n,s_{t_n})\equiv(t_n,s_n)$ on which $H(t)$ is a constant (i.e. $s_nF(s_n)$).
Therefore

$$\int_{T_1} \gamma(t)dt = \Sigma s_nF(s_n)\int_{t_n}^{s_n}\gamma(t)dt$$

$$= \Sigma s_nF(s_n)(\psi^{-1}(s_n)-\psi^{-1}(t_n)) = \infty.$$

We cannot assume $t_n\geq s_{n-1}$, but we can select a subset of the intervals $(t_{n_i},s_{n_i})\equiv(a_i,b_i)$ such that $a_i\geq b_{i-1}$ and

$$\Sigma b_iF(b_i)(\psi^{-1}(b_i)-\psi^{-1}(a_i)) = \infty.$$

Furthermore since

$$(\psi^{-1}(b_i)-\psi^{-1}(a_i))b_iF(b_i) \leq \frac{b_i}{a_i}\int_{a_i}^{b_i}t\gamma(t)F(t)dt$$

we are able to select the intervals so that $\lim_{i\to\infty} b_i/a_i=\infty$.

We obtain the desired sequence by defining $N_k=[b_i]$ for

$$\psi^{-1}([b_i]) \geq k > \psi^{-1}([b_{i-1}]).$$

and observe that

$$\Sigma N_kF(N_k) \geq \Sigma(\psi^{-1}([b_i])-\psi^{-1}([b_{i-1}])-1)\cdot[b_i]F([b_i])$$

$$\geq \Sigma(\psi^{-1}([b_i])-\psi^{-1}(a_i)-1)\cdot[b_i]F([b_i]) = \infty.$$

The first part of Corollary I is immediate and also the sufficiency of $t_{k-1}-t_k\to 0$ in the second. For the necessity, choose $0<\kappa(t)\uparrow\infty$ such that

$$\int_0^\infty t\gamma^*(t)F(t)dt \equiv \int_0^\infty t\kappa(t)\gamma(t)F(t)dt < \infty. \tag{2.8}$$

Define ψ^* as the inverse of $\int_0^t\gamma^*(s)ds$. It is not difficult to see that κ may be chosen such that γ^* and ψ^* satisfy the conditions on γ and ψ. Define $t_k=\psi^{-1}(\psi^*(k))$, $N_k=[\psi(t_k)]=[\psi^*(k)]$. Then (1.3) holds and (2.8), $N_k/\psi^*(k)\to 1$ implies (1.2). Finally,

$$t_{k+1}-t_k = \int_{\psi^*(k)}^{\psi^*(k+1)} \gamma(t)dt = o\Big(\int_{\psi^*(k)}^{\psi^*(k+1)} \gamma^*(t)dt\Big) = o(1).$$

For Corollary II, define $\Gamma_k=\{n:\psi(k)-1\le n<\psi(k+1)\}$. Let N_k, $N_k'\in\Gamma_k$ satisfy

$$p(N_k',\varepsilon) \le p(n,\varepsilon) \le p(N_k,\varepsilon)$$

for all $n\in\Gamma_k$. Let $a_k^n=(\psi^{-1}(n+1)\wedge(k+1))\vee k-(\psi^{-1}(n)\wedge(k+1))\vee k$. Note that

$$\Gamma_k = \{n:a_k^n>0\},\quad \sum_k a_k^n = \psi^{-1}(n+1)-\psi^{-1}(n),\quad \sum_n a_k^n = 1.$$

From this we have

$$\begin{aligned}\sum_k p(N_k',\varepsilon) &\le \sum_{k,n} a_k^n p(n,\varepsilon)\\ &= \sum_n(\psi^{-1}(n+1)-\psi^{-1}(n))\,p(n,\varepsilon) \le \sum_k p(N_k,\varepsilon).\end{aligned}\tag{2.9}$$

Since

$$\psi(k)-1 \le N_k \le \psi(k+1)$$

there exist $k\le t_k\le k+1$ such that $\lim_{k\to\infty} N_k/\psi(t_k)=1$. Since $\inf_k t_k-t_{k-2}\ge 1$ and $\sup_k t_k-t_{k-1}\le 2$, Theorem A applies to the right hand side of (2.9) and similarly to the left.

It only remains to verify the claims in the remark following

Theorem B. Note first that (1.8)⇒(1.4) since

$$c^{-1}(t-1)\gamma(t) \le \int_1^t \left(\frac{t}{s}\right)^{\theta}\gamma(s)ds =$$

$$\left[\left(\frac{t}{s}\right)^{\theta}\psi^{-1}(s)\right]_1^t+\theta t^{\theta}\int_1^t s^{-\theta-1}\psi^{-1}(s)ds \le$$

$$\psi^{-1}(t)+\theta t^{\theta}\psi^{-1}(t)\int_1^t s^{-\theta-1}ds \le 2\psi^{-1}(t)$$

(1.8)⇒(1.7) now follows from

$$H(t) \le \int_t^{\infty} s\,d|F|(s) = \int_t^{\infty} F(s)ds+tF(t),$$

$$\int_0^{\infty}(\psi^{-1}(t)-\psi^{-1}(0))F(t)dt = \int_0^{\infty}\gamma(t)\int_t^{\infty}F(s)ds\,dt.$$

Finally, it ψ is concave then (1.4)⇒(1.8) since

$$t\gamma(t) \ge \int_0^t \gamma(s)ds = \psi^{-1}(t)-\psi^{-1}(0),$$

and if $\psi(x)=x^{\beta}$ then $t\gamma(t)=\psi^{-1}(t)/\beta$.

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