OC and ASN of the SPRT for the Poisson Process

Preprint
April
1976

8

Institute of Mathematical Statistics
University of Copenhagen
OC AND ASN OF THE SPRT FOR THE POISSON PROCESS

Preprint 1976 No. 8

INSTITUTE OF MATHEMATICAL STATISTICS
UNIVERSITY OF COPENHAGEN

April 1976
Summary
The paper presents an algorithm for the operating characteristic (OC) and the average sample number (ASN) functions of the sequential probability ratio test (SPRT) of the mean of a Poisson process.

As a basis of the algorithm an appendix gives a discussion of the accuracy of different formulas for the OC and ASN.

Key words
Poisson process, sequential probability ratio test, operating characteristic, average sample number, algorithm, approximation.
Purpose:
In a Poisson process where $\lambda$ denotes the mean occurrence rate per observational unit, we want to test the hypothesis $\lambda = \lambda_1$ against the alternative $\lambda = \lambda_2$, $\lambda_2 > \lambda_1 > 0$. The sequential probability ratio test (SPRT) has a continuation region of the form

$$- a + \lambda_{12} t < x_t < r + \lambda_{12} t \quad \text{for } t > 0,$$

where $x_t$ denotes the number of events in $[0, t]$, $\lambda_{12} = (\lambda_2 - \lambda_1) / (\ln(\lambda_2 / \lambda_1))$.

$a$ and $r$ could be determined as shown by Hald and Møller (1976).

For given $v = \lambda / \lambda_{12} \geq 0$ and $a, r > 0$ the procedure computes the probability of accepting $\lambda = \lambda_1$ ($OC(\lambda)$) and the average sample number ($ASN(\lambda)$).

Tables of the $OC$ and $ASN$ for selected values of $(v, a, r)$ have been given by Kiefer and Wolfowitz (1956).
Method:

The OC and ASN functions have been derived by Bartky (1943), Burman (1946) and Dvoretzky et al. (1953).

With OC (λ) = L(a,v)/L(a+r,v) and ASN(λ) = OC(λ)S(a+r,v) - S(r,v)
we compute L(y,v) and S(y,v) according to the value of y.
If y is less than 8 we must use the exact formulas rearranged in this way

\[
L(y,v) = e^{yv} \sum_{j=0}^{[y]} \left\{ (j-y)v \right\}^{j} / j!
\]

\[
S(y,v) = \sum_{i=j+1}^{[y]} \left( \sum_{j=0}^{[y]} (i-y)v \right) j! / j! * e^{(y-i)v} - [y] - 1
\]

which makes computation rather easy.
Otherwise we find the solution, t, to the equation
\[
v = t / (e^t - 1)
\]
and use the approximation formulas from Bartky (1943):

\[
L(y,v) \approx \begin{cases} 1/(1-v) + e^{-yt}/(1-v-t) & v \neq 1 \\ 2(y + 1/3) & v = 1 \end{cases}
\]

\[
S(y,v) \approx \begin{cases} (y+\delta_v)v/(1-v) - L(y,v)(t+v)/t & v \neq 1 \\ (y + 1/3)(y - 5/3) - 1/18 & v = 1 \end{cases}
\]

with \( \delta_v = 1/t - 1/4v/(1-v) \).

Structure:

real procedure sprtoa(v,a,r,asn);
Formal parameters:

\[ v \quad \text{Real value: } v = \frac{\lambda}{\lambda_{12}} \]

\[ a \quad \text{Real value: } a \]

\[ r \quad \text{Real value: } r \]

\[ \text{asn} \quad \text{Real output: the average sample number} \]

Failure indications:

If \( v, a, r \) are outside the allowed range indicated in the Purpose section, \( sprtoc \) and \( asn \) will be set to \(-1\).

Restrictions:

None.

Accuracy:

For a computer working with at least 11 significant digits, the algorithm will normally work with a smallest relative accuracy at \(10^{-8}\). The only exception is when \( a \) is small and \( r \) is large at the same time, in this case the relative accuracy of \( ASN \) may be as small as \(10^{-3}\).

If the computer works with less significant digits, the accuracy will be less and we may use the approximation formulas for smaller values of \( y \).

Given the number of significant digits at the computer we are able to give a rough guide for the accuracy of the result using the exact formulas by means of the following table. This could again lead to a new limit on \( y \) for using the approximation formulas.
The number of correct digits computing $L(y,1)$ by Bartky's approximation formula is approximately $y + 1$.

**REFERENCES:**


<table>
<thead>
<tr>
<th>$y$</th>
<th>Number of lost digits computing $L(y,1)$ by the exact formula.</th>
<th>$a+r$</th>
<th>Number of lost digits computing ASN by the exact formula with $v = 1$ and $\frac{1}{2} &lt; r/a &lt; 2$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>1.3</td>
<td>6</td>
<td>1.1</td>
</tr>
<tr>
<td>6</td>
<td>1.8</td>
<td>7</td>
<td>1.5</td>
</tr>
<tr>
<td>7</td>
<td>2.2</td>
<td>8</td>
<td>1.9</td>
</tr>
<tr>
<td>8</td>
<td>2.7</td>
<td>9</td>
<td>2.3</td>
</tr>
<tr>
<td>9</td>
<td>3.2</td>
<td>10</td>
<td>2.7</td>
</tr>
<tr>
<td>10</td>
<td>3.7</td>
<td>12</td>
<td>3.7</td>
</tr>
<tr>
<td>12</td>
<td>4.7</td>
<td>14</td>
<td>4.6</td>
</tr>
<tr>
<td>14</td>
<td>5.7</td>
<td>16</td>
<td>5.6</td>
</tr>
<tr>
<td>16</td>
<td>6.7</td>
<td>18</td>
<td>6.5</td>
</tr>
</tbody>
</table>
REAL PROCEDURE SPRTOC(V, A, R, ASN);
VALUE V, A, R,
REAL V, A, R, ASN;

COMMENT COMPUTES OC AND ASN OF THE SPRT FOR A POISSON PROCESS;

BEGIN
REAL T, DELTA, AR, TERM, EPS, LR, LAR, SR, SAR, V1, THIRD, D18,
     HALF, ONE, TWO, FOUR;
INTEGER IR, IAR, I, J;

EPS := 1.0E-8;          COMMENT SMALL NUMBER;
D18 := 0.0555555555555555556; COMMENT 1/18;
THIRD := 0.333333333333333333; COMMENT 1/3;
HALF := 0.5;            COMMENT 1/2;
ONE := 1.0;             COMMENT 1;
TWO := 2.0;             COMMENT 2;
FOUR := 4.0;            COMMENT 4;

COMMENT CHECK PARAMETERS;
IF V LSS 0.0 OR A LEQ 0.0 OR R LEQ 0.0 THEN
BEGIN
COMMENT INVALID PARAMETERS;
SPRTOC := ASN := -ONE;
GOTO EXIT
END;

IR := ENTIER(R - EPS);
AR := A + R;
IAR := ENTIER(AR - EPS);

IF IAR GEQ 8 THEN
BEGIN
COMMENT COMPUTE PARAMETERS FOR APPROXIMATION FORMULAS:
SOLUTION, T, TO V = T/(EXP(T) - 1)
AND DELTA = 1/T - 0.5xV/(1-V);

V1 := ONE - V;  TERM := V1 x THIRD;
T := V1 x (TWO + TERM x(TWO + TERM x(FOUR + TERM x(8.8 + TERM x 20.8))));
IF ABS(V1) GTR 0.01 THEN
BEGIN
REAL DERIV, TOLD, FUN, ET1;
COMMENT FOR V ≠ 1 USE NEWTON ITERATION;

LOOP:
ET1 := EXP(T) - ONE;
FUN := T / ET1;
DERIV := (ONE - FUN)/ET1 - FUN;
TOLD := T;
T := T - (FUN-V)/DERIV;
IF ABS(T-TOLD) GTR EPS x ABS(T) THEN GOTO LOOP
END;
DELTA := IF ABS(T) GTR EPS THEN ONE/T - HALF x V/V1 ELSE THIRD
END APPROXIMATION PARAMETERS;

COMMENT COMPUTE LR AND SR;
IF IR LSS 8 THEN
BEGIN
REAL ARRAY EXPS, BASE, POTS(0:IAR);
REAL DIDJ, EXPNV, EXPNAV;

COMMENT COMPUTE $L(R,V)$ AND $S(R,V)$ BY SUMMATION;

EXPNV := EXP(-V);
EXPNAV := EXP(-A×V);
EXPS(0) := TERM := EXP(R×V);

FOR I := 1 STEP 1 UNTIL IR DO
BEGIN
TERM := TERM × EXPNV; EXPS(I) := TERM;
COMMENT EXPS(I) = EXP(R-I);
END I;

TERM := - R × V;
LR := EXPS(0);
SR := IR + 1;
FOR I := 1 STEP 1 UNTIL IR DO
BEGIN
TERM := TERM + V;
BASE(I) := TERM;
POTS(I) := ONE;
COMMENT BASE(I) = (I-R)×V;
SR := SR - EXPNS(I)
END I;

FOR J := 1 STEP 1 UNTIL IR DO
BEGIN
DIDJ := ONE / J;
LR := LR + BASE(J)×POTS(J)×EXPS(J)×DIDJ;
FOR I := J+1 STEP 1 UNTIL IR DO
BEGIN
POTS(I) := TERM := BASE(I)×POTS(I)×DIDJ;
COMMENT POTS(I) = ((I-R)×V)××J/FAK(J);
SR := SR - TERM×EXPS(I)
END I;
END J;

IF IAR LSS 8 THEN
BEGIN

COMMENT COMPUTE $L(A+R,V)$ AND $S(A+R,V)$ BY SUMMATION;

TERM := EXPS(IR);
FOR I := IR+1 STEP 1 UNTIL IAR DO
BEGIN
TERM := TERM × EXPNV; EXPS(I) := TERM;
COMMENT EXPS(I) = EXP(R-I);
END I;

TERM := - AR × V;
LAR := EXPS(0);
SAR := - (IAR + 1) × EXPNAV;
FOR I := 1 STEP 1 UNTIL IAR DO
BEGIN
TERM := TERM + V;
BASE(I) := TERM;
POTS(I) := ONE;
COMMENT BASE(I) = (I-(A+R)) × V;
SAR := SAR + EXPS(I)
END I;
FOR $J := 1$ STEP 1 UNTIL IAR DO
BEGIN
    DIDJ := ONE / J;
    LAR := LAR + BASE(J) x POTS(J) x EXPS(J) x DIDJ;
END I;
END J;

TERM := LR / LAR;
SPRTOC := TERM x EXPNAV;
ASN := TERM x SAR + SR;
COMMENT A+R <= 8 FINISHED;
END
ELSE
BEGIN
COMMENT COMPUTE L(A+R,V) AND S(A+R,V) BY APPROXIMATION;

IF ABS(T) LSS EPS THEN
BEGIN
    TERM := AR + THIRD;
    LAR := TWO x TERM;
    SAR := TERM x (TERM - TWO) - D18;
END
ELSE
BEGIN
    LAR := ONE / V1 + EXP(-AR x T) / (V1 - T);
    SAR := (AR + DELTA) x V / V1 - LAR x (T + V) / T
END;
SPRTOC := TERM := LR / LAR;
ASN := TERM x SAR + SR
END;
COMMENT R <= 8 FINISHED;
END
ELSE
BEGIN
COMMENT USE APPROXIMATION FOR ALL TERMS;

IF ABS(T) LSS EPS THEN
BEGIN
    SPRTOC := (R+THIRD) / (AR+THIRD);
    ASN := A x (R+THIRD + D18/(AR+THIRD))
END
ELSE
BEGIN
    TERM := (V1-T) / V1;
    SPRTOC := TERM := (TERM + EXP(-R x T)) / (TERM + EXP(-AR x T));
    ASN := ((AR+DELTA) x TERM - R-DELTA) x V / V1
END
END APPROXIMATION;
EXIT:
END SPRTOC;
APPENDIX

THE ACCURACY OF THE FORMULAS FOR THE OC AND ASN.

1. The exact formulas

It is well-known – see e.g. Burman (1946) – that the OC of the SPRT is

\[ P(\lambda) = H(a,r,v) = L(r,v)/L(a+r,v), \quad v = \lambda/\lambda_{12} \]  

with

\[ L(y,v) = e^{yv} \sum_{i=0}^{[y]} \left\{ (i-y)e^{-v}\right\}^i/i! \]  

where \( a,r > 0, \quad v > 0 \) and \([y]\) denotes the integral part of \( y \).

It is also known that the ASN function may be found by means of

\[ \text{ASN}(\lambda) = M(a,r,v)v = H(a,r,v) \left\{ \sum_{i=1}^{[a+r]} L(a+r-i,v)-[a+r]-1 \right\} - \left\{ \sum_{i=1}^{[r]} L(r-i,v)-[r]-1 \right\} \]  

It is easy to verify that the definition of \([y]\) could be changed to

\[ [y] = \max \{ i \in \mathbb{Z} | i < y \}, \]

which is more convenient for computational purposes.

For small values of \( a \) and \( r \) these formulas give accurate results within a reasonable amount of computation. But as \( a \) and \( r \) become larger this is not necessarily true.

It is easy to show that for fixed \( y \), the minimum for the sum in (2) is found for \( v = 1 \). This means that in investigations of the accuracy it is reasonable to consider \( L(y,1) \). From approximations we know that for large values of \( y \) we will have \( L(y,1) \approx 2(y+1/3) \) and this means that the sum in (2) will be approximately \( 2ye^{-y} \). The first term in the alternating sum is \( 1 \) and the next are of the same order of magnitude. For increasing values of \( y \) the accuracy of (2) and (3) is thus decreasing rapidly.
At UNIVAC 1110 where double precision real numbers have 17-18 correct decimals we may use (1) and (3) for $a + r < 20$ yielding at least 6-7 significant digits. If we want the same accuracy for larger values of $a$ and $r$, we must use an approximation formula with this property.

**WALD'S APPROXIMATION**

The most well-known approximation is given by Jones (1952) adapting the formulas found by Wald (1947) for the approximate values of $(a, r)$ to the exact $(a, r)$.

$$P(\lambda) \approx P_W(\lambda) = \begin{cases} \frac{e^{(r+1/3)t - 1}}{e^{(r+1/3)t} - e^{-at}} & v \neq 1 \\ \frac{(r+1/3)}{(a+r+1/3)} & v = 1 \end{cases}$$  

where $t$ is the solution of $v = t / (e^t - 1)$, $v \neq 1$.  

This approximation could also be written in terms of

$$L(y, v) \approx L_W(y, v) = \begin{cases} \frac{e^{(y+1/3)t - 1}}{(1-v-t)} & v \neq 1 \\ \frac{2(y+1/3)}{1-v} & v = 1 \end{cases}$$

The approximation to the ASN is found from

$$M(a, r, v) \approx M_W(a, r, v) = \begin{cases} \frac{(a+r+1/3)P_W(\lambda) - (r+1/3)}{(1-v)} & v \neq 1 \\ a(r+1/3) & v = 1 \end{cases}$$

Table 1 shows a summary of the investigation of the accuracy of (4) and (7) for $0.001 \leq P(\lambda) \leq 0.999$. For both formulas the accuracy is shown in the case where it is smallest, which is $P(\lambda) = 0.999$ for (4) and at $P(\lambda) = 0.001$ for (7). The accuracy is measured in terms of

$$-\log_{10}\left\{\frac{f_W(\lambda) - f(\lambda)}{f(\lambda)}\right\},$$

which gives the number of correct significant digits in the approximation. Investigating the OC we use the tail probability for $f$.  

Table 1a.

Table of $-\log_{10}\{ (P_W(\lambda) - P(\lambda)) / (1-P(\lambda)) \}$ for $P(\lambda) = 0.999$.

<table>
<thead>
<tr>
<th>$r$</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>1.46</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>1.46</td>
<td>1.98</td>
<td>2.52</td>
<td></td>
<td></td>
</tr>
<tr>
<td>a+r</td>
<td>12</td>
<td>1.46</td>
<td>1.98</td>
<td>2.50</td>
<td>2.86</td>
</tr>
<tr>
<td>16</td>
<td>1.46</td>
<td>1.98</td>
<td>2.50</td>
<td>2.86</td>
<td>3.14</td>
</tr>
<tr>
<td>20</td>
<td>1.46</td>
<td>1.98</td>
<td>2.50</td>
<td>2.86</td>
<td>3.13</td>
</tr>
</tbody>
</table>

Table 1b.

Table of $-\log_{10}\{ (ASN_W(\lambda) - ASN(\lambda)) / ASN(\lambda) \}$ for $P(\lambda) = 0.001$.

<table>
<thead>
<tr>
<th>$r$</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>1.34</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>1.82</td>
<td>2.03</td>
<td>1.95</td>
<td></td>
<td></td>
</tr>
<tr>
<td>a+r</td>
<td>12</td>
<td>2.09</td>
<td>2.29</td>
<td>2.34</td>
<td>2.30</td>
</tr>
<tr>
<td>16</td>
<td>2.17</td>
<td>2.46</td>
<td>2.54</td>
<td>2.57</td>
<td>2.55</td>
</tr>
<tr>
<td>20</td>
<td>2.27</td>
<td>2.58</td>
<td>2.68</td>
<td>2.74</td>
<td>2.75</td>
</tr>
</tbody>
</table>
BARTKY'S APPROXIMATION

Bartky (1943) has given approximations to the OC for the binomial SPRT from which we can derive for $P(\lambda) \propto P_B(\lambda) = L_B(r,v)/L_B(a+r,v)$

\[
L_B(y,v) = \begin{cases} 
1/(1-v) + e^{-y/(1-v-t)} & \text{if } v \neq 1 \\
2(y+1/3) & \text{if } v = 1 
\end{cases}
\]

(8)

where $t$ is given in (5).

For the ASN we get

\[
\sum_{i=1}^{[y]} L(y-i-[y]-1) \propto \begin{cases} 
(y+\delta_v)v/(1-v) - L_B(y,v)^{(t+v)/t} & \text{if } v \neq 1 \\
(y+1/3)(y - 5/3) - 1/18 & \text{if } v = 1 
\end{cases}
\]

with $\delta_v = 1/t - \frac{1}{3} v/(1-v)$.

This leads to

\[
M(a,r,v) \propto M_B(a,r,v) = \begin{cases} 
\{(a+r+\delta_v)P_B(\lambda) - (r+\delta_v)\}/(1-v) & \text{if } v \neq 1 \\
a\{r + 1/3 + 1/(18(a+r+1/3))\} & \text{if } v = 1 
\end{cases}
\]

(10)

In the same way as for Wald's approximation we show the relative accuracy of Bartky's formulas in table 2.
### Table 2a.

Table of $-\log_{10}\{((P_B(\lambda)-P(\lambda))/(1-P(\lambda))\}$ for $P(\lambda) = 0.999$.

<table>
<thead>
<tr>
<th></th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>1.02</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>1.02</td>
<td>3.05</td>
<td>5.32</td>
<td></td>
<td></td>
</tr>
<tr>
<td>a+r</td>
<td>12</td>
<td>1.02</td>
<td>3.05</td>
<td>5.35</td>
<td>7.29</td>
</tr>
<tr>
<td></td>
<td>16</td>
<td>1.02</td>
<td>3.05</td>
<td>5.35</td>
<td>7.29</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>1.02</td>
<td>3.05</td>
<td>5.35</td>
<td>&gt; 7</td>
</tr>
</tbody>
</table>

### Table 2b.

Table of $-\log_{10}\{(ASN_B(\lambda) - ASN(\lambda))/ASN(\lambda)\}$ for $P(\lambda) = 0.001$.

<table>
<thead>
<tr>
<th></th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>1.79</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>2.55</td>
<td>3.75</td>
<td>4.61</td>
<td></td>
<td></td>
</tr>
<tr>
<td>a+r</td>
<td>12</td>
<td>2.75</td>
<td>4.38</td>
<td>5.99</td>
<td>6.74</td>
</tr>
<tr>
<td></td>
<td>16</td>
<td>2.84</td>
<td>4.63</td>
<td>6.50</td>
<td>8.20</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>2.89</td>
<td>4.75</td>
<td>6.75</td>
<td>&gt; 8</td>
</tr>
</tbody>
</table>