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MULTITYPE BRANCHING DIFFUSIONS

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We review some of the basic limit theorems for Markov branching processes in the framework of multitype branching diffusions on bounded domains with mixed boundary conditions. This setting allows to exhibit methods of the limit theory for general Markov branching processes without having to impose technical conditions.

1. THE MODEL

Let Ω be the union of K connected open sets Ω_ν, $\nu = 1,\ldots,K$, in an N-dimensional, orientable manifold of class C^∞, let the closures $\bar{\Omega}_\nu$ be compact and pairwise disjoint, and let the boundary $\partial\Omega$ consist of a finite number of simply connected (N-1)-dimensional hypersurfaces of class C^3. Let X be the union of K Borel sets X_ν such that

$$\Omega_\nu \subset X_\nu \subset \overline{\Omega}_\nu, \quad \nu = 1,\ldots,K,$$

in a way to be determined, and suppose to be given a uniformly elliptic differential operator $A|D(A)$, represented in local coordinates on X by

$$A: = \sum_{i,j=1}^{N} \frac{1}{\sqrt{a(x)}} \frac{\partial}{\partial x^i} a^{ij}(x)\sqrt{a(x)} \frac{\partial}{\partial x^j} + \sum_{i=1}^{N} b^i(x) \frac{\partial}{\partial x^i}$$

$$D(A): = \{u|_X : u \in C^2(\overline{\Omega}) \wedge (\alpha u + \beta \tfrac{\partial u}{\partial n})|_{\partial\Omega} = 0\},$$

where (a^{ij}) and (b^i) are the restrictions to X of a symmetric, second-order, contravariant tensor of class $C^{2,\lambda}(\overline{\Omega})$ and a first-order, contravariant tensor of class $C^{1,\lambda}(\overline{\Omega})$,

$$a: = \det(a^{ij})^{-1},$$

$$0 \leq \alpha, \beta \in C^{2,\lambda}(\partial\Omega), \quad \alpha+\beta \equiv 1,$$

$$\overline{\Omega} \setminus X: = \{\beta=0\}.$$

By $\frac{\partial}{\partial n}$ we denote the exterior normal derivative according to (a^{ij}) at $\partial\Omega$.

Define B as the Banach algebra of all complex-valued, bounded, Borel-measurable functions on X with supremum-norm $\|\cdot\|$, B_+ as the cone of all non-negative functions in B, further

$$C^\ell: = \{u|_X : u \in C^\ell(\overline{\Omega})\},$$

$$C_0^\ell: = \{u|_X : u \in C^\ell(\overline{\Omega}) \wedge u|_{\overline{\Omega}\setminus X} \equiv 0\}.$$

As is wellknown, the closure of $A|\{\xi \in D(A): A\xi \in C_0^0\}$ in B is the C_0^0-generator of a contraction semigroup $\{T_t\}_{t\in\mathbb{R}_+}$ in B, which is non-negative respective B_+, stochastically continuous in $t\geq 0$ on B, and strongly continuous in $t\geq 0$ on C_0^0, with $T_tB \subseteq C_0^2$ for $t>0$. This semigroup determines a conservative, continuous, strong Markov process $\{x_t, P^x\}$ on $X \cup \{\partial\}$, where ∂ is a trap.

Suppose to be given a $k \in B_+$, and define $\bar{k}(x) := k(x)$ for $x \in X$, $\bar{k}(\partial) := 0$, and

$$\delta_t := \exp\{-\int_0^t \bar{k}(x_s)ds\}.$$

Let $\{x_t^0, P_0^x\}$ be the δ_t-subprocess of $\{x_t, P^x\}$, defined as a conservative process on $X \cup \{\partial\} \cup \{\Delta\}$, where Δ is a trap corresponding to the stopping by δ_t. For $\xi \in B$ define $\xi_0(x) := \xi(x)$, if $x \in X$, and $\xi_0(\partial) := \xi_0(\Delta) := 0$.

Then

$$T_t^0 \xi(x) := E_0^x \xi_0(x_t^0), \quad x \in X, \ t \geq 0,$$

defines a non-negative contraction semigroup $\{T_t^0\}_{t \in \mathbb{R}_+}$ on B. It is the unique solution of

$$(1.1) \quad T_t^0 = T_t - \int_0^t T_s k T_{t-s}^0 ds, \quad t \geq 0,$$

and it is stochastically continuous in $t \geq 0$ on B and strongly continuous in $t \geq 0$ on C_0^0, with $T_t^0 B \subseteq C_0^1$ for $t \geq 0$.

Let $X^{(n)}, n \geq 1$, be the symmetrization of the direct product of n disjoint copies of X, $X^{(0)} := \{\theta\}$ with some extra point θ.

Define

$$\hat{X} := \bigcup_{n=0}^{\infty} X^{(n)},$$

and let $\hat{A}$ be the σ-algebra on $\hat{X}$ induced by the Borel algebra on X.

Define

$$\hat{x}[\xi] := 0, \qquad \hat{x} = \theta,$$

$$:= \sum_{\nu=1}^{n} \xi(x_\nu), \hat{x} = \langle x_1, \ldots, x_n \rangle \in X^{(n)}, \quad n > 0$$

for every finite-valued Borel-measurable ξ on X. Suppose to be given a stochastic kernel $\pi | X \otimes \hat{A}$

such that

$$m\xi(x) := \int_{\hat{X}} \hat{x}[\xi]\pi(x,d\hat{x}), \quad \xi\in B, \ x\in X,$$

defines a bounded operator m on B and the K×K-matrix with elements

$$m_{\nu\mu} := \int_{X_\nu} k(x) m 1_{X_\mu}(x)dx, \quad \nu,\mu=1,\ldots,K,$$

is irreducible.

More explicitly, we assume that either

$$m\xi(x) = \int_X m(x,y)\xi(y)dy, \quad \xi\in B, \ x\in X,$$

$$m(x,y) \leq \overline{m}(x,y), \quad (x,y) \in X\otimes X,$$

$$\overline{m} \in C^1(\overline{\Omega}\otimes\overline{\Omega}), \quad \overline{m}(\cdot,x) = \overline{m}(x,\cdot) \equiv 0, \quad x \in \overline{\Omega}\setminus X,$$

(1.2) $$dy := \sqrt{a(y)}\, dy^1\ldots dy^N,$$

where $y^1,\ldots,y^N$ are local coordinates of y, or that the connected components $X_\nu, \nu = 1,\ldots,K$, of X are congruent and

$$\pi(x,\hat{A}) = p_{0\ldots0}(x)1_{\hat{A}}(\theta) + \sum_{\substack{n_1\geq 0,\ldots,n_K\geq 0 \\ n_1+\ldots+n_K>0}} p_{n_1\ldots n_K}(x)$$

$$\times\, 1_{\hat{A}}(\overbrace{\kappa_k x,\ldots,\kappa_1 x}^{n_1},\ldots,\overbrace{\kappa_K x,\ldots,\kappa_K x}^{n_K}), \quad x\in X, \ \hat{A}\in\hat{\mathcal{A}}$$

where $1_{\hat{A}}$ is the indicator function of $\hat{A}$, $\{p_{n_1\ldots n_K}(x)\}$ a probability distribution on $\mathbb{Z}_+^K$ for every $x\in X$, and $\kappa_\nu x$ the picture of x produced in X_ν by the given congruence. In the second case

$$m\xi(x) = \sum_{\nu=1}^{K} m_\nu(x)\xi(\kappa_\nu x), \quad \xi\in B, x\in X,$$

$$m_\nu := \sum_{n_1\geq 0,\ldots,n_K\geq 0} n_\nu p_{n_1\ldots n_K}, \quad \nu=1,\ldots,K.$$

The pair (x_t^0,π) determines a conservative, right-continuous strong Markov process $\{\hat{x}_t,P^{\hat{x}}\}$ on $(\hat{X},\hat{A})$, constructed according to the following intuitive rules: All particles at a time move independently of each other, each according to $\{x_t^0,P_0^x\}$. A particle hitting ∂ disappears, a particle hitting Δ is replaced by a population of new particles according to $\pi(x_{t_\Delta -},\cdot)$, where $x_{t_\Delta -}$ is the left limit of the path at the hitting time of Δ, cf.[9], [17].

A simple example for the first kind of branching law we have admitted is the following: A branching event at x results with probability $p_{n_1..n_K}(x)$ in $n_1+..+n_K$ new particles, n_ν of them in X_ν, $\nu = 1,..,K$. The places of birth are distributed independently, a location in X_ν with the distribution density $f_\nu(x,\cdot)$, $\nu = 1,..,K$. Then

$$m(x,y) = \sum_{\nu=1}^{K} 1_{X_\nu}(y) f_\nu(x,y) \sum_{n_1,..,n_K \geq 0} n_\nu p_{n_1,..,n_K}(x).$$

The idea behind the second type of branching law is this: There are K different kinds of particles moving on the same physical domain. To the kind ν we assign X_ν as abstract domain of diffusion, $\nu = 1,..,K$. In the physical domain new particles are always born at the termination point (left limit) of their immediate ancestor.

In terms of the generating functional

$$F_t(\hat{x},\eta) := E^{\hat{x}}\tilde{\eta}(\hat{x}_t),$$

$$\tilde{\eta}(\hat{x}) := 1, \qquad \hat{x} = \theta,$$

$$:= \prod_{\nu=1}^{n} \eta(x_\nu), \quad \hat{x} = \langle x_1,..,x_n\rangle,$$

$t \geq 0$, $\hat{x} \in \hat{X}$, $\eta \in \bar{S} := \{\xi \in B: \|\xi\| \leq 1\}$,

the assumption of independent motion and branching takes the form

$$\text{(F.1)} \quad F_t(\hat{x},\eta) = 1, \qquad \hat{x} = \theta,$$

$$= \prod_{\nu=1}^{n} F_t(\langle x_\nu\rangle,\eta), \quad \hat{x} = \langle x_1,..,x_n\rangle, \quad n>0.$$

Defining $F_t: \bar{S} \to \bar{S}$ by $F_t[\cdot](x) := F_t(\langle x\rangle,\cdot)$, $x \in X$, (F.1) combined with the Chapman-Kolmogorov equation yields

$$\text{(F.2)} \quad F_{t+s}[\cdot] = F_t[F_s[\cdot]], \quad t,s \geq 0.$$

For every $t>0$ define $\hat{x}_{t-}$ on $\hat{Y}$ with $Y: = X\cup\{\partial\}$, and let A_0 be the set of open spheres intersected with X. Define

$$\tau: = \inf\{t>0: \exists U\in A_0: \hat{x}_{t-}[1_U]\neq\hat{x}[1_U]\}.$$

It follows from the strong Markov property of $\{\hat{x}_t, P^{\hat{x}}\}$ that for every $\eta\in\bar{S}$ the function $F_t[\eta](x)$, $t\geq 0$, $x\in X$, solves

$$\text{(IF)}\qquad u_t(x) = E^{\langle x\rangle}\tilde{\eta}(\hat{x}_t)1_{\{t<\tau\}} + E^{\langle x\rangle}(E^{\hat{x}_\tau}\tilde{\eta}(\hat{x}_{t-s}))\big|_{s=\tau}1_{\{\tau<s\}}$$

$$= T_t^0\,\eta(x) + P_0^x(x_\tau^0=\partial, \tau\leq t)$$

$$+ \int_0^t\int_X P_0^x(x_\tau^0=\Delta,\ x_{\tau-}^0\in dy,\ \tau\in ds)\int_{\hat{X}}\pi(y,d\hat{x})F_{t-s}(\hat{x},\eta)$$

$$= T_t^0\eta(x) + H_t(x) + \int_0^t T_s^0\{kf[u_{t-s}]\}(x)ds,$$

$$H_t(x): = 1 - T_t^0(x) - \int_0^t T_s^0 k(x)ds,$$

$$f[\eta](x): = \int_{\hat{X}}\pi(x,d\hat{x})\tilde{\eta}(\hat{x}).$$

The uniqueness of the solution is easily verified by use of

$$\|f[\eta] - f[\xi]\| \leq \|m\|\,\|\eta-\xi\|.$$

The assumptions guarantee that for every $t\geq 0$

$$M_t\xi(x): = E^{\langle x\rangle}\hat{x}_t[\xi],\quad \xi\in B,\ x\in X,$$

defines a non-negative, linear-bounded operator M_t on B. It follows from (F.1) that

$$\text{(F.3)}\qquad E^{\hat{x}}\xi(\hat{x}_t) = \hat{x}[M_t\xi],\quad \hat{x}\in\hat{X},\ \xi\in B,\ t\geq 0,$$

and from (F.2) that $\{M_t\}_{t\in\mathbb{R}_+}$ is a semigroup: Simply set $\eta = \zeta+\lambda\xi$, differentiate with respect to λ at $\lambda=0$ and let $\zeta\to 1$, using dominated convergence. Similarly, (IF) implies that for every $\xi\in B$ the function $M_t\xi(x)$, $t\geq 0$, $x\in X$, solves

(IM) $$v_t(x) = T_t^0\xi(x) + \int_0^t T_s^0\{kmv_{t-s}\}(x)ds.$$

Again, the solution is unique.

Throughout this paper $c_\nu > 0$, $\nu \in \mathbb{N}$, will be suitable real constants.

2. POSITIVITY THEOREM

To obtain a satisfactory limit theory we need a positivity result which is stronger than the conventional Kreĭn-Rutman theorem. Define

$$D_0^+ := \{u|_X : u\in C^1(\overline{\Omega}), u>0 \text{ on } X, u=0 \wedge \frac{\partial u}{\partial n}<0 \text{ on } \overline{\Omega}\setminus X\}.$$

Theorem 1 ([6],[7]). The moment semigroup $\{M_t\}_{t\geq 0}$ is stochastically continuous in $t\geq 0$ on B, strongly continuous in $t\geq 0$ on C_0^0 with $M_t B \subseteq C_0^1$ for $t>0$. It can be represented in the form

(M) $$M_t = \rho^t \varphi\Phi^* + \Delta_t, \quad t>0,$$

$$\Phi^*[\xi] = \int_X \varphi^*(x)\xi(x)dx, \quad \xi\in B,$$

where $0<\rho \in \mathbb{R}$, $\varphi\in D_0^+$, $\varphi^*\in D_0^+$, $\Phi^*[\varphi] = 1$,

and Δ_t: $B\to B$ such that for all $t>0$

$$\varphi\Phi^*\Delta_t = \Delta_t\varphi\Phi^* = 0,$$

$$-\alpha_t\varphi\Phi^* \leq \Delta_t \leq \alpha_t\varphi\Phi^* \quad [B_+],$$

with $\alpha.: \mathbb{R}_+\to \mathbb{R}_+$ satisfying $\rho^{-t}\alpha_t \downarrow 0$ as $t\uparrow\infty$.

Remark. Notice that there are no continuity requirements for k, m(x,y), or $m_\nu(x)$. We do not consider $\{M_t\}$ as extension of a semigroup generated on C_0^0, but as restriction of a semigroup generated on L^2.

Proof. We modify the procedure of [7]. Define $L^2:=L^2(X)$ respective (1.2). Let $\overline{T}_t$ be the extension of T_t to L^2 and $\overline{T}_t^*$ the adjoint of $\overline{T}_t$. Then

$$T_t\xi(x) = \int_X p_t(x,y)\xi(y)dy \quad [B],$$

$$\overline{T}_t^*\xi(x) = \int_X p_t(y,x)\xi(y)dy \quad [L^2],$$

where $p_t(x,y)$ is the fundamental solution of $\partial p_t/\partial t=Ap_t$. That is, $p_t(x,y)$ is given as a continuous function on $\{t>0\}\otimes\overline{\Omega}\otimes\overline{\Omega}$, continuously differentiable in x and y for $t>0$, such that for $0<t\leq t_0$, t_0 arbitrary but fixed,

$$(2.1)\quad \begin{aligned} &p_t(x,y)>0, \quad (x,y)\in X_\nu\otimes X_\nu, \quad \nu=1,..,K,\\ &p_t(x,y)=0, \quad (x,y)\in X_\nu\otimes X_\mu, \quad \nu\neq\mu, \end{aligned}$$

$$(2.2)\quad p_t(x,\cdot)=p_t(\cdot,x)\equiv 0, \quad x\in\overline{\Omega}\setminus X,$$

$$(2.3)\quad \begin{aligned} &\frac{\partial p_t}{\partial n_x}(x,y)<0, \quad (x,y)\in(\overline{\Omega}_\nu\setminus X_\nu)\otimes X_\nu,\\ &\frac{\partial p_t}{\partial n_y}(x,y)<0, \quad (x,y)\in X_\nu\otimes(\overline{\Omega}_\nu\setminus X_\nu), \quad \nu=1,..,K, \end{aligned}$$

$$(2.4)\quad \sup_{x,y\in X}\{|\frac{\partial p_t}{\partial x^i}(x,y)|+|\frac{\partial p_t}{\partial y^i}(x,y)|\} = O(t^{-(N+1)/2}), \quad i=1,..,N,$$

$$(2.5)\quad \sup_{x\in X}\int_X\{|\frac{\partial p_t}{\partial x^i}(x,y)|+|\frac{\partial p_t}{\partial x^i}(y,x)|\}dy = O(t^{-1/2}), \quad i=1,..,N,$$

$$(2.6)\quad p_t(x,y) = c_1 t^{-N/2}\sum_j 1_{Y_j}(x)\exp\{-c_2 t^{-1}\sum_{i=1}^N|x_j^i-y_j^i|^2\} \text{ on } X\otimes X,$$

where $\{Y_j\}$ is a finite covering of $\overline{\Omega}$ by canonical coordinate neighbourhoods and $x_j^1,..,x_j^N$ are the coordinates of x in the coordinate system associated with Y_j, cf. [11], [16]. As an immediate consequence,

$$T_t^+\xi(x): = \int_X p_t(x,y)\xi(y)dy \quad [B]$$

defines a restriction of $\overline{T}_t^*$ to B, whose norm $\|T_t^+\|$ is bounded on bounded t-intervals.

Let $\overline{m}$, $\overline{b}$, and $\overline{T}_t^0$ be the extensions to L^2 of m,

$$b: = km + \|k\| - k$$

and T_t^0. The closure of A+k(m-1) in L^2 generates a semigroup $\overline{M}_t$, which is the unique solution of

$$\overline{M}_t = \overline{T}_t^0 + \int_0^t \overline{T}_s^0 \, k\overline{m}\overline{M}_{t-s} ds \qquad [L^2]$$

and thus is identical with the extension of M_t to L^2. Similary the semigroup

$$\overline{N}_t: = e^{\|k\| t} \overline{M}_t,$$

generated by the closure of A+b in L^2, is the extension to L^2 of the unique solution N_t of

$$(2.7) \quad N_t = T_t + \int_0^t T_s bN_{t-s} ds \qquad [B].$$

Since $T_t B \subseteq C_0^0$, $t>0$, it follows from (1.1),(IM), and (2.7) by use of $\|T_t\| \leq 1$, the boundedness of k and m, and dominated convergence that $T_t^0 B \subseteq C_0^0$, $M_t B \subseteq C_0^0$, and $N_t B \subseteq C_0^0$ for $t>0$. Hence also

$$N_t = e^{\|k\| t} M_t.$$

From (2.7)

$$(2.8) \quad N_t = \sum_{k=0}^{\infty} T_t^{(n)}, \quad t \geq 0, \qquad [B]$$

$$T_t^{(0)}: = T_t, \quad T_t^{(n+1)}: = \int_0^t T_s b T_{t-s}^{(n)} ds, \quad n \geq 0 \qquad [B].$$

From this, in particular,

$$(2.9) \quad \|N_t\| \leq e^{\|b\| t}, \quad t \geq 0.$$

Given (2.7) with the bounds for T_t, b, and N_t, strong continuity of $T_t|C_0^0$ implies strong continuity of $N_t|C_0^0$ in $t \geq 0$, and recalling

$N_tB\subseteq C_0^0$, $t>0$, stochastic continuity of $T_t|B$ implies stochastic continuity of $N_t|B$ in $t\geq 0$. Using $p_t(\cdot,y)\in C^1$, (2.4), (2.5), and dominated convergence, we also get $N_tB\subseteq C^1$ for $t\geq 0$.

By continuity of $p_t(x,y)$ in (x,y) with (2.2), $T_t|B$ and $T_t|C_0^0$ are compact, if $t>0$. For $0<\varepsilon<t$ rewrite (2.7) as

$$N_t = T_t+T_\varepsilon\int_\varepsilon^t T_{s-\varepsilon}bN_{t-s}ds+\int_0^\varepsilon T_sbN_{t-s}ds$$

and note that the integrals on the right are bounded operators, the norm of the second being $O(\varepsilon)$ if t is fixed. That is, compactness of T_t implies compactness of N_t for $t>0$. The cone B_+ and its dual B_+^* are closed, have a non-empty interior, and span B and its dual B^*. By (2.1) the spectral radius of T_t and thus the spectral radius σ_t of N_t are positive. Hence, the spectrum of N_t is purely discrete, each non-zero eigenvalue has finite multiplicity, and there exist non-trivial $\varphi_t\in B_+$, $\Phi_t^*\in B_+^*$ such that

$$N_t\varphi_t = \sigma_t\varphi_t, \quad \Phi_t^*N_t = \sigma_t\Phi_t^*,$$

cf.[15]. The same holds for $N_t|C_0^0$. Since $T_sB\subseteq C_0^0$, $s>0$, the spectral radius is the same, and we can take the same $\varphi_t\in C_0^0\cap B_+$ in both cases. For $\varepsilon>0$, $n\leq\ell$, $t>(\ell+1)\varepsilon$ define

$$T_t^{(0,\varepsilon)}: = T_{t-\varepsilon}, \quad T_t^{(n,\varepsilon)}: = \int_\varepsilon^{t-n\varepsilon} T_{s-\varepsilon}bT_\varepsilon T_{t-s}^{(n-1,\varepsilon)}ds.$$

By (2.8)

$$N_t \geq T_\varepsilon\sum_{n=0}^{\ell} T_t^{(n,\varepsilon)} \qquad [B_+].$$

Fixing t and choosing any $\xi\in B_+$ which is positive on a set of positive measure, we can by (2.1) and the irreducibility assumption on $(m_{\nu\mu})$ find $\varepsilon>0$ and $\ell\in\mathbb{N}$ such that $t>(\ell+1)\varepsilon$ and

$$\sum_{n=0}^{\ell} T_t^{(n,\varepsilon)}\xi \geq c_3 > 0 \text{ on } \Omega' = \cup_{\nu=1}^{K}\Omega'_\nu,$$

where all the $\Omega'_\nu \subset X_\nu$, $\nu=1,..,K$, have positive measure. By (2.1), (2.3), (2.4), and l'Hospital's rule

$$\sup_{x\in X}(T_\varepsilon 1_{X\setminus\Omega'}(x)/T_\varepsilon 1_{\Omega'}(x)) < \infty.$$

That is, there exists a $\delta\in\mathbb{R}_+$, $\delta>0$, such that $T_\varepsilon 1_{\Omega'}\geq\delta T_\varepsilon 1_X$ and hence

$$N_t\xi \geq c_3\delta T_\varepsilon 1_X.$$

On the other hand $T_\varepsilon 1_X \in D_0^+$, so that by (2.4), (2,5), (2,7), (2.9, and l'Hospital's rule there exist a $c_4>0$ such that

$$N_t\xi \leq c_4 T_\varepsilon 1_X.$$

Consequently, σ_t is a simple eigenvalue of $N_t|C_0^0$, larger in absolute value that any other eigenvalue, cf.[14, Chapter 2]. Once again referring to $N_t B\subseteq C_0^0$, $t>0$, the same is true for $N_t|B$. The $(T_\varepsilon 1_X)$ - boundedness from below, $T_\varepsilon 1_X\in D_0^+$, and $N_t B\subseteq C_0^1$ imply $\varphi_t\in D_0^+$ and thus $\Phi_s^*[\varphi_t]>0$ for $s,t>0$. Using the semigroup property of N_t, it follows that $\sigma_r=:\sigma^r$ for all rational r and, since σ^r is simple, that $\varphi_r=:\varphi$ for all rational r. By continuity of N_t therefore $\sigma_t=\sigma^t$ and $\varphi_t=\varphi$ for all $t>0$.

Now consider the problem in L^2. Again, $\overline{N}_t$ is compact, the spectral radius $\overline{\sigma}_t$ of $\overline{N}_t$ is in the spectrum of $\overline{N}_t$, and there exist non-trivial, non-negative $\overline{\varphi}_t, \overline{\varphi}_t^*\in L^2$ such that

$$\overline{N}_t\overline{\varphi}_t = \overline{\sigma}_t\overline{\varphi}_t, \quad \overline{N}_t^*\overline{\varphi}_t^* = \overline{\sigma}_t\varphi_t^*,$$

where $\overline{N}_t^*$ is the adjoint of $\overline{N}_t$. However,

$$\overline{\sigma}_t\int_X\overline{\varphi}_t^*(x)\varphi(x)dx = \int_X\overline{N}_t^*\overline{\varphi}_t^*(x)\varphi(x)dx$$

$$= \int_X\overline{\varphi}_t^*(x)N_t\varphi(x)dx = \sigma^t\int_X\overline{\varphi}_t^*(x)\varphi(x)dx>0.$$

That is, $\bar{\sigma}_t \equiv \sigma^t$, and we can take $\bar{\varphi}_t \equiv \varphi$. Viewing $\bar{T}_t$ as the perturbed semigroup with $\bar{N}_t$ as the unperturbed semigroup and adjoining the corresponding perturbation equation, or simply considering $\bar{N}_t^*$ as generated by the adjoint of $A+\bar{b}$, we get

$$\bar{N}_t^* = \bar{T}_t^* + \int_0^t \bar{T}_s^* \bar{b}^* \bar{N}_{t-s}^* ds \qquad [L^2].$$

Since $\|T_t^+\|$ is bounded on bounded t-intervals and $\bar{b}^*$ has a bounded restriction b^+ to B by assumption, the unique solution of

$$N_t^+ = T_t^+ + \int_0^t T_s^+ b^+ N_{t-s}^+ ds \qquad [B]$$

is a restriction of $\bar{N}_t^*$ to B. It can be written in the form

$$N_t^+ = \sum_{n=0}^{\infty} T_t^{+(0)}, \quad T_t^{+(0)} := T_t^+, \quad T_t^{+(n)} = \int_0^t T_s^+ b^+ N_{t-s}^{+(n-1)} ds,$$

which implies, in particular, that $\|N_t^+\|$ is bounded on bounded t-intervals. We can now repeat for N_t^+ the argument used for N_t and obtain

$$\Phi_t^*[\xi] \equiv \Phi^*[\xi] = \int_X \varphi^*(x)\xi(x)dx, \quad \xi \in B,$$

with $\varphi^* \in D_0^+$. We normalize $\Phi^*[\varphi]=1$.

Summing up, we have shown that

$$N_t = \sigma^t \varphi \Phi^* + \Gamma_t, \quad t>0,$$

with $\Gamma_t : B \to B$ such that

$$\varphi \Phi^* \Gamma_t = \Gamma_t \varphi \Phi^* = 0, \quad t>0,$$

$$\|\Gamma_{n\varepsilon}\| = 0(\vartheta_\varepsilon^n), \quad n \in \mathbb{N},$$

where $\vartheta_\varepsilon \in \mathbb{R}_+$, $\vartheta_\varepsilon < \sigma^\varepsilon$, for every $\varepsilon>0$. Since $\{\Gamma_t\}$ is a semigroup

$$\|\Gamma_t\| \le \sup_{0 \le s \le \varepsilon} \|\Gamma_s\| \, \|\Gamma_{[t/\varepsilon]\varepsilon}\|$$

$$\le \left(\sup_{0 \le s \le \varepsilon} \|N_s\| + \max\{1,\sigma^\varepsilon\} \|\varphi\| \, \Phi^*[1] \right) \|\Gamma_{[t/\varepsilon]\varepsilon}\|, \quad t>0.$$

That is,

$$\|\Gamma_t\| = O(\vartheta^t), \quad t>0,$$

with some $\vartheta\in\mathbb{R}_+, \vartheta<\sigma$. If $0<2\varepsilon<t$,

$$|\Gamma_t\xi(x)| \leq \sup_{\|\eta\|=1} |\Gamma_\varepsilon\eta(x)| \; \|\Gamma_{t-2\varepsilon}\| \; \|\Gamma_\varepsilon\xi\|$$

$$\leq (N_\varepsilon 1(x)+\sigma^\varepsilon\varphi(x)\Phi^*[1])\|\Gamma_{t-2\varepsilon}\|$$

$$\times \; (\|N_\varepsilon\xi\| + \sigma^\varepsilon\|\varphi\| \; \Phi^*[|\xi|]).$$

Since $N_\varepsilon 1\in D_0^+$, as well as $\varphi\in D_0^+$, we have $N_\varepsilon 1 \leq C_\varepsilon\varphi$, where C_ε depends only on ε. As

$$|\int_Y N_\varepsilon\xi(x)dx| \leq \int_X |\xi(y)| N_\varepsilon^+ 1_Y(y)dy, \quad Y\in A,$$

all that is left to be shown is

(2.10) $N_\varepsilon^+ 1_Y \leq C_\varepsilon^*\varphi^* \int_Y dx, \quad Y\in A,$

with C_ε^* depending only on ε.

Using (2.4) and $\varphi^*\in D_0^+$,

$$T_\varepsilon^{+(0)} 1_Y \leq c_5\varepsilon^{-(N+1)/2}\varphi^* \int_Y dx, \quad Y\in A.$$

For $n\geq 1$

$$T_t^{+(0)} = \int_0^t T_s^+ \tilde{T}_{t-s}^{(n)} ds, \quad \tilde{T}_t^{(1)} := b^+ T_t^+, \quad \tilde{T}_t^{(n+1)} := b^+ \int_0^t T_s^+ \tilde{T}_{t-s}^{(n)} ds.$$

By (2.6) we have $p_t(x,y) \leq c_6\tilde{p}_t(x,y)$ with

$$\int_X \tilde{p}_s(x,z)\tilde{p}_{t-s}(z,y)dz \leq \tilde{p}_t(x,y), \quad 0\leq s\leq t,$$

and, if the X_ν are congruent,

$$\tilde{p}_t(\kappa_\nu x, \kappa_\nu y) = \tilde{p}_t(x,y), \quad \nu=1,..,K.$$

Using these properties, it is verified by induction that

(2.11) $\tilde{T}_t^{(n)} 1_Y(x) \leq c_7^n \dfrac{t^{n-1}}{(n-1)!} \left(\int_Y dy + \int_Y p_t(x,y)dy\right)$

where $\int_Y dy$ occurs only with the first kind of branching law.
From (2.11) by (2.4), (2.5), and $\varphi^*\in D_0^+$

$$T^{+(n)}1_Y(x) = \frac{c_7^n}{(n-1)!}\left(\int_0^{\varepsilon}\int\int_X p_s(z,x)(\varepsilon-s)^{n-1}dzds\int_Y dy\right.$$

$$\left.+\left\{\int_0^{\varepsilon/2}+\int_{\varepsilon/2}^{\varepsilon}\right\}\int_X p_s(z,x)(\varepsilon-s)^{n-1}\int_Y \tilde{p}_{\varepsilon-s}(z,y)dydzds\right)$$

$$\leq \frac{c_7^n}{(n-1)!}\left[c_8\varepsilon^{n-1/2}+c_9\varepsilon^{-(N-1)/2}\left(\frac{\varepsilon}{2}\right)^{n-1}\right.$$

$$\left.+c_{10}\varepsilon^{-(N+1)/2}\frac{1}{n}\left(\frac{\varepsilon}{2}\right)^{n}\right]\varphi^*(x)\int_Y dz,$$

which implies (2.10).

3. THREE LEMMATA

By first-order Taylor expansion

(FM) $1-F_t[\xi] = M_t[1-\xi]-R_t(\xi)[1-\xi], \quad \xi\in\bar{S},$

$$R_t(\eta)\zeta(x) := E^{\langle x\rangle}\omega(\eta,\zeta,\hat{x}_t),$$

$$\omega(\eta,\zeta,\hat{x}) := 0, \quad \hat{x}[1]\leq 1,$$

$$:= \sum_{\nu=1}^{n}\zeta(x_\nu)\left(1-\int_0^1\prod_{\mu\neq\nu}[1-\lambda(1-\eta(x_\mu))]d\lambda\right),$$

$$\hat{x} = \langle x_1,\ldots,x_n\rangle, \; n>1.$$

The mapping $R_t(\cdot)[\cdot]:\bar{S}\otimes B\to B$ is sequentially continuous respective the product topology on bounded regions, non-increasing in the first and linear-bounded in the second variable, and it satisfies

(RM) $0=R_t(1)\zeta \leq R_t(\eta)\zeta \leq M_t\eta, \quad (\eta,\zeta)\in\bar{S}_+\otimes B_+,$

where $\bar{S}_+ := \bar{S}\cap B_+$.

<u>Lemma 1</u>. ([5],[8]). For every $t>0$ there exists a mapping $g_t:\bar{S}_+\to B_+$ such that

(R)
$$R_t(\xi)[1-\xi] = g_t[\xi]\rho^t\Phi^*[1-\xi]\varphi, \quad \xi\in\bar{S}_+,$$

$$\lim_{\|1-\xi\|\to 0}\|g_t[\xi]\| = 0,$$

where the convergence is uniform in t on any closed, bounded interval $[a,b]$ with $a>0$.

Proof. It follows from (IF), (IM), (FM), and the corresponding expansion for f,

$$1-f[\xi] = m[1-\xi]-r(\xi)[1-\xi], \quad \xi\epsilon\overline{S},$$

that for every $\varepsilon>0$ and $\xi\epsilon\overline{S}_+$ the function $R_t(\xi)[1-\xi](x)$, $t\geq\varepsilon$, $x\epsilon X$, solves

$$(3.1) \qquad w_t(x) = A_t(x)+B_t^\varepsilon(x)+\int_0^{t-\varepsilon} T_s^0\{kmw_{t-s}\}(x)dx,$$

$$A_t(x): = \int_0^t T_s^0\{kr(F_{t-s}[\xi])[1-F_{t-s}[\xi]]\}(x)ds,$$

$$B_t^\varepsilon(x): = \int_0^\varepsilon T_{t-s}^0\{kmR_s(\xi)[1-\xi]\}(x)ds.$$

As is easily verified, $R_t(\xi)[1-\xi]$ is the only solution bounded in $[\varepsilon,\varepsilon+\lambda]$ for any $\lambda>0$, and it thus equals the limit of the (non-decreasing) iteration sequence $(w_t^{(\nu)}(x))_{\nu\epsilon\mathbb{Z}_+}$, $w_t^{(0)}\equiv 0$.

Suppose $0<\delta<\varepsilon/2$ and $\xi\epsilon\overline{S}_+$. By (FM) and (RM) there exist a $c_{11}>0$ such that for $\delta\leq s\leq t-\delta$ and $t\leq\varepsilon+\lambda$

$$(3.2) \qquad F_{t-s}[\xi] \geq 1-c_{11}\|1-\xi\|.$$

By (IM)

$$(3.3) \qquad T_t^0 \leq M_t \qquad [B_+].$$

Further

$$(3.4) \qquad 0=r(1)\eta \leq r(\zeta)\eta \leq m\eta, \quad (\zeta,\eta)\epsilon S_+\otimes B_+.$$

Finally, recalling the assumptions on m and the fact that $\varphi,\varphi^*\epsilon D_0^+$, there exist constants c and c^* such that

$$(3.5) \qquad km\varphi \leq c\varphi,$$

$$(3.6) \qquad \Phi^*[km\eta] \leq c^*\Phi^*[\eta], \quad \eta\epsilon B_+.$$

Using (3.2-6) and (M),

$$A_t \leq \left\{\int_0^\delta + \int_{t-\delta}^t\right\} M_s[kmM_{t-s}[1-\xi]]ds$$

$$+ \int_\delta^{t-\delta} M_s[kr(1-c_{11}\|1-\xi\|)M_{t-s}[1-\xi]]ds$$

$$\leq \delta(c+c^*)(1+\rho^{-\varepsilon/2}\alpha_{\varepsilon/2})\rho^t\Phi^*[1-\xi]\varphi$$

$$+ t(1+\rho^{-\delta}\alpha_\delta)(1+\rho^{-\varepsilon/2}\alpha_{\varepsilon/2})\|k\varphi\|$$

$$\times \Phi^*[r(1-c_{11}\|1-\xi\|)[\varphi]]\rho^t\Phi^*[1-\xi]\varphi.$$

Since

$$\lim_{\|1-\xi\|\to 0} \Phi^*[r(1-c_{11}\|1-\xi\|)\varphi] = 0,$$

we can for every $\varepsilon'>0$ first fix $\delta>0$ such that

$$\left\{\int_0^\delta + \int_{t-\delta}^t\right\}(\cdots)ds \leq \frac{1}{2}t\varepsilon'\Phi^*[1-\xi]\varphi,\ \xi\in\overline{S}_+,$$

and then choose a $\delta'>0$ such that

$$\int_\delta^{t-\delta}(\cdots)ds \leq \frac{1}{2}t\varepsilon'\Phi^*[1-\xi]\varphi,\quad \|1-\xi\|<\delta'.$$

That is,

$$A_t \leq t\Theta_{\varepsilon,\lambda}[\xi]\rho^t\Phi^*[1-\xi]\varphi,\quad \varepsilon\leq t\leq\varepsilon+\lambda,$$

(3.7)
$$\lim_{\|1-\xi\|\to 0}\Theta_{\varepsilon,\lambda}[\xi] = 0,\quad \varepsilon,\lambda>0.$$

Using (RM) and the fact that $T^0_{t-s}\leq M_{t-\varepsilon}M_{\varepsilon-s}$ on B_+,

(3.8) $$B^\varepsilon_t(x)\leq M_{t-\varepsilon}[\int_0^\varepsilon M_{\varepsilon-s}[kmM_s[1-\xi]](x)ds] =: \overline{B}^\varepsilon_t(x),\quad t>\varepsilon.$$

By (IM), (M), and (3.6)

(3.9) $$\int_0^{t-\varepsilon} T^0_s\{km\overline{B}^\varepsilon_{t-s}\}ds \leq \overline{B}^\varepsilon_t \leq \varepsilon c^*(1+\rho^{-t+\varepsilon}\alpha_{t-\varepsilon})\rho^t\Phi^*[1-\xi]\varphi.$$

From (3.7-9)

$$\lim_{\nu\to\infty} w_t^{(\nu)} \le \{e^{ct}t\theta_{\varepsilon,\lambda}[\xi]+\varepsilon c^*(1+\rho^{-t+\varepsilon}\alpha_{t-\varepsilon})\}\rho^t\Phi^*[1-\xi]\varphi,$$

for $\varepsilon<t\le\varepsilon+\lambda$. Since $\varepsilon,\lambda>0$ were arbitrary, this implies (R).□

Let $P(\cdot,\cdot)$ be any stochastic kernel on $X\otimes\hat{A}$ such that

$$M\xi(x):=\int_{\hat{X}}\hat{x}[\xi]P(x,d\hat{x}),$$

defines a bounded operator M on B. Let $F[\cdot](x)$ be the generating functional of $P(x,\cdot)$, and as in (FM) expand

$$1-F[\xi]=M[1-\xi]-R(\xi)[1-\xi], \quad \xi\in\overline{S}.$$

Let Ψ^* be a non-negative, linear-bounded functional on B, sequentially continuous with respect to the product topology on bounded regions, and let $\psi\in\overline{S}_+$ be positive on X, possibly with $\inf\psi=0$.

<u>Lemma 2</u> ([5],[8]). Suppose $\lambda\in(0,1)$. Then

$$(3.10)\quad \sum_{\nu=1}^{\infty}\Psi^*[R(1-\lambda^\nu\psi)\psi] < \infty$$

if and only if

$$(3.11)\quad \Psi^*[\int_{\hat{X}}\hat{x}[\psi]\log\hat{x}[\psi]P(\cdot,d\hat{x})] < \infty.$$

<u>Proof</u>. We have

$$\int_0^\infty\Psi^*[R(1-\lambda^t\psi)\psi]dt-\Psi^*[M\psi]\le$$
$$\le\sum_{\nu=1}^{\infty}\Psi^*[R(1-\lambda^\nu\psi)\psi]\le\int_0^\infty\Psi^*[R(1-\lambda^t\psi)\psi]dt.$$

Substituting $s=s(\hat{x},t):=-\hat{x}[\log(1-\lambda^t\psi)]/\hat{x}[\psi]$, we get

$$\int_0^\infty\Psi^*[R(1-\lambda^t\psi)\psi]dt$$
$$=\Psi^*[\int_{\hat{X}}\int_0^\infty(\exp\{\hat{x}[\log(1-\lambda^t\psi)]\}-1+\lambda^t\hat{x}[\psi])\lambda^{-t}dtP(\cdot,d\hat{x})$$
$$=\Psi^*[\int_{\hat{X}}\int_0^{s(\hat{x},0)}\{s^{-2}(\exp\{-\hat{x}[\psi]s\}-1+\hat{x}[\psi]s)$$
$$+a(\hat{x},s)\}b(\hat{x},s)dsP(\cdot,d\hat{x})],$$

$$a(\hat{x},s(\hat{x},t)):=s^{-2}(\lambda^{t}-s)\hat{x}[\psi]=\frac{\hat{x}[\lambda^{t}\psi]-\hat{x}[|\log(1-\lambda^{t}\psi)|]}{(\hat{x}[\log(1-\lambda^{t}\psi)]/\hat{x}[\psi])^{2}},$$

$$b(\hat{x},s(\hat{x},t)):=-\lambda^{-t}s^{2}\left(\frac{\partial s}{\partial t}\right)^{-1}=\frac{1}{|\log\lambda|}\;\frac{(\hat{x}[\log(1-\lambda^{t}\psi)])^{2}}{\hat{x}[\lambda^{t}\psi]\hat{x}[\lambda^{t}\psi/(1-\lambda^{t}\psi)]},$$

Since $a(\hat{x},s(\hat{x},t))$ and $b(\hat{x},s(\hat{x},t))$ are bounded as functions of $(\hat{x},t)\in\hat{X}\otimes\mathbb{R}_{+}$, even if $\inf\psi=0$, the substitution $u:=\hat{x}[\psi]s$ leads to the equivalence of (3.1o) and

$$(3.12)\quad \Psi^{*}[\int_{\hat{X}}\hat{x}[\psi]\int_{0}^{\hat{x}[|\log(1-\psi)|]}u^{-2}(e^{-u}-1+u)\,du\,P(\cdot,d\hat{x})]<\infty.$$

For all $v>0$

$$0<c_{11}\leq[\log(1+v)]^{-1}\int_{0}^{v}u^{-2}(e^{-u}-1+u)\,du\leq c_{12}<\infty.$$

Hence (3.12) is equivalent to

$$\Psi^{*}[\int_{\hat{X}}\hat{x}[\psi]\log(1+\hat{x}[|\log(1-\psi)|])P(\cdot,d\hat{x})]<\infty,$$

which in turn is equivalent to (3.11).□

The independence property (F.1) can also be expressed in the following way. Let F_t be the σ-algebra generated on the sample space by $\{\hat{x}_s;s\leq t\}$. For $0\leq s\leq t$ and every non-negative, A-measurable η

$$(3.13)\quad \hat{x}_t[\eta]=\sum_{i=1}^{\hat{x}_s[1]}\hat{x}_t^{s,i}[\eta]\quad\text{a.s. }[P^{\hat{x}}],$$

where the $\hat{x}_t^{s,i}$, $i=1,..,\hat{x}_s[1]$, are F_t-measurable and independent conditioned on F_s, and for every $\hat{A}\in\hat{A}$

$$P^{\hat{x}}(\hat{x}_t^{s,i}\in\hat{A}|F_s)=P^{\langle x_i\rangle}(\hat{x}_{t-s}\in\hat{A})\quad\text{a.s. }[P^{\hat{x}}]$$

with $\hat{x}_s^{s,i}=\langle x_i\rangle$. The sample space may not be large enough to allow (3.13) for all $s\leq t$. However, we shall need this representation only for fixed s, or for t,s restricted to sets of the form $\{n\delta: n=0,1,2,..\}$, $\delta>0$. In both cases there exist processes equivalent to $\{\hat{x}_t, P^{\hat{x}}\}$ which satisfy (3.13). Hence we can use (3.13) for the process itself without loss of generality.

Lemma 3 ([1]). Let $\chi: \mathbb{R}_+ \to \mathbb{R}_+$ be concave with $\chi(0)=0$. Then for every $t>0$

$$(3.14)\quad \Phi^*[E^{\langle\cdot\rangle}\hat{x}_t[\varphi]\chi(\hat{x}_t[\varphi])] < \infty$$

if and only if

$$(3.15)\quad \Phi^*[k\int_{\hat{X}}\hat{x}[\varphi]\chi(\hat{x}[\varphi])\pi(\cdot,d\hat{x})] < \infty.$$

Remark. While $\log x$ does not satisfy the assumptions on χ, (3.14) with

$$\chi(x) = 1_{[0,e)}(x)x/e+1_{[e,\infty)}(x)\log x$$

is equivalent to

$$\Phi^*[E^{\langle\cdot\rangle}\hat{x}_t[\varphi]\log\hat{x}_t[\varphi]] < \infty,$$

and the same applies to (3.15).

Proof. We first assume (3.15). Let $0<\tau_1\leq\tau_2\leq..$ be the branching times of $\{\hat{x}_t, P^{\hat{x}}\}$, i.e. the times of discontinuities in $\hat{x}_t$ not caused by absorption via ∂. Define

$$\tilde{I}_t(\hat{x}) := E^{\hat{x}}\hat{x}_t[\varphi]\chi(\hat{x}_t[\varphi]), \qquad I_t(x) := \tilde{I}_t(\langle x\rangle),$$

$$\tilde{I}_{t,n}(\hat{x}) := E^{\hat{x}}\hat{x}_t[\varphi]\chi(\hat{x}_t[\varphi])1_{\{\tau_{n+1}>t\}}, I_{t,n}(x) := \tilde{I}_{t,n}(\langle x\rangle).$$

Then

$$(3.16)\quad I_{t,n+1}(x) = \int_0^t T_s^0\{k\int_{\hat{X}}\pi(x,d\hat{x})\tilde{I}_{t-s,n}(\hat{x})\}(x)ds + I_{t,o}(x).$$

Let τ_n^i, $n=1,2,..$, be the branching times of $\{\hat{x}_t^{0,i}, P^{\hat{x}}\}$, $i=1,..,\hat{x}_0[1]$. Then

$$(3.17)\quad \hat{x}_t[\varphi]1_{\{\tau_{n+1}>t\}} \leq \sum_{i=1}^{\hat{x}_0[1]} \hat{x}_t^{0,i}[\varphi]1_{\{\tau_{n+1}^i>t\}}$$

If S_r is the sum of r independent, non-negative random variables z_i, then by use of Jensen's inequality

$$(3.18)\quad ES_r\chi(S_r) \le \sum_{i=1}^{r}\{EZ_i\chi(\sum_{j\neq i} EZ_j)+EZ_i\chi(Z_i)\}$$

$$\le ES_r\chi(ES_r)+\sum_{i=1}^{r} EZ_i\chi(Z_i).$$

Applying this to (3.17), we have for $0\le t\le t_0$, t_0 arbitrary but fixed,

$$(3.19)\quad \tilde{I}_{t,n}(\hat{x}) \le \rho^t\hat{x}[\varphi]\chi(\rho^t\hat{x}[\varphi])+\hat{x}[I_{t,n}],$$

$$\int_{\hat{X}}\pi(x,d\hat{x})\tilde{I}_{t,n}(\hat{x}) \le c_{12}+c_{13}\iota(x)+mI_{t,n}(x)$$

$$\iota(x) := \int_{\hat{X}}\pi(x,d\hat{x})\hat{x}[\varphi]\chi(x[\varphi]).$$

Inserting (3.19) into (3.16) and using (3.3), (3.6), we get

$$\Phi^*[I_{t,n+1}] \le c_{14}+c_{15}\Phi^*[k\iota]+c_{16}t\sup_{0\le s\le t}\Phi^*[I_{t,n}],$$

where $\|I_{t,0}\| = \sup_{x\in X}\varphi(x)\chi(\varphi(x))$ has been absorbed into c_{14}. From this, for $0\le t\le t_0$ with $c_{16}t<1$,

$$\Phi^*[I_t] = \sup_n \sup_{0\le s\le t}\Phi^*[I_{s,n}] < \infty.$$

Applying (3.18) to $Z_i=\hat{x}_{t+s}^{t,i}[\varphi]$, $i=1,..,\hat{x}_t[1]$, $t,s\le t_0$,

$$I_{t+s}(x) = E^{\langle x\rangle}E(\hat{x}_{t+s}[\varphi]\chi(\hat{x}_{t+s}[\varphi])\,|\,F_t)$$

$$\le E^{\langle x\rangle}\{\rho^s\hat{x}_t[\varphi]\chi(\rho^s\hat{x}_t[\varphi])+\hat{x}_t[I_s]\}$$

$$\le c_{17}+c_{18}I_t(x)+c_{19}\Phi^*[I_s]\varphi(x).$$

Thus (3.14) holds for all $t\ge 0$.

Now suppose (3.14) holds for some t. By (1.3) and (M) the process $\{\rho^{-t}\hat{x}_t[\varphi],F_t,P^{\hat{x}}\}$ is a martingale. Since $u\chi(u)$ is convex, this implies

$$(3.2o)\quad \tilde{I}_s(\hat{x}) \le c_{2o}+c_{21}\tilde{I}_t(\hat{x}),\quad 0\le s\le t.$$

We have

$$I_s(x) \geq E^{\langle x \rangle} \hat{x}_s[\varphi] \chi(\hat{x}_s[\varphi]) 1_{\{\tau_1 < s\}}$$

$$= \int_0^s T_u^0 \{k \int_{\hat{X}} \pi(\cdot, d\hat{x}) \tilde{I}_{s-u}(\hat{x})\}(x) du, \quad s \leq t$$

From (IM) and (3.6)

$$(3.21) \quad \Phi^*[T_s^0 \xi] \geq (1-c^* s) \rho^s \Phi^*[\xi], \quad s \geq 0,$$

for every non-negative A-measurable ξ. Hence, for $s \leq 1/c^*$

$$\Phi^*[I_s] \geq c_{22} s \Phi^*[k\iota] - c_{23},$$

which implies (3.15).□

4. THE SUBCRITICAL CASE

Note that $P^{\langle x \rangle}(x_t = \theta) = F_t[0](x)$, $t>0$, $x \in X$. Since $F_t[0]$ is non-decreasing in t by (F.2),

$$q(x) := \lim_{t \to \infty} P^{\langle x \rangle}(\hat{x}_t = \theta), \quad x \in X,$$

exists and satisfies $q = F_t[q]$, $t>0$. From (IF)

$$(4.1) \quad 1 - F_t[\xi] = T_t^0(1-\xi) + \int_0^t T_s^0 \{k(1 - f[F_{t-s}[\xi]])\} ds.$$

If $\xi = 1$ a.e., then $F_t[\xi] \equiv 1$, $t>0$. However, if $\xi \in \bar{S}_+$ such that $\xi < 1$ on a set of positive measure, it follows from (2.1) and the irreducibility assumption on m by iteration of (4.1) that

$$F_t[\xi](x) < 1, \quad x \in X, \quad t>0.$$

Theorem 2 ([5],[8]). Suppose $\rho < 1$. Then $q = 1$, and there exists a constant $\gamma \in \mathbb{R}_+$ such that

$$(4.2) \quad \lim_{t \to \infty} \rho^{-t} P^{\hat{x}}(\hat{x}_t \neq \theta) = \gamma \hat{x}[\varphi]$$

uniformly in $\hat{x} \in X^{(n)}$ for every $n>0$. We have $\gamma > 0$ if and only if for some (and thus all) $t>0$

$$(\text{X LOG X}) \quad \Phi^*[E^{\langle \cdot \rangle} \hat{x}_t[\varphi] \log \hat{x}_t[\varphi]] < \infty.$$

Moreover, there exists a probability measure P on $(\hat{X},\hat{A})$ such that

(4.3) $$\lim_{t\to\infty} P^{\hat{y}}(\hat{x}_t[1_{A_\nu}]=n_\nu;\nu=1,..,j\,|\,\hat{x}_t\neq\theta) = P(\hat{x}[1_{A_\nu}]=n_\nu;\nu=1,..,j)$$

for each finite, measurable decomposition $\{A_\nu\}_{1\leq\nu\leq j}$ of X and uniformly in $\hat{y}\in X^{(n)}$ for every n>0. If $\gamma>0$, then

(4.4) $$\int_{\hat{X}}\hat{x}[\xi]P(d\hat{x}) = \gamma^{-1}\Phi^*[\xi], \quad \xi\in B.$$

If $\gamma=0$, then

(4.5) $$\int_{\hat{X}}\hat{x}[\xi]P(d\hat{x}) = \infty$$

for every $\xi\in B_+$ positive on a set of positive measure.

Remark. By Lemma 3 and the remark following it, (X LOG X) is equivalent to

(x log x) $\Phi^*[k\int_{\hat{X}}\pi(\cdot,d\hat{x})\hat{x}[\varphi]\log\hat{x}[\varphi]] < \infty$,

and this in turn is equivalent to

$$\int_X\xi(x)k(x)\int_{\hat{X}}\pi(x,d\hat{x})\hat{x}[\xi]\log\hat{x}[\xi]dx,$$

where ξ is any continuous, positive function on X which concides near $\bar{\Omega}\setminus X$ with a function in D_0^+.

Proof. From (FM), (RM), and (M) with $\rho<1$

(4.6) $$||1-F_t[\xi]|| \leq ||1-F_t[0]|| + ||F_t[|\xi|]-F_t[0]||$$

$$\leq 2\,||1-F_t[0]|| \leq \rho^t(1+\rho^{-t}\alpha_t)\Phi^*[1]\,||\varphi|| \to 0, \quad t\to\infty,$$

uniformly in $\xi\in\bar{S}$. To continue we need

Lemma 4. Given that $||1-F_t[0]||\to 0$, as $t\to\infty$, there exists for every t>0 a mapping $h_t:\bar{S}_+\to B$ such that

(4.7) $$1-F_t[\xi]=(1+h_t[\xi])\Phi^*[1-F_t[\xi]]\varphi, \quad t>0, \ \xi\in\bar{S}_+,$$
$$\lim_{t\to\infty}||h_t[\xi]|| = 0 \text{ uniformly in } \xi\in\bar{S}_+.$$

Proof. If $\xi=1$ a.e., then $F_t[\xi]\equiv 1$, and we may take $h_t[\xi]\equiv 0$. Now suppose $\xi<1$ on a set of positive measure, i.e. $F_t[\xi]<1$ on X. From (F.2) and (FM)

$$1-F_t[\xi]=M_s[1-F_{t-s}[\xi]]-R_s(F_{t-s}[\xi])[1-F_{t-s}[\xi]], t>s>0, \xi\in\overline{S}.$$

From this by (M) and (R)

$$(1-\rho^{-s}\alpha_s-\|g_s[F_{t-s}[\xi]]\|)\rho^s\Phi^*[1-F_{t-s}[\xi]]\varphi$$

$$\leq 1-F_t[\xi] \leq (1+\rho^{-s}\alpha_s)\rho^s\Phi^*[1-F_{t-s}[\xi]]\varphi .$$

Combining these inequalities with those obtained by applying Φ^* to them,

$$-\frac{2\rho^{-s}\alpha_s+\|g_s[F_{t-s}[\xi]]\|}{1+\rho^{-s}\alpha_s}\varphi \leq \frac{1-F_t[\xi]}{\Phi^*[1-F_t[\xi]]}-\varphi$$

$$\leq \frac{2\rho^{-s}\alpha_s+\|g_s[F_{t-s}[\xi]]\|}{1-\rho^{-s}\alpha_s-\|g_s[F_{t-s}[\xi]]\|}\varphi$$

for $t\geq t^*(s)$ and $s\geq s^*$ with some $t^*(s)<\infty, s^*<\infty$. Now use $\rho^{-s}\alpha_s\to 0$, $s\to\infty$, (R), and $\|1-F_t[\xi]\| \leq \|1-F_t[0]\|\to 0$, $t\to\infty$.□

<u>Proof of Theorem 2 continued</u>. Using (F.2), (FM), (M), and (RM),

$$(4.8)\quad 0 \leq \rho^{-t-s}\Phi^*[1-F_{t+s}[\xi]]$$

$$= \rho^{-t}\Phi^*[1-F_t[\xi]]-\rho^{-t-s}\Phi^*[R_s(F_t[\xi])[1-F_t[\xi]]$$

$$\leq \rho^{-t}\Phi^*[1-F_t[\xi]] \leq \rho^{-t}\Phi^*[1-F_t[0]].$$

Hence, there exists a non-negative, non-increasing functional $\gamma[\cdot]$ on $\overline{S}_+$ such that

$$(4.9)\quad \rho^{-t}\Phi^*[1-F_t[\xi]]\downarrow\gamma[\xi],\ t\uparrow\infty,\quad \xi\in\overline{S}_+.$$

Combined with (F.1), written in the form

$$(4.10)\ F_t(\langle x_1,..,x_n\rangle,\xi) = \prod_{\nu=1}^{n}(1-(1-F_t[\xi](x_\nu))),$$

this implies (4.2) with $\gamma:=\gamma[0]$. From (4.8) and (4.7)

$$\rho^{-n}\Phi^*[1-F_n[0]] =$$

$$= \rho^{-1}\Phi^*[1-F_1[0]] \prod_{\nu=1}^{n-1} \{1-\rho^{-1}\Phi^*[R_1(F_\nu[0])[(1+h_\nu[0])\varphi]]\}.$$

That is, $\gamma>0$ if and only if

$$\sum_{\nu=1}^{\infty} \Phi^*[R_1(F_\nu[0])[(1+h_\nu[0])\varphi]] < \infty.$$

If $\gamma>0$, there exists by (4.6) a positive real $\varepsilon<\|\varphi\|^{-1}$ such that $1-F_\nu[0]\geq\varepsilon\rho^\nu\varphi$ for all sufficiently large ν, so that

$$(4.11) \quad \sum_{\nu=1}^{\infty} \Phi^*[R_1(1-\varepsilon\rho^\nu\varphi)\varphi] < \infty,$$

in view of (RM). On the other hand, if $\gamma=0$, there is for every $\varepsilon>0$ a ν_0 such that $1-F_\nu[0]\leq\varepsilon\rho^\nu\varphi$ for all $\nu\geq\nu_0$, and (4.11) cannot hold. That is, $\gamma>0$ if and only if (4.11) is satisfied for some $\varepsilon<\|\varphi\|^{-1}$.

Now recall Lemma 2.

The generating functional of $P^{\hat{x}}(\hat{x}_t\in\cdot\,|\hat{x}_t\neq\theta)$ is given by

$$G_t(\hat{x},\xi):= \frac{F_t(\hat{x},\xi)-F_t(\hat{x},0)}{1-F_t(\hat{x},0)} = 1 - \frac{1-F_t(\hat{x},\xi)}{1-F_t(\hat{x},0)}.$$

Define G_t: $\bar{S}\to\bar{S}$ by $G_t[\cdot](x):=G_t(\langle x\rangle,\cdot)$. If there exists a functional G on $\bar{S}_+$ such that

$$(4.12) \quad \lim_{t\to\infty}|G_t[\xi](x)-G[\xi]| = 0, \quad \xi\in S_+,\ x\in X,$$

and for every sequence $(\xi_n)_{n\in\mathbb{N}}$ in $\bar{S}_+$ with $\xi_n(x)\to1$, $n\to\infty$, $x\in X$,

$$(4.13) \quad \lim_{n\to\infty} G[\xi_n] = 1,$$

then G is the restriction to $\bar{S}_+$ of the generating functional of a probability distribution P on $(\hat{X},\hat{A})$, cf. [3],[18], using (4.10)

$$\lim_{t\to\infty} G_t(\hat{x},\xi) = G[\xi], \quad \xi\in\bar{S}_+,$$

and from this, by setting $\xi=\Sigma_\nu 1_{A_\nu}\lambda_\nu$, $|\lambda_\nu|\leq1$, $\nu=1,..,j$, and appealing to the continuity theorem for generating functions, (4.3). Uniformity of (4.12) in x entails the proposed uniformity of (4.3). We now prove the existence of a G satisfying (4.12), (4.13), uniformly in x.

If $\gamma>0$, then (4.12) with $G[\xi]=1-\gamma[\xi]/\gamma$ and uniformity in x follows from (4.9) and Lemma 4, and (4.13) is obtained from

$$0 \leq \gamma[\xi_n] \leq \rho^{-t}\Phi^*[1-F_t[\xi_n]]\to 0,\ n\to\infty,$$

using dominated convergence. In the following we admit $\gamma=0$.

Lemma 5. For every $t>0$ and $\xi\in\bar{S}_+$ the function $(1-F_t[\xi])/\varphi$ has a continuous extension to $\bar{\Omega}$.

Proof. Recall (4.1). Since all quantities in the integrand are uniformly bounded and $T_s^0 B\subseteq C_0^0$, $s>0$, we have $1-F_t[\xi]\in C_0^0$, $t>0$. That is, $(1-F_t[\xi])/\varphi$ is continuous on X. The continuous extendability to $\bar{\Omega}$ follows by use of $\varphi\in D_+^0$, $T_s^0 B\subseteq C_0^1$, $s>0$, (2.5) and l'Hospital's rule.□

Proof of Theorem 2 continued. Fix $\xi\in\bar{S}_+$. By Lemma 5 the function $h_t[\xi](x)$ of Lemma 4 has a continuous extension $\bar{h}_t[\xi](x)$ to $\bar{\Omega}$ for every $t>0$. Hence, there exists a t_0 such that $G_t[\xi](x)$ has a continuous extension $\bar{G}_t[\xi](x)$ to $\bar{\Omega}$ for every $t\geq t_0$. Since $\bar{\Omega}$ is compact, there must then for each $t\geq t_0$ exist an $\bar{x}_t\in\bar{\Omega}$ such that $\bar{G}_t[\xi](\bar{x}_t)=\|G_t[\xi]\|$. It follows by the same argument as in [12, p.421] that $\bar{G}_t[\xi](\bar{x}_t)$ is decreasing, as $t\to\infty$. Thus

(4.14) $G[\xi]:=\lim_{t\to\infty}\bar{G}_t[\xi](\bar{x}_t)$

exists. However, for all $t\geq t_0$,

$$1-G_t[\xi]=\frac{1+h_t[\xi]}{1+h_t[0]}\cdot\frac{1+\bar{h}_t[0](\bar{x}_t)}{1+\bar{h}_t[\xi](\bar{x}_t)}(1-\bar{G}_t[\xi](\bar{x}_t)),$$

so that (4.14) and Lemma 4 imply (4.12) with uniformity in x.

Using Lemma 4, (F.2), (FM), and (4.1),

(4.15) $$1-G[F_t[\xi]] = \lim_{s\to\infty}\frac{\Phi^*[1-F_t[F_s[\xi]]]}{\Phi^*[1-F_s[0]]} =$$

$$= \rho^t(1-G[\xi]) - \lim_{s\to\infty}\Phi^*\left[R_t(F_s[0])\ \frac{1-F_s[\xi]}{\phi^*[1-F_s[0]]}\right]$$

$$= \rho^t(1-G[\xi]),\quad t>0,\quad \xi\in\overline{S}_+.$$

In particular,

(4.16) $G[F_t[0]] = 1-\rho^t.$

Now let $(\xi_n)_{n\in\mathbb{N}}$ be any sequence in $\overline{S}_+$ with $\xi_n(x)\to 1$, $n\to\infty$, $x\in X$. Fix $\delta>0$, $s>0$, $n_0>0$ such that

$$\rho^{-\delta}\alpha_\delta<1,$$

$$c_{24}:=\sup_{\xi\in\overline{S}_+:\|1-\xi\|\leq\|1-F_s[0]\|}\|g_\delta[\xi]\|<1-\rho^{-\delta}\alpha_\delta,$$

$$(\rho+\alpha_1)\Phi^*[1-\xi_n]\leq\rho^\delta(1-\rho^{-\delta}\alpha_\delta-c_{24})\Phi^*[1-F_s[0]],\ n\geq n_0.$$

By (M), (4.1), and (R) this is clearly possible. In view of (4.1), the monotony of $F_t[0]$, (F.2), (FM), (MR), (M), and (R), there exists a sequence of integers $(\ell(n))_{n\in\mathbb{N}}$ such that $\ell(n)\geq s$ if $n\geq n_0$, $\ell(n)\to\infty$ as $n\to\infty$, and

$$1-F_1[\xi_n]\leq(\rho-\alpha_1)\Phi^*[1-\xi_n]\varphi$$

$$\leq\rho^\delta(1-\rho^{-\delta}\alpha_\delta-c_{24})\Phi^*[1-F_{\ell(n)}[0]]\varphi$$

$$\leq 1-F_{\delta+\ell(n)}[0],\quad n\geq n_0.$$

Hence, by (4.15), (4.16),

$$1\geq G[\xi_n]=1-\rho^{-1}(1-G[F_1[\xi_n]])$$

$$\geq 1-\rho^{-1}(1-G[F_{\delta+\ell(n)}[0]])$$

$$=1-\rho^{\delta+\ell(n)-1},\quad n\geq n_0,$$

which implies (4.13).

To derive (4.4), suppose $\gamma>0$. Then by (4.2)

$$\lim_{t\to\infty} E^{\langle x\rangle}(\hat{x}_t[1]\,|\,\hat{x}_t\neq\theta) = \gamma^{-1}\Phi^*[1]<\infty.$$

Hence, G has a bounded first moment functional M. From (4.15)

$$M[M_t\xi] = \rho^t M\xi.$$

By (M) therefore $M=\varepsilon\Phi^*$, ε a positive real number. Using (4.16) and expanding G similarly as F_t in (FM),

$$1 = \rho^{-t}(1-G[F_t[0]])$$

$$= M[\rho^{-t}(1-F_t[0])]-R(F_t[0])[\rho^{-t}(1-F_t[0])],$$

where $R(\zeta)[\xi]$ is linear-bounded in ξ and tends to 0, as $\|1-\zeta\|\to 0$. From this, by (4.2), $1=M[\gamma\varphi]$. That is, $\varepsilon=\gamma^{-1}$.

Now suppose $\gamma=0$, and define

$$\varepsilon_n := \Phi^*[1-F_n[0]]/\Phi^*[1], \quad n\in\mathbb{N}.$$

By (4.1) and the monotony of $F_n[0]$, $0<\varepsilon_n\downarrow 0$, as $n\uparrow\infty$. Fix $t>0$, $n_1>0$, $s>0$ such that

$$\rho^{-t}\alpha_t < 1,$$

$$\rho^{-s}\alpha_s<1, \quad (\rho^t-\alpha_t-\rho^t c_{25})/(\rho^s+\alpha_s)\geq 1,$$

$$c_{25} := \sup_{n\geq n_1} \|g_t[1-\varepsilon_n]\|.$$

Due to (M) with $\rho<1$ and (R) this is possible. Then, using (FM), (R), and (M)

$$1-F_t[1-\varepsilon_n]\geq(\rho^t-\alpha_t-\rho^t c_{25})\Phi^*[\varepsilon_n]\varphi\geq 1-F_s[F_n[0]], \quad n\geq n_1.$$

Applying (4.15) and (F.2),

$$(1-G[1-\varepsilon_n])/\varepsilon_n=\rho^{-t}(1-G[F_t[1-\varepsilon_n]])/\varepsilon_n$$

$$\geq\rho^{-t}(1-G[F_s[F_n[0]]])/\varepsilon_n=\rho^{s-t+n}\Phi^*[1]/\Phi^*[1-F_n[0]], \quad n\geq n_1.$$

If $\gamma=0$, the last expression tends to ∞, as $n\to\infty$, by (4.2). That is, in this case G cannot have a bounded first moment functional.

5. THE CRITICAL CASE

For $t>0$ define

$$\mu(t):=\frac{1}{2t}\Phi^*[E^{\langle\cdot\rangle}\{\hat{x}_t[\varphi]^2-\hat{x}_t[\varphi^2]\}]\leq\infty.$$

<u>Proposition</u>. If $\rho=1$, then

$$\mu(t)\equiv\mu:=\frac{1}{2}\Phi^*[k\int_{\hat{X}}\pi(\cdot,d\hat{x})\{\hat{x}[\varphi]^2-\hat{x}[\varphi^2]\}],\ t>0.$$

<u>Proof</u>. Extend T_t^0, m, M_t, and Φ^* set of all non-negative, not necessarily bounded, A-measurable functions, and define

$$M_t^{(2)}[\xi](x):=E^{\langle x\rangle}\{\hat{x}_t[\xi]^2-\hat{x}_t[\xi^2]\},\ t>0,\ \xi\in B_+,\ x\in X.$$

From (F.2)

$$M_{t+s}^{(2)}[\xi]=M_t^{(2)}[M_s\xi]+M_tM_s^{(2)}[\xi],\ s,t>0,\ \xi\in B_+$$

Applying (M) with $\rho=1$,

(5.1) $$\Phi^*[M_t^{(2)}[\varphi]]=t\Phi^*[M_1^{(2)}[\varphi]]$$

for all rational t, further

$$\Phi^*[M_t^{(2)}[\varphi]]\geq\Phi^*[M_s^{(2)}[\varphi]],\ t\geq s.$$

That is, (5.1) holds for all $t>0$.

By (IF), the function $M_t^{(2)}[\xi](x)$, $t\geq 0$, $x\in X$, finite or not, solves

$$z_t(x)=\int_0^t T_s^0\{kmz_{t-s}+km^{(2)}[M_{t-s}\xi]\}(x)ds,$$

$$m^{(2)}[\xi](x):=\int_{\hat{X}}\pi(x,d\hat{x})\{\hat{x}[\xi]^2-\hat{x}[\xi^2]\},\ \xi\in B_+,\ x\in X.$$

Using (M) with $\rho=1$, (3.3), (3.6), and (3.21),

$$0\leq\Phi^*[\int_0^t T_s^0\{kmM_{t-s}^{(2)}[\varphi]\}ds\leq tc^*\sup_{0\leq s\leq t}\Phi^*[M_s^{(2)}[\varphi]]=2c^*t^2\mu,$$

$$t(1-c^*t)\Phi^*[km^{(2)}[\varphi]]\leq\Phi^*[\int_0^t T_s^0\{km^{(2)}[\varphi]\}ds]\leq t\Phi^*[km^{(2)}[\varphi]],\ t>0$$

Divide by t and let $t\downarrow 0$. □

<u>Theorem 3</u> ([4],[8]). Suppose $\rho=1$. Then either $\mu=0$ and

$\hat{x}_t[1]=\hat{x}_0[1]$ a.s. for all $t\geq 0$, or $\mu>0$ and $q\equiv 1$. If $0<\mu<\infty$ then

$$(5.2)\quad \lim_{t\to\infty} tP^{\hat{x}}(\hat{x}_t\neq\theta) = \mu^{-1}\hat{x}[\varphi]$$

uniformly in $\hat{x}\in X^{(n)}$ for each $n>0$, and for every finite, measurable decomposition $\{A_\nu\}_{1\leq\nu\leq j}$ of X and any $\hat{x}\neq\theta$

$$(5.3)\quad \lim_{t\to\infty} P^{\hat{x}}(t^{-1}\hat{x}_t[1_{A_\nu}]\leq\lambda_\nu;\nu=1,..,j\,|\,\hat{x}_t\neq\theta)$$

$$= \begin{cases} 0, & \min_\nu \lambda_\nu\leq 0 \\ 1-\exp\{-\min_\nu[(\mu\Phi^*[1_{A_\nu}])^{-1}\lambda_\nu]\}, & \min_\nu\lambda_\nu>0 \end{cases}$$

uniformly in $(\lambda_1,..,\lambda_j)\in \mathbb{R}^j$. For $\xi\in B$

$$(5.4)\quad \lim_{t\to\infty} t^{-1}E^{\langle x\rangle}(\hat{x}_t[\xi]\,|\,\hat{x}_t\neq\theta)=\mu\Phi^*[\xi].$$

Remarks. (a) If $\hat{x}_t[1]=\hat{x}_0[1]$ a.s. for all $t>0$, then it follows by (FM) and (M) that φ is constant and, with $\varphi\equiv 1$,

$$\lim_{t\to\infty} P^{\langle x\rangle}(\hat{x}_t[1_A]=1)=\Phi^*[1_A],\ x\in X,\ A\in\mathcal{A}.$$

This case occurs if and only if

$$\int_{\partial\Omega}\alpha(\tilde{y})\,d\tilde{y}+\int_X k(x)\,\pi(x,\{\hat{x}[1]\neq 1\})\,dx = 0,$$

where $d\tilde{y}$ is the differential surface element of $\partial\Omega$.

(b) As in the case of (x log x) the condition $\mu<\infty$ is equivalent to the condition obtained by substituting for φ and φ^* some continuous positive function which near $\overline{\Omega}\setminus X$ behaves as a function in D_0^+.

(c) A more intuitive way of expressing (5.3) is the following: The conditional d.f. of the vector $t^{-1}(\hat{x}_t[1_{A_1}],..,\hat{x}_t[1_{A_j}])$, given $\hat{x}_t\neq\theta$, converges to the d.f. of a vector of the form $(\Phi^*[1_{A_1}],..,\Phi^*[1_{A_j}])w$ with $P(w>\lambda)=\exp\{-\lambda/\mu\}$, $\lambda\geq 0$.

Lemma 6. For any finite collection $\{Y_\nu\}_{1\leq\nu\leq j}$ of sets in $\mathcal{A}$

the function $P^{\langle x\rangle}(\hat{x}_t[1_{Y_\nu}]=n_\nu;\nu=1,..,j)$ is continuous in $x\in X$ for every $t>0$ and continuous in $t>0$ for every $x\in X$.

Proof. It suffices to prove the lemma for finite decompositions of X. For any such decomposition

$$
\begin{aligned}
P^{\langle x\rangle}(\hat{x}_t[1_{Y_\nu}]=n_\nu;\nu=1,..,j) &= H_t(x)+I_t(x), && \sum_\nu n_\nu=0\\
&= \sum_\nu 1_{n_\nu=1} T_t^0 1_{Y_\nu}(x), && \sum_\nu n_\nu=1\\
&= I_t(x), && \sum_\nu n_\nu>1
\end{aligned}
$$

$$I_t(x):=\int_0^t T_s^0\{k\int_{\hat{X}}\pi(\cdot,d\hat{x})P^{\hat{x}}(\hat{x}_{t-s}[1_{Y_\nu}]=n_\nu;\nu=1,..,j)\}(x)ds.$$

This follows from (IF). The continuity of $H_t(x)$ and $T_t^0 1_{Y_\nu}(x)$ in x and t and that of $I_t(x)$ in x follows immediately from $\| T_t^0 \|\leq 1, T_t^0 B\subset C_0^0$, $t>0$, and the continuity of T_t^0 in t. As for the continuity of I_t in t, note that

$$\| I_{t+\delta}-I_t\| \leq \| T_\delta^0(T_\varepsilon^0 I_{t-\varepsilon})-T_\varepsilon^0 I_{t-\varepsilon}\| +3\| k\| \varepsilon,$$

$$\| I_{t-\delta}-I_t\| \leq \| T_{\varepsilon-\delta}^0(T_\varepsilon^0 I_{t-2\varepsilon})-T_\varepsilon^0(T_\varepsilon^0 I_{t-2\varepsilon})\| +4k\| \varepsilon\| ,$$

whenever $0<2\delta<2\varepsilon<t$.□

Proof of Theorem 3. Since $\varphi>0$ on X, $\mu=0$ if and only if $\Phi^*[1-P^{\langle\cdot\rangle}(\hat{x}_t\in X^{(1)})]=0$, $t>0$, i.e. $P^{\langle x\rangle}(\hat{x}_t\in X^{(1)})=1$, $x\in X$, $t>0$, by continuity. Now suppose $\mu>0$. Then $P^{\langle x\rangle}(\hat{x}_t\in X^{(1)})\neq 1$ on an x-set of positive measure depending on t. Since by (FM) and (M) with $\rho=1$

$$\Phi^*[P^{\langle\cdot\rangle}(\hat{x}_t=\theta)]=\Phi^*[R_t(0)1],\ t>0,$$

this implies $P^{\langle x\rangle}(\hat{x}_t=\theta)>0$ on a set of positive measure, depending on t. Define

$$N(t):=\{x\in X:P^{\langle x\rangle}(\hat{x}_t=\theta)=0\},\ t>0.$$

Since $P^{\langle x\rangle}(\hat{x}_t=\theta)$ is continuous, N(t) is compact. If $\Phi^*[1_{N(t)}]=0$ for some $t>0$, then $\Phi^*[1-q]= 0$ as in [3;III,12,13].

By $q=F_t[q]$ and Lemma 5, or (FM) and (M) with $\rho=1$, $\Phi^*[1-q]$ implies $q\equiv 1$. Suppose $\Phi^*[1_{N(t)}]>0$ for all $t>0$. Fix s so that $\alpha_s<1$ and define

$$N:=\bigcap_{n\in\mathbb{N}} N(ns).$$

A routine extension of [3;II,6], using compactness of $N(t)$ and thus N and continuity of $P^{\langle x\rangle}(\hat{x}_{2s}[1_N]>1)$ in x, shows that

$$\inf_{x\in N} P^{\langle x\rangle}(\hat{x}_{2s}[1_N]>1)>0$$

and that due to this $\{0<\hat{x}[1]\leq d\}$, $0<d<\infty$, is a transient event of $\{\hat{x}_{2ns},P^{\langle x\rangle},n\in\mathbb{Z}_+\}$. Given $\rho=1$, this again implies $q\equiv 1$.

Lemma 7. If $\rho=1$ and $\mu<\infty$, then for every $\delta>0$

$$\lim_{\mathbb{N}\ni n\to\infty}\frac{1}{n\delta}\{\Phi^*[1-F_{n\delta}[\xi]]^{-1}-\Phi^*[1-\xi]^{-1}\}=\mu$$

uniformly in $\xi\in\overline{S}_+$ with $\xi<1$ on a set of positive measure.

Proof. Fix ξ as required in the lemma. Then $1-F_t[\xi]>0$ on X for all $t>0$. Using (F.2)

$$\frac{1}{n\delta}\{\Phi^*[1-F_n[\xi]]^{-1}-\Phi^*[1-\xi]^{-1}\}$$

$$=\frac{1}{n}\sum_{\nu=0}^{n-1}\frac{1}{\delta}\{\Phi^*[1-F_\delta[F_{\nu\delta}[\xi]]]^{-1}-\Phi^*[1-F_{\nu\delta}[\xi]]^{-1}\}$$

$$=\frac{1}{n}\sum_{\nu=0}^{n-1}\frac{1}{\delta}(1-\Phi^*[1-F_{\nu\delta}[\xi]]\Lambda_\delta[F_{\nu\delta}[\xi]])^{-1}\Lambda_\delta[F_{\nu\delta}[\xi]],$$

$$\Lambda_\delta[\zeta]:=\Phi^*[1-\zeta]^{-2}\{\Phi^*[1-\zeta]-\Phi^*[1-F_\delta[\zeta]]\}.$$

If $\mu<\infty$, then for $\zeta=1-\eta\varphi\in\overline{S}_+$ with $\eta\in B_+$ and $\xi\in\overline{S}_+$

$$\Phi^*[1-F_t[\zeta]]=\Phi^*[M_t\zeta]-\tfrac{1}{2}\Phi^*[M_t^{(2)}[1-\zeta]]+\tfrac{1}{2}\Phi^*[R_t^{(2)}(\zeta)[1-\zeta]],$$

$$R_t^{(2)}(\xi)[1-\zeta](x):=E^{\langle x\rangle}\omega^{(2)}(\xi,\zeta,\hat{x}_t),$$

$$\omega^{(2)}(\xi,\zeta,\hat{x}):=0,\quad \hat{x}[1]\leq 2$$

$$:= \frac{1}{(n-2)!} \sum_{(i_1,..,i_n)} \zeta(x_{\nu_1})\zeta(x_{\nu_2})$$

$$\times \left(1-2\int_0^1(1-\lambda)\prod_{\kappa=3}^{n}[1-\lambda(1-\xi(x_{i_\kappa}))]d\lambda\right),$$

$$\hat{x}=\langle x_1,..,x_n\rangle,\ n>2.$$

By dominated convergence, $\Phi^*[R_t^{(2)}(\cdot)[\cdot]]$ is sequentially continuous on bounded regions in $\overline{S}_+\otimes\{\xi=\eta\varphi:\eta\in B_+\}$, and we have

$$0 = \Phi^*[R_t^{(2)}(1)[\eta\varphi]] \leq \Phi^*[R_t^{(2)}(\xi)[\eta\varphi]]$$
$$\leq \Phi^*[M_t^{(2)}[\eta\varphi]] \leq 2t\mu\|\eta\|^2$$

for $t\geq 0$, $(\xi,\eta)\in\overline{S}_+\otimes B_+$. Using (M) with $\rho=1$ and Lemma 4,

$$\Lambda_\delta[F_t[\xi]]=\tfrac{1}{2}\Phi^*[M_\delta^{(2)}[(1+h_t[\xi])\varphi]]$$
$$-\tfrac{1}{2}\Phi^*[R_\delta^{(2)}(F_t[\xi])[(1+h_t[\xi])\varphi]].$$

Since $1\geq F_t[\xi]\geq F_t[0]\uparrow 1$, as $t\uparrow\infty$,

$$\lim_{t\to\infty}\Lambda_\delta[F_t[\xi]]=\delta\mu$$

uniformly in ξ.□

<u>Proof of Theorem 3 continued</u>. Lemma 7, Lemma 4, and (F.1) written in the form (4.1o) yield (5.2) with t restricted to sets of the form $\{n\delta;n\in\mathbb{N}\}$, $\delta>0$. Since $P^{\hat{x}}(\hat{x}_t=\theta)$ is monotone in t, this implies (5.2) with $t\in\mathbb{R}_+$.

The Laplace transform $L_t^{\hat{x}}(s_1,..,s_j)$ of $Q_t^{\hat{x}}(\lambda_1,..,\lambda_j):=P^{\hat{x}}(t^{-1}\hat{x}_t[1_{A_\nu}]\leq\lambda_\nu;\nu=1,..,j)$ is given by

$$L_t^{\hat{x}} = \frac{F_t(\hat{x},\xi_t)-F_t(\hat{x},0)}{1-F_t(\hat{x},0)} = 1-\frac{1-F_t(\hat{x},\xi_t)}{1-F_t(\hat{x},0)},$$

$$\xi_t:=e^{-\xi/t},\quad \xi:=\sum_{\nu=1}^{j}s_\nu 1_{A_\nu}.$$

Note that

$$t\Phi^*[1-\xi_t]\to\Phi^*[\xi],\quad t\to\infty.$$

Using this, it follows again from Lemma 7, Lemma 4, and (F.1) that

$$= \nu\delta(1-F_{\nu\delta}(\hat{x},\xi_{\nu\delta})) \to (1+\mu\Phi^*[\xi])^{-1}\Phi^*[\xi]\hat{x}[\varphi], \quad \mathbb{N}\ni\nu\to\infty.$$

From this by (5.2)

$$\lim_{\mathbb{N}\ni\nu\to\infty} L^{\hat{x}}_{\nu\delta}=(1+\mu\Phi^*[\xi])^{-1}, \quad \delta>0.$$

The expression on the right is the Laplace transform of the limit d.f. proposed in (5.3). Denote this d.f. by Q_∞. By the continuity theorem $Q^{\hat{x}}_{\nu\delta}\to Q_\infty$, $\nu\to\infty$, and since Q_∞ is continuous, we have uniform convergence. Hence, we have convergence respective the metric

$$d(Q_1,Q_2):=\inf\{\varepsilon: Q_1(\lambda_1-\varepsilon,..,\lambda_j-\varepsilon)-\varepsilon\le Q_2(\lambda_1,..,\lambda_j) \le Q_1(\lambda_1+\varepsilon,..,\lambda_j+\varepsilon)+\varepsilon, \lambda_\nu\in[0,\infty), \nu=1,..,j\},$$

defined for all pairs of j-dimensional distribution functions Q_1,Q_2 with $Q_1(0,..,0)=Q_2(0,..,0)=0$. Writing

$$Q^{\hat{x}}_t(\lambda_1,..,\lambda_j)=\sum_{\substack{n_\nu\le t\lambda_\nu;\nu=1,..,j\\ n_1+..+n_j>0}} \frac{P^{\hat{x}}(\hat{x}_t[1_{A_\nu}]=n_\nu;\nu=1,..,j)}{P^{\hat{x}}(\hat{x}_t\neq\theta)},$$

it follows from Lemma 6 and (F.1) that $Q^{\hat{x}}_t$ is continuous in $t>0$ respective d. By the Croft-Kingman lemma [13] therefore

$$\lim_{\mathbb{R}_+\ni t\to\infty} d(Q^{\hat{x}}_t,Q_\infty)=0$$

which implies (5.3).

Concerning (5.4), note that

$$\hat{x}[M_t\xi]=E^{\hat{x}}\hat{x}_t[\xi]=P^{\hat{x}}(\hat{x}_t\neq\theta)E^{\hat{x}}(\hat{x}_t[\xi]\,|\,\hat{x}_t\neq\theta),$$

and apply (M) and (5.2).

6. THE SUPERCRITICAL CASE

By the martingale convergence theorem there exists a random variable W with $E^{\hat{x}}W\le\hat{x}[\varphi]$ such that

$$W=\lim_{t\to\infty}\rho^{-t}\hat{x}_t[\varphi] \qquad \text{a.s. } [P^{\hat{x}}].$$

Theorem 4 ([1]). Suppose $\rho>1$. Then $1-q\in D_0^+$, and for every almost everywhere continuous $\eta\in B$

$$\lim_{t\to\infty} \rho^{-t}\hat{x}_t[\eta]=\Phi^*[\eta]W \qquad \text{a.s.}[P^{\hat{x}}].$$

We have $E^{\hat{x}}W=\hat{x}[\varphi]$, $\hat{x}\in\hat{X}$, if and only if for some (and thus all) $t>0$

(X LOG X) $\Phi^*[E^{\langle\cdot\rangle}\hat{x}_t[\varphi]\log\hat{x}_t[\varphi]]<\infty$,

otherwise $W=0$ a.s. $[P^{\hat{x}}]$.

Remark. There exist a normalization sequence $\gamma_t=L(\rho^{-t})\rho^{-t}$, $L(s)$ slowly varying as $s\to 0$, and a random variable $\tilde{W}$ such that

$$P^{\hat{x}}(\tilde{W}<\infty)=1,\ P^{\hat{x}}(\tilde{W}=0)=\tilde{q}(\hat{x}),\ \hat{x}\in\hat{X},$$

and for every almost everywhere continuous $\eta\in B$

(6.1) $$\lim_{t\to\infty}\gamma_t\hat{x}_t[\eta]=\Phi^*[\eta]\tilde{W} \qquad \text{a.s.}[P^{\hat{x}}].$$

To obtain (6.1) for $\eta=\varphi$, extend [9] by use of (M), (R), and Sevastyanov's transformation. To get from there to (6.1) with a general η, proceed as below, but with $\beta_t=\hat{x}_t[\varphi]$. A detailed treatment of this and other problems will be given in a separate paper jointly with Fred Hoppe.

Proof. This proof differs in parts from the proof given in [1]. For the moment fix $t>0$. By (FM), (M) with $\rho>1$, and (R) we can find an $\varepsilon>0$ such that $\Phi^*[1-F_t[1-\xi]]>\Phi^*[\xi]$ whenever $\|\xi\|<\varepsilon$. Suppose $\Phi^*[1-q]=0$. Then $\Phi^*[1-F_s[0]]\to 0$, as $s\to\infty$. By (F.2), (FM), (RM), and (M) there must then exist an $s>0$ such that $\|1-F_s[0]\|<\varepsilon$ and consequently $\Phi^*[1-F_{t+s}[0]]>\Phi^*[1-F_s[0]]$. But this contradicts the fact that $F_s[0]$ is non-decreasing. Hence, $q<1$ on a set of positive measure. From (IF) and $q=F_t[q]$, $t>0$,

$$1-q=T_t^0(1-q)+\int_0^t T_s^0\{k(1-f[q])\}ds.$$

By (2.1) and the irreducibility assumption on m, iteration of this equation yields $q<1$ on X, and using $T_s^0 B \subseteq C_0^0$, $s>0$, and (2.3-5) we get $1-q \in D_0^+$.

Next we turn to the degeneracy question for W. Define

$$\psi_t(\lambda)(x) := E^{\langle x \rangle} \exp\{-\rho^{-t} \hat{x}_t[\varphi]\lambda\} = F_t[\exp\{-\rho^{-t}\varphi\lambda\}](x),$$

$$\psi(\lambda)(x) := E^{\langle x \rangle} e^{-W\lambda}, \quad \lambda \geq 0, \; x \in X.$$

Then

(6.2) $$\psi_{t+s}(\lambda) = F_t[\psi_s(\rho^{-t}\lambda)], \quad t,s>0, \; \lambda \geq 0,$$

$$\psi(\lambda) = F_t[\psi(\rho^{-t}\lambda)], \quad \lambda>0, \; t>0.$$

The last equation implies

$$E^{\langle x \rangle} W = M_t[\rho^{-t} E^{\langle \cdot \rangle} W](x), \quad x \in X, \; t>0.$$

By (M) we therefore have either $E^{\langle x \rangle} W = \varphi(x)$, $x \in X$, or $E^{\langle x \rangle} W = 0$, $x \in X$. Given this alternative, the first occurs if and only if

(6.3) $$\lim_{n \to \infty} \Phi^*[1-\psi_n(1)] > 0.$$

We show that (6.3) is equivalent to (X LOG X). By (FM) and (6.2)

$$\Phi^*[1-\psi_n(1)] = \Phi^*[1-F_1[\psi_{n-1}(\rho^{-1})]]$$

$$= \rho\Phi^*[1-\psi_{n-1}(\rho^{-1})] \left\{ 1-\Phi^*\left[R_1(\psi_{n-1}(\rho^{-1})) \frac{1-\psi_{n-1}(\rho^{-1})}{\Phi^*[1-\psi_{n-1}(\rho^{-1})]} \right] \right\}$$

$$= \rho^{n-1}\Phi^*[1-\psi_1(\rho^{-n+1})] \prod_{\nu=1}^{n-1} \left\{ 1-\Phi^*\left[R_1(\psi_{n-\nu}(\rho^{-\nu})) \frac{1-\psi_{n-\nu}(\rho^{-\nu})}{\Phi^*[1-\psi_{n-\nu}(\rho^{-\nu})]} \right] \right\}$$

Using (FM), (M), and (R),

$$\lim_{n \to \infty} \rho^{n-1}\Phi^*[1-\psi_1(\rho^{-n+1})] = 1,$$

and there exist $\varepsilon>0$, $\varepsilon'>0$, and $n'>0$ such that

$$1-\varepsilon\rho^{-\nu}\varphi \leq \psi_{n-\nu}(\rho^{-\nu}) \leq 1-\varepsilon'\rho^{-\nu}\varphi, \quad n \geq n', \; \nu \leq n.$$

Hence, (6.3) is equivalent to

$$\sum_{\nu=1}^{\infty} \Phi^*[R_1(1-\vartheta\rho^{-\nu}\varphi)\varphi]<\infty$$

with some $\vartheta>0$. Now recall Lemma 2.

Lemma 8. For $0<\delta\in\mathbb{R}_+$ let $Y^\delta_{n,i}, Z^\delta_{n,i}, \beta^\delta_n$, $i=1,..,\hat{x}_{n\delta}[1]$, $n=0,1,2,..$, be random variables such that

$$0\le Y^\delta_{n,i}\le Z^\delta_{n,i},\quad \beta^\delta_n\ge 0 \qquad \text{a.e.}[P^{\hat{x}}].$$

Suppose the $Y^\delta_{n,i}$ are independent conditioned on $F_{n\delta}$, the same is true of the

$$\tilde{Y}^\delta_{n,i}:=Y^\delta_{n,i}1_{\{Z^\delta_{n,i}\le\beta^\delta_{n-1}\}},\quad i=1,..,\hat{x}_{n\delta}[1],$$

and the distribution $G^\delta_{\langle x\rangle}$ of $Z^\delta_{n,i}$ depends only on $\langle x_i\rangle:=\hat{x}^{n\delta,i}_{n\delta}$,

$$\Phi^*[\int^\infty \lambda dG^\delta_{\langle\cdot\rangle}(\lambda)]<\infty.$$

Suppose further β^δ_n is $F_{n\delta}$-measurable, $\{\beta^\delta_n>0\}\supset\{\beta^\delta_{n+1}>0\}$,

(6.4) $\lim_{n\to\infty} (\beta^\delta_n)^{-1}\beta^\delta_{n+1}>1$ a.e. on $\Gamma_\delta:=\bigcap_{n\in\mathbb{N}}\{\beta^\delta_n>0\}$,

and $(\beta^\delta_n)^{-1}\hat{x}_{n\delta}[\varphi]1_{\{\beta^\delta_n>0\}}$ is bounded a.e. $[P^{\hat{x}}]$. Define

$$S^\delta_n:=1_{\Gamma_\delta}(\beta^\delta_{n-1})^{-1}\sum_{i=1}^{\hat{x}_{n\delta}[1]} Y^\delta_{n,i},\quad \tilde{S}^\delta_n:=1_{\Gamma_\delta}(\beta^\delta_{n-1})^{-1}\sum_{i=1}^{\hat{x}_{n\delta}[1]}\tilde{Y}^\delta_{n,i}$$

Then

$$\lim_{n\to\infty}\{S^\delta_n-E^{\hat{x}}(\tilde{S}^\delta_n|F_{n\delta})\}=0 \qquad \text{a.s.}[P^{\hat{x}}].$$

Proof. Omitting the superscripts $\hat{x}$ and δ, setting $\delta=1$ elsewhere, and using (1.3), (M), and (6.4),

$$\sum_{n=1}^{\infty} E\{[\tilde{S}_n-E(\tilde{S}_n|F_n)]^2|F_{n-1}\}$$

$$\le\sum_{n=1}^{\infty} E\Big\{(\beta_{n-1})^{-2}\sum_{i=1}^{\hat{x}_n[1]} E(\tilde{Y}^2_{n,i}|F_n)|F_{n-1}\Big\}1_{\{\beta_{n-1}>0\}}$$

$$=\sum_{n=1}^{\infty}(\beta_{n-1})^{-2}\hat{x}_{n-1}[M_1[\int_0^{\beta_{n-1}}\lambda^2 dG_{\langle\cdot\rangle}(\lambda)]]1_{\{\beta_{n-1}>0\}}$$

$$\leq C_1 \sum_{n=1}^{\infty} \beta_{n-1}^{-1} \int_0^{\beta_{n-1}} \lambda^2 d\Phi^*[G_{\langle\cdot\rangle}(\lambda)] 1_{\{\beta_{n-1}>0\}}$$

$$\leq C_2 \int_0^{\infty} \lambda d\Phi^*[G_{\langle\cdot\rangle}(\lambda)] + C_3,$$

$$\sum_{n=1}^{\infty} P\{S_n \neq \tilde{S}_n | F_{n-1}\}$$

$$= \sum_{n=1}^{\infty} E\{\sum_{i=1}^{\hat{x}_n[1]} P(Y_{n,i} > \beta_{n-1} | F_n) | F_{n-1}\} 1_{\{\beta_{n-1}>0\}}$$

$$\leq \sum_{n=1}^{\infty} \hat{x}_{n-1}[M_1[\int_{\beta_{n-1}}^{\infty} dG_{\langle\cdot\rangle}(x)]] 1_{\{\beta_{n-1}>0\}}$$

$$\leq C_4 \sum_{n=1}^{\infty} \beta_{n-1} \int_{\beta_{n-1}}^{\infty} d\Phi^*[G_{\langle\cdot\rangle}(\lambda)] 1_{\{\beta_{n-1}>0\}}$$

$$\leq C_5 \int_0^{\infty} \lambda d\Phi^*[G_{\langle\cdot\rangle}(\lambda)] + C_6.$$

The $C_1, \ldots, C_6$ are finite, but in general random. Chebychev's inequality and the conditional Borel-Cantelli lemma complete the proof.□

Proof of Theorem 4 continued. For $\eta \in B_+$, $0 < \delta \in \mathbb{R}_+$, and $n, r \in \mathbb{N}$ set

$$Y_{n,i}^{\delta} := Z_{n,i}^{\delta} := \hat{x}_{(n+r)\delta}^{n\delta,i}[\eta], \quad \beta_n^{\delta} := \rho^{(n+1)\delta}.$$

In the notation of Lemma 8,

$$\rho^{-(n+r)\delta} \hat{x}_{(n+r)\delta}[\eta] = \rho^{-r\delta}\{S_n^{\delta} - E^{\hat{x}}(\tilde{S}_n^{\delta} | F_{n\delta}) + \rho^{-r\delta} E^{\hat{x}}(S_n^{\delta} | F_{n\delta}) - \rho^{-r\delta} \varepsilon_n^{\delta},$$

$$\varepsilon_n^{\delta} := E^{\hat{x}}(S_n^{\delta} - \tilde{S}_n^{\delta} | F_{n\delta}).$$

By (1.3) and (M)

$$(1 - \rho^{-r\delta} \alpha_{r\delta}) \Phi^*[\eta] \rho^{-n\delta} \hat{x}_{n\delta}[\varphi] \leq \rho^{-r\delta} E^{\hat{x}}(S_n^{\delta} | F_{n\delta})$$

$$\leq (1 + \rho^{-r\delta} \alpha_{r\delta}) \Phi^*[\eta] \rho^{-n\delta} \hat{x}_{n\delta}[\varphi].$$

That is, if $\varepsilon_n^{\delta} \to 0$ a.s., $n \to \infty$, for every r, then by Lemma 8

$$\lim_{n\to\infty} \rho^{-n\delta}\hat{x}_{n\delta}[\eta]=\Phi^*[\eta]W \qquad \text{a.s.}[P^{\hat{x}}].$$

We now prove $\varepsilon_n^\delta\to 0$. First, note that

$$\varepsilon_n^\delta \le \rho^{-n\delta}\hat{x}_{n\delta}[E^{\langle\cdot\rangle}\hat{x}_{r\delta}[\eta]]$$

$$\le(\rho^{r\delta}+\alpha_{r\delta})\Phi^*[\eta]\rho^{-n\delta}\hat{x}_{n\delta}[\varphi],$$

so that in any case

$$\limsup_n \rho^{-n\delta}\hat{x}_{n\delta}[1]<\infty \quad \text{a.s.}$$

Secondly

$$\varepsilon_n^\delta \le \|\eta\|\rho^{-n\delta}\hat{x}_{n\delta}[1]\sup_x \int_{\rho^{n\delta}} \lambda dP^{\langle x\rangle}(\hat{x}_{r\delta}[1]\le\lambda).$$

From (IF), for $y>1$,

$$\int_y^\infty \lambda dP^{\langle x\rangle} \;(\hat{x}_t[1]\le\lambda)=\int_0^t T_s^0\{kN_{t-s}^y\}(x)ds,$$

$$N_{t-s}^y(x):=\int_{\hat{X}}\sum_{n\ge y} n\pi(x,d\hat{x})P^{\hat{x}}(\hat{x}_{t-s}[1]=n).$$

We have $N_s^y(x)\le m[M_s[1]]\le\|m\|e^{\|km\|s}$ and $N_s^y(x)\to 0$, $y\to\infty$, for all x and s. Using $\|T_s^0\|\le 1$, $s>0$, boundedness of $p_t(x,y)$ on $[\varepsilon,t]\otimes X\otimes X$ for every $\varepsilon>0$, and dominated convergence, this implies

$$\sup_x \int_y^\infty \lambda dP^{\langle x\rangle}(\hat{x}_{r\delta}[1]\le\lambda)\to 0, \quad y\to\infty.$$

Hence, $\varepsilon_n^\delta\to 0$, $n\to\infty$, for every r.

<u>Lemma 9</u>. If $\{\beta_t, t\in\mathbb{R}_+\}$ is a rightcontinuous process such that the $\beta_n^\delta:=\beta_{n\delta}$, $n\in\mathbb{N}$, $\delta>0$, satisfy the assumptions of the preceding lemma, with

$$\lim_{t\to\infty}\beta_t^{-1}\beta_{t+s}=\beta^s \qquad \text{a.s. on } \Gamma=\bigcap_{t\ge 0}\{\beta_t>0\},\ s>0,$$

and $\tilde{W}$ a random variable such that

$$\text{(6.5)} \qquad \lim_{\mathbb{N}\ni n\to\infty}\beta_{n\delta}^{-1}\hat{x}_{n\delta}[\xi]=\Phi^*[\xi]\tilde{W} \quad \text{a.s. on } \Gamma$$

for every $\delta>0$ and $\xi\in B_+$, then

$$\lim_{t\to\infty} \beta_t^{-1}\hat{x}_t[\eta]=\Phi^*[\eta]\tilde{W} \quad \text{a.s. on } \Gamma$$

for any almost everywhere continuous $\eta\in B$.

Proof. For every $U\in A$ define

$$\xi_U^\delta(x):=P^{\langle x\rangle}(\hat{x}_t[1_U]=\hat{x}_t[1]\,\forall t\in[0,\delta]).$$

Clearly, $\xi_U^\delta(x)\uparrow 1_U(x)$, as $\delta\downarrow 0$, for every $x\in X$. Set

$$Y_{n,i}^\delta:=1_{\{\hat{x}_t^{n\delta,i}[1_U]=\hat{x}_t^{n\delta,i}[1]\forall t\in[n\delta,(n+1)\delta]\}}, \qquad Z_{n,i}^\delta=1.$$

Then

$$\hat{x}_t[1_U]\geq S_n^\delta, \quad t\in[n\delta,(n+1)\delta],$$

and by Lemma 8 and (6.5)

$$\begin{aligned}(6.6)\quad \liminf_t \beta_t^{-1}\hat{x}_t[1_U]&\geq\beta^{-\delta}\liminf_n S_n^\delta\\ &=\beta^{-\delta}\liminf_n E^{\hat{x}}(\tilde{S}_n^\delta|F_{n\delta})=\beta^{-\delta}\liminf_n E^{\hat{x}}(S_n^\delta|F_{n\delta})\\ &=\beta^{-\delta}\liminf_n \beta_{n\delta}^{-1}\hat{x}_{n\delta}[\xi_U^\delta]=\beta^{-\delta}\Phi^*[\xi_U^\delta]\tilde{W}\uparrow\Phi^*[1_U]\tilde{W},\ \delta\downarrow 0 \text{ a.e.on}\Gamma.\end{aligned}$$

Next, set

$$Y_{n,i}^\delta=Z_{n,i}^\delta=\hat{x}_{(n+1)\delta}^{n\delta,i}[1]+\#\{t:\hat{x}_{t-}^{n\delta,i}[1]>\hat{x}_t^{n\delta,i}[1],n\delta<t\leq(n+1)\delta\}.$$

Then

$$E^{\langle x\rangle}Y_{o,1}^\delta\leq e^{\alpha\delta},\alpha=\|k\|\cdot(\|m\|+1),$$

and again by Lemma 8 and (6.5)

$$\begin{aligned}(6.7)\quad \limsup_t \beta_t^{-1}\hat{x}_t[1]&\leq\beta^{-\delta}\limsup_n S_n^\delta\\ &=\beta^{-\delta}\limsup_n E^{\hat{x}}(\tilde{S}_n^\delta|F_{n\delta})\leq\beta^{-\delta}\limsup_n E^{\hat{x}}(S_n^\delta|F_{n\delta})\\ &\leq e^{\alpha\delta}\limsup_n \beta_{n\delta}^{-1}\hat{x}_{n\delta}[1]=e^{\alpha\delta}\Phi^*[1]\tilde{W}\downarrow\Phi^*[1]\tilde{W} \quad \text{a.e.}\Gamma,\ \delta\downarrow 0.\end{aligned}$$

From (6.6) and (6.7) with U=X

$$\lim_{t\to\infty}\beta_t^{-1}\hat{x}_t[1]=\Phi^*[1]\tilde{W} \quad \text{a.e. on } \Gamma,$$

and from this and (6.6) for any U with a boundary of measure zero

$$\limsup_t \beta_t^{-1}\hat{x}_t[1_U] = \Phi^*[1]\tilde{W} - \liminf_t \beta_t^{-1}\hat{x}_t[1_U]$$

$$\leq \Phi^*[1_U]\tilde{W} \quad \text{a.e. on } \Gamma.$$

Now take an appropriate denumerable class of such U's and apply Theorem 2.2 of [2]. □

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