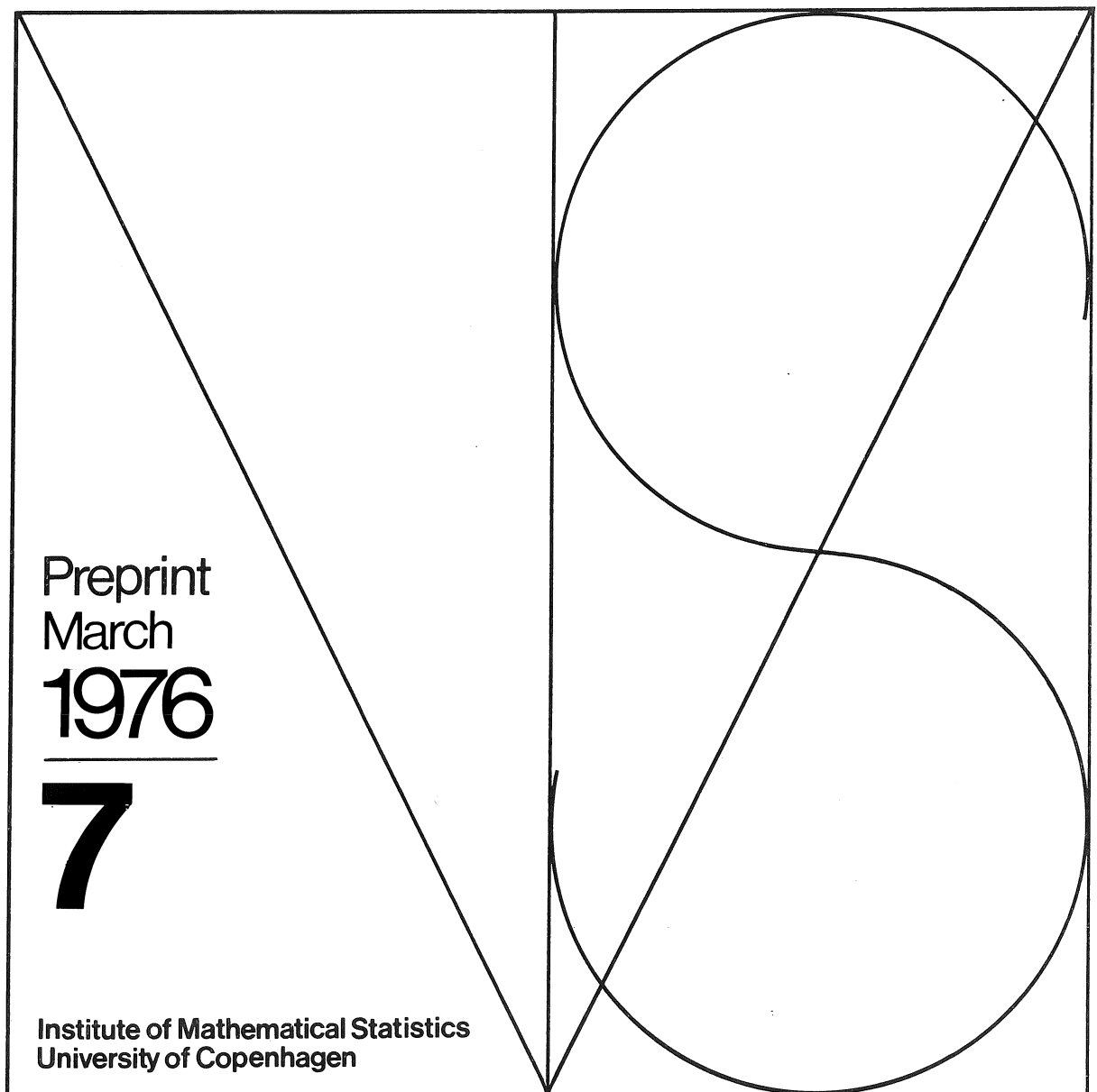


Niels Keiding

Inference and Tests for Fit in the  
Birth-Death-Immigration Process



Niels Keiding

INFERENCE AND TESTS FOR FIT IN THE  
BIRTH-DEATH-IMMIGRATION PROCESS:  
COMMENTS ON A PAPER BY J.E. COHEN  
AND ON THE USE OF THE JACKKNIFE

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## ABSTRACT

This report consists of two parts. We first (p.1) reconsider the statistical aspects of a paper by Cohen (1969) who applied a simple birth - immigration - death - emigration (BIDE) process to the population dynamics of natural primate troops. The attention is focused on the statistical analysis of Altmann's records over a year of the vital events of a baboon troop. Cohen studied this data via the "jackknife". We point out a theoretical disadvantage of the jackknife method in these circumstances, correct some mistakes in the computations and proceed to present an alternative analysis via maximum likelihood. We also suggest some ways of testing for fit of the BIDE model. Secondly, the jackknife method is studied in more detail (p.16). It is shown that for estimation of the parameter in the pure (linear) birth process, a jackknife method based on a fine division of the time interval under observation is asymptotically equivalent to the maximum likelihood method.

Statistical comments to a paper by  
J.E. Cohen on the application of a simple  
stochastic population model to natural primate troops

Cohen (1969) gave some examples of application of a simple stochastic birth-immigration-death-emigration (BIDE) model to describe the natural dynamics of primate troops. It is the purpose of this note to reconsider the statistical aspects of that paper and to present some statistical methods which might also be useful in other similar studies.

Cohen compared the theory to two types of data: first, size distributions of troops assumed to be in an equilibrium state, and second, a set of observations (by S.A. Altmann) of all vital events in a baboon troop over a year. Distributions of troop sizes for howler monkeys, gibbons, colobus and langurs, and baboons were analyzed. For the gibbons we propose here an alternative and possibly more natural interpretation of the BIDE model which yields a considerably better fit of the equilibrium distribution.

The statistical analysis of the actual dynamics over one year of the baboon troop may benefit from the advances in statistical theory since the time of Cohen's paper (1969). We present here a direct (maximum likelihood) attack on the inference problem as an alternative to Cohen's "jackknife" approach. (It may in fact be shown that the two approaches are approximately equivalent). A reiteration of the jackknife calculations shows that some numerical slips must have occurred in Cohen's computations and we present the corrected jackknife analysis as well.

The problem of testing the fit of the BIDE model is considered in some detail.

The analyses suggest that the dynamics is reasonably well described by the

BIDE model and that the stationary distribution corresponding to the dynamical data is compatible with that of the other troops observed by Altmann (this is opposite to Cohen's conclusion). On the other hand we also point out that the dynamically studied troop has a stationary expectation well below its actual size. This may suggest that the dynamics of the troop concerns a "demographically active subgroup" or that some other form of heterogeneity within the troop has to be included into the description. A similar conclusion was reached by Cohen (1975) in a recent study of the social structure of orang-utans. (The BIDE model assumes equal birth, death, immigration and emigration rates for all individuals in the population and "averaging" of different individual rates over the troop such as suggested by Cohen (1969, p. 458) will not lead to a BIDE process for the whole troop).

#### Equilibrium distributions of gibbon troops

Cohen (1969, p. 464) suggested the following mechanism for gibbon troop formation: Each troop consists of a father and a mother and a variable number of children. Therefore the births to a troop happen with intensity independent of troop size and may therefore be described as "immigrations". In other words, the birth parameter  $\lambda$  in the BIDE process should be set equal to zero, and the equilibrium distribution is then Poisson. Cohen interpreted the size of the troop, including the parents, as governed by the BIDE process, and accordingly fitted a 0,1-truncated Poisson distribution to the observed data. Another possibility would be to describe the number of children by a BIDE process and thus fit a complete Poisson distribution to the number of children, that is, to troop size minus two. This gives a much nicer fit, cf. Table 1, although this fact should be considered in the light of the extra flexibility that the shifting of the size distribution allows.

Table 1

Observed and fitted distributions of numbers of children in Gibbon troops.

Size	Hylobates lar			Total		
	Obs	Cohen's fit	Poisson	Obs	Cohen's fit	Poisson
0	8	10.9	8.4	12	14.4	11.4
1	15	12.5	14.9	17	15.3	18.3
2	12	10.7	13.0	14	12.2	14.8
3	9	7.3	7.6	9	7.8	8.0
$\geq 4^*$	5	7.6	5.0	5	7.2	4.6
$\bar{x}$	1.756			1.614		

\* interpreted as 4 when computing  $\bar{x}$  but as  $\geq 4$  in computing the expected values.

#### Dynamics of baboon troops

Cohen analyzed a set of observations on a baboon troop obtained in the Amboseli reserve, Kenya, by Dr. S.A. Altmann.

The data are given in Table 2 (identical to Cohen's Table 8). It is desired to describe this by a BIDE process with intensities  $\lambda$  (birth),  $\mu$  (death + emigration), and  $\nu$  (immigration). Let us first estimate these parameters assuming the model to hold true and later develop some tests of the adequacy of the model.

Table 2  
 BIRTH (B), IMMIGRATION (I), DEATH (D), AND EMIGRATION (E)  
 IN ALTMANN'S MAIN STUDY TROOP\*

Row j	After this many days	at this troop size	this event occurred.
1 .....	41	40	B
2 .....	5	41	B
3 .....	22	42	B
4 .....	2	43	D
5 .....	17	42	D
6 .....	26	41	I
7 .....	0	42	I
8 .....	55	43	B
9 .....	35	44	I
10 .....	20	45	E
11 .....	5	44	D
12 .....	6	43	E
13 .....	32	42	D
14 .....	4	41	D
15 .....	0	40	D
16 .....	22	39	D
17 .....	10	38	B
18 .....	0	39	B
19 .....	7	40	D
20 .....	4	39	B
21 .....	17	40	D
22 .....	11	39	E
23 .....	3	38	B
24 .....	4	39	D
25 .....	8	38	D
26 .....	2	37	D
27 .....	5	36	B
28 .....	10	37	B
.....	final	38	...

\* From Cohen (1969)

The maximum likelihood estimators (MLE), which were already given by Cohen, and estimates of their variances are derived in Appendix 1. The numerical results are presented in Table 3 along with the jackknife estimates computed according to the description by Cohen (the latter are different from Cohen's as regards  $\mu$  and  $p$ ).

Table 3

Maximum likelihood and jackknife estimates

	ML	s.d.	Jackknife	s.d.
$\lambda$	$6.5 \cdot 10^{-4}$	$2.1 \cdot 10^{-4}$	$6.5 \cdot 10^{-4}$	$2.9 \cdot 10^{-4}$
$\mu$	$9.7 \cdot 10^{-4}$	$2.5 \cdot 10^{-4}$	$9.7 \cdot 10^{-4}$	$3.3 \cdot 10^{-4}$
$\nu$	$8.0 \cdot 10^{-3}$	$4.6 \cdot 10^{-3}$	$8.0 \cdot 10^{-3}$	$5.7 \cdot 10^{-3}$
$r = \nu/\lambda$	12.4	8.2	7.8	12.5
$p = 1-\lambda/\mu$	.33	.27	.40	.37
correlation ( $r,p$ )		.37		

It is observed in the second part of this report for occurrence/exposure rates such as the ML estimators of  $\lambda$  and  $\mu$  that if the number of subdivisions on which the jackknife method is based is large, the jackknife estimator will converge towards the occurrence/exposure rate itself and the usual jackknife variance estimate will converge to an obvious variance estimate based on ML theory. In accordance with this result the ML and jackknife estimates of  $\lambda$  and  $\mu$  are seen to be identical to two significant digits, and this trivially holds true for  $\hat{\nu}$ , being a simple average.



For  $p$  and  $r$  the difference between MLE and jackknife is still easily within the standard error.

With the MLE estimates, or the revised jackknife, there is no significant difference between the present estimates of  $r = v/\lambda$  and  $p = 1 - \lambda/\mu$  and those ( $r = 2.99$ ,  $p = 0.06$ ) obtained by Cohen from fitting the stationary negative binomial distribution to the sizes of 51 baboon troops observed by Altmann. This conclusion is opposite to Cohen's.

#### Test for fit of the BIDE model

We approach the problem of evaluating the fit of the model from three different viewpoints. First, the time intervals between events are compared to their expected exponential distribution, second, the pattern of events (births, deaths and immigrations) is studied, and finally the expected stationary distribution is computed.

First, as explained in Appendix 2, the normalized waiting times

$$(\tau_{n+1} - \tau_n) \left( X_{\tau_n} + \frac{v}{\lambda + \mu} \right) \approx (\tau_{n+1} - \tau_n) (X_{\tau_n} + 5) = Y_n$$

should be exponentially distributed with expectation  $(\lambda + \mu)^{-1}$  and independent. Fig. 1 shows, on a logarithmic ordinate, one minus the observed and expected distribution functions of the  $Y_n$ 's, indicating a fair fit. Also, under this hypothesis,  $2 \sum Y_n/s$ , where  $s$  is the standard deviation of the  $Y_n$ 's, should be  $\chi^2$  distributed with 56 degrees of freedom. The observed value is 52.3, also indicating a good fit.

A useful way of getting an overall impression on the constancy of the intensities consists in plotting Aalen's (1975) nonparametric estimates of the integrated intensity. The plots are explained in Appendix 3 and shown in Fig. 2 for each of the three types of event : immigrations, births, and deaths + emigrations. It is seen that although the fit to the exponential inter-event times was seen to be satisfactory above, a definite seasonality in the vital events is obvious, in particular concerning deaths and emigrations for which many more take place in the latter half of the year.

We next propose a check on the sequence of vital events, without regard to calendar time. It will be very difficult to make any checks concerning the immigrations since there were only observed three of those, If immigrations are disregarded, there results a linear birth-and-death process which may be considered as made up of waiting times and a random walk of births (B) and losses (D and E). cf. Moran (1951,1953). This imbedded random walk is here given by

BBB B BB BB B BB  
 DD EDEDDDD D DE DDD .

We first tabulate numbers of adjacent pairs of events (D meaning D or E)

B B	B D
D B	D D

as

4	5	9
5	10	15
9	15	24

The  $\chi^2$  - test of independence gives  $\chi^2 = 0.296$ ,  $f = 1$ , showing no deviation from the random walk hypothesis.

Finally we calculate the expectation and variance of the stationary negative binomial distribution corresponding to the observed dynamical data. With  $q = 1 - p$ , this gives expectation  $rq/p$ , estimated at 24.8, and standard de-

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viation  $(rq)^{1/2}/p \approx 8.6$ . Although the standard error of the stationary expectation may be estimated as 14.3 (in a similar fashion as explained in Appendix 1) it is still striking that the process revolves slowly around the observed time average of 41.3, that is, under the model the process spends the time of observation in a rather extreme part of its stationary distribution. An exact use of this fact as counterevidence to the model would, however, need a study of the transient behaviour of the BIDE process beyond the scope of the present note. Another explanation would be that there is some heterogeneity in the troop such that the vital events governing the BIDE process correspond to what is going on in a "demographically active" subtroop, cf. the fact that the stationary expectation is lower than the time average.

As a general conclusion from these considerations we may state that the observed process is fairly well compatible with the BIDE hypothesis.

#### Acknowledgements

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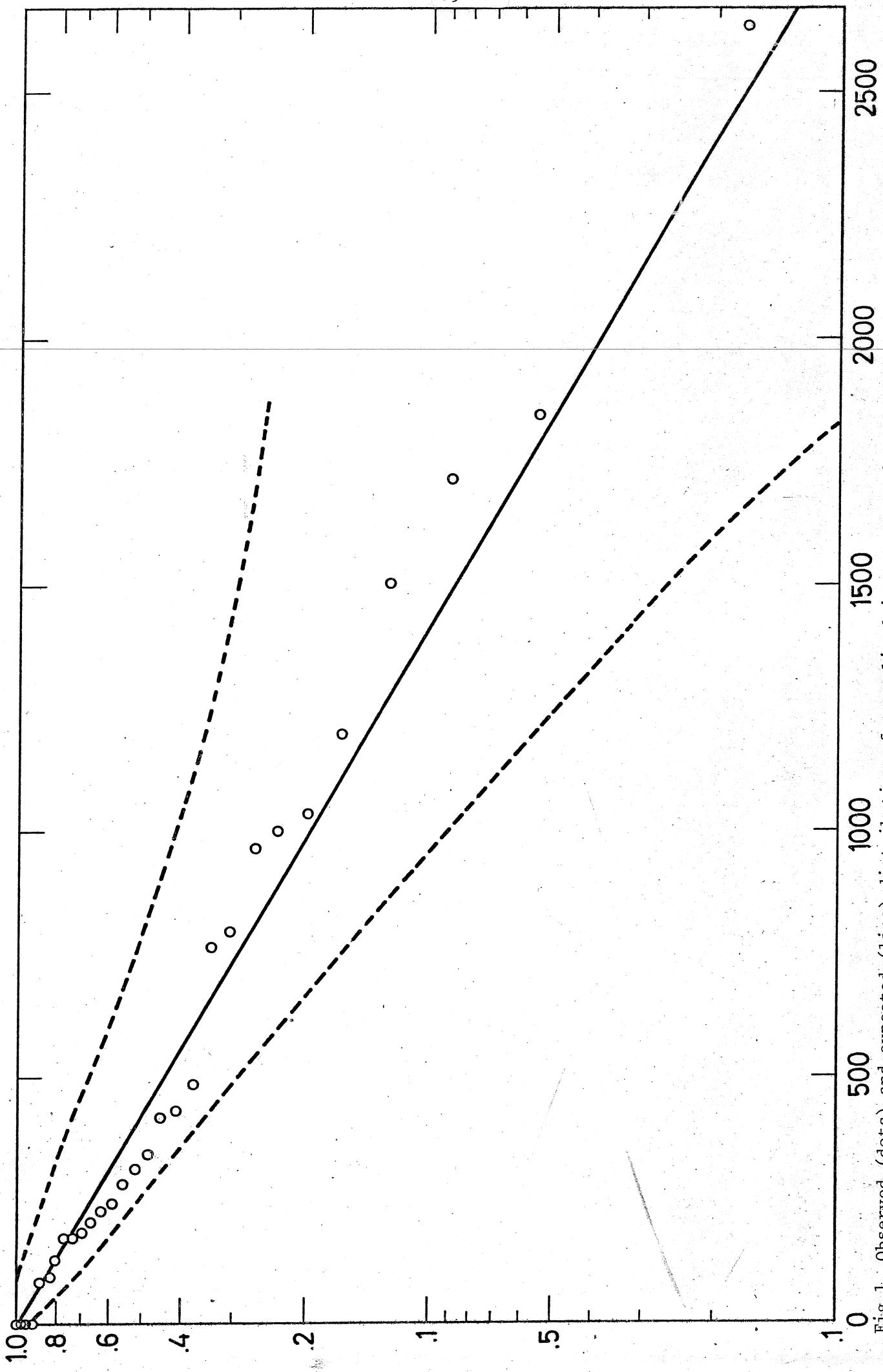


Fig 1. Observed (dots) and expected (line) distribution of normalized inter-event times with 95% confidence limits.

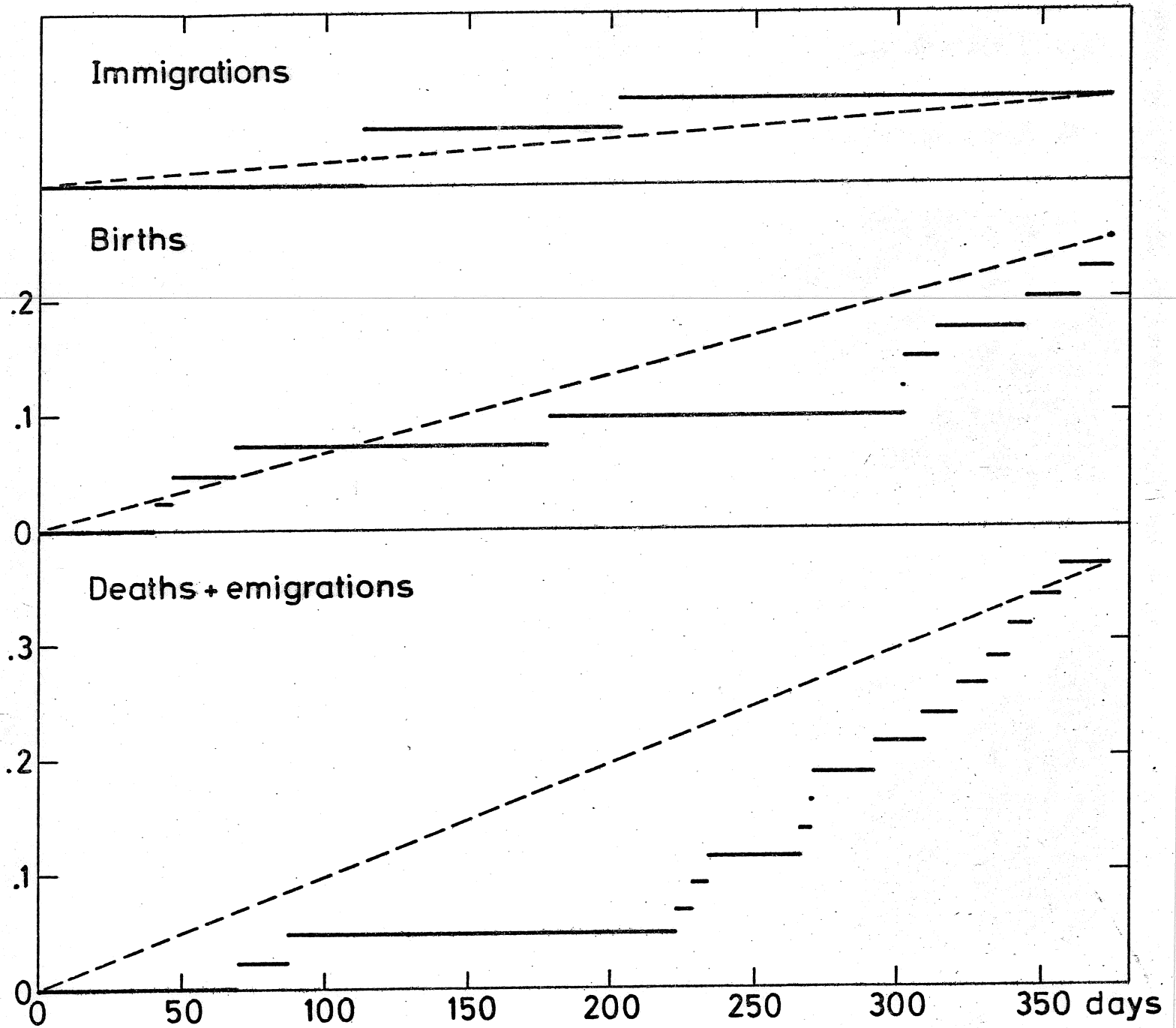


Fig. 2. Aalen's nonparametric estimator of the integrated intensities compared with the expected straight lines.

Appendix 1

Estimation in the BIDE process

We refer to standard probabilistic treatments like that of Bailey (1964, pp.91 - 101). Maximum likelihood estimation in an immigration - death process was studied by Benes<sup>V</sup> (1957) and in a linear birth - and - death process by Keiding (1975), cf. also his references. With notation as above, the likelihood function based on continuous observation of a BIDE process  $\{X_u\}$  in a time interval  $[0, t]$  is

$$e^{-v I_t - \lambda B_t - \mu D_t - (\lambda + \mu) S_t}$$

where  $I_t$  = no. of immigrations,  $B_t$  = no. of births,  $D_t$  = no. of losses and  $S_t = \int_0^t X_u du$  = time at risk.

This gives (exactly as Cohen, (1969, pp.470-471)) the ML estimators

$$\hat{v} = I_t/t, \quad \hat{\lambda} = B_t/S_t, \quad \hat{\mu} = D_t/S_t,$$

$$\hat{r} = \hat{v}/\hat{\lambda} = (I_t S_t)/t B_t, \quad \hat{q} = \hat{\lambda}/\hat{\mu} = B_t/D_t.$$

Since we have a population size of about 40 and  $\lambda t$  and  $\mu t$  around .3, the small-sample study of Beyer, Keiding and Simonsen (1976) will indicate that a large population approximation will be suitable for deriving asymptotic distributions. In fact, if both population size and  $vt$  are large,  $(\hat{\lambda}, \hat{\mu}, \hat{v})$  is asymptotically normal with asymptotically independent components (since the likelihood factorizes).

Now  $\hat{v} = I_t/t$  is Poisson with expectation  $E(\hat{v}) = v$  and variance  $\sigma_v^2 = v/t$ .

For  $\hat{\lambda}$  and  $\hat{\mu}$ , we get

$$-D_{\lambda}^2 \log L = B_t / \lambda^2$$

$$-D_{\mu}^2 \log L = D_t / \mu^2$$

and an approximation to the variance  $\sigma_{\lambda}^2$  of  $\lambda$  is then given by  $(E [-D_{\lambda}^2 \log L])^{-1}$ .

It is possible to derive  $E(B_t)$ , but since this will have to be estimated anyway, one might as well estimate  $\sigma_{\lambda}^2$  by  $\hat{\lambda}^2 / B_t = B_t / S_t^2$  and  $\sigma_{\mu}^2$  by  $D_t / S_t^2$ .

By the usual theorem on differentiable transformations we get, approximately,

$$\text{Var}(\hat{r}) = \frac{v^2}{\lambda^4} \sigma_{\lambda}^2 + \frac{1}{\lambda^2} \sigma_v^2$$

$$\text{Var}(\hat{q}) = \text{Var}(\hat{p}) = \frac{1}{\mu^2} \sigma_{\lambda}^2 + \frac{\lambda^2}{\mu^4} \sigma_{\mu}^2$$

$$\text{Cov}(\hat{r}, \hat{p}) = -\text{Cov}(\hat{r}, \hat{q}) = -\frac{v}{\mu\lambda^2} \sigma_{\lambda}^2$$

The numerical values of all these quantities in the present case are quoted in Table 3. Finally, one may similarly derive

$$\text{Var} \left( \frac{\hat{r}\hat{q}}{\hat{p}} \right) = \text{Var} \left( \frac{\hat{v}}{\hat{\mu} - \hat{\lambda}} \right) = \frac{1}{(\mu - \lambda)^2} \sigma_v^2 + \left( \frac{v}{\mu - \lambda} \right)^2 (\sigma_{\mu}^2 + \sigma_{\lambda}^2)$$

where  $r\hat{q}/\hat{p}$  is the stationary expectation.

Appendix 2

The distribution of the inter-event times of a BIDE process

Let  $0 = \tau_0 < \tau_1 < \dots < \tau_n = t$  be the times at which events happen in the BIDE process  $\{X_u\}$ . It is then well-known and follows in particular directly from the minimal construction of the process (Feller, 1971, p. 326 ff.) that given the development of the process up to the random time  $\tau_i$ , the distribution of the inter-event time  $\tau_{i+1} - \tau_i$  is exponential with expectation  $(\lambda + \mu)X_{\tau_i} + \nu$ . It follows that the random variables  $(\tau_{i+1} - \tau_i)[X_{\tau_i} + \nu/(\lambda + \mu)]$ ,  $i = 0, \dots, n-1$ , are independently identically exponentially distributed with expectation  $(\lambda + \mu)^{-1}$ . References to this fact for  $\nu = 0$ , that is, for the linear birth - and - death process, and extensive applications to inference problems were given by Athreya and Keiding (1976). It is therefore reasonable to assume that since  $\nu/(\lambda + \mu)$  is estimated by  $\hat{\nu}/(\hat{\lambda} + \hat{\mu}) = 4.96 \approx 5$ ,  $Y_i = (\tau_{i+1} - \tau_i)[X_{\tau_i} + 5]$  will be approximately i.i.d. exponential with expectation  $(\lambda + \mu)^{-1}$ . This yields a way of checking the inter-event time structure of the BIDE process. We do this by plotting the distribution of  $(Y_i)$  and by computing the sum  $\sum Y_i$  and standard deviation  $s$  of the  $Y_i$ 's. Each  $Y_i$  is approximately  $\chi^2$ -distributed with 2 degrees of freedom and scale parameter  $[2(\lambda + \mu)]^{-1}$ , and  $s$  is a consistent estimator of  $(\lambda + \mu)^{-1}$  so that  $2 \sum Y_i/s$  is approximately  $\chi^2$  with  $2n = 56$  d.f.



Appendix 3

Aalen's nonparametric estimator of the integrated intensities.

Recently Aalen (1975) proposed a nonparametric estimator of the integrated intensity of a counting process  $\{N_t\}$  generalizing the well-known Kaplan-Meier nonparametric method of estimating a distribution function from censored data. Aalen's method assumes that the intensity has the multiplicative form  $Y_t \alpha_t$ , and in that case

$$\int_0^t \frac{1}{Y_s} dN_s$$

in an unbiased estimator of

$$\int_0^t \alpha_s ds.$$

(The interpretation of the estimator is as follows. Each time  $t$  the counting process  $\{N_s\}$  makes a jump, an amount  $Y_t^{-1}$  is added to the estimator.)

In the particular case of a constant intensity  $\alpha$ , a plot of Aalen's estimator will therefore, by comparison with the straight line  $\alpha t$ , supply a check on the model.

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Jackknifing occurrence/exposure rates.

## 1. INTRODUCTION

Occurrence/exposure rates are natural estimators of the transition intensities in many simple continuous-time Markov chain models such as used in life-testing and medical follow-up studies (Littell 1952, Epstein and Sobel 1953), demography (Sverdrup 1965, Hoem 1971), or queuing theory (Cox 1965). When observation is performed in a fixed time interval, the distributional properties of these rates are rather messy (Beyer, Keiding and Simonsen 1976) and the question may therefore be raised whether the jackknife methodology could be of any help in reducing bias and/or providing approximate confidence intervals. In fact Cohen (1969) modelled the dynamics of a baboon troop by a birth-immigration-death-emigration process, using jackknifed occurrence/exposure rates as estimators of the birth, immigration, and death + emigration intensities.

A concise review of jackknife methods was given by Miller (1974) and Bissell and Ferguson (1975) discussed applications and gave worked examples.

The purpose of this note is to point out that the jackknife method is in a certain sense asymptotically equivalent to maximum likelihood estimation. Using Gaver and Hoel's (1970) idea of an increasingly finer partition of the time interval under observation, we show in the simple example of the pure (linear) birth process first that the jackknifed rate converges towards the occurrence/exposure rate itself and second, that the normalized variance of the pseudovalues (which ordinarily is used as estimator of the variance of the jackknifed estimator) converges to an obvious variance estimate obtained from ML theory.

These theoretical results conform well with the numerical results presented in the first part of this report.

It should finally be mentioned that Iglehart (1975) has used jackknife methods for estimation in stationary point processes and that Miller (1975) recently successfully attempted the application of jackknife methods to the related area of nonparametric estimation of survival functions based on censored data.

## 2. ESTIMATION IN THE PURE (LINEAR) BIRTH PROCESS

Keiding (1974) considered estimation of the parameter  $\lambda$  of the pure birth process  $X_t$  with infinitesimal transition probabilities

$$P\{X_{t+h} = j \mid X_t = i\} = \begin{cases} i\lambda h + o(h), & j = i + 1 \\ 1 - i\lambda h + o(h), & j = i \\ o(h) & \text{otherwise} \end{cases}$$

with  $X_0 = x_0$  fixed and observed in a given time interval  $[0, t]$ . Then the maximum likelihood estimator is  $\hat{\lambda} = B_t/S_t$  with  $B_t = X_t - x_0$ , the number of births to the population in  $[0, t]$  and  $S_t = \int_0^t X_u du$ , the total time under exposure.

Following the approach adopted by Gaver and Hoel (1970) in the case of the Poisson process, partition the time interval  $[0, t]$  into  $n$  subintervals

$[0, \tau], \dots, ((n-1)\tau, t]$ ,  $\tau = t/n$ . Define  $\hat{\lambda}_i$  as the occurrence/exposure rate based on observation of  $X_u$  in the intervals  $[0, (i-1)\tau]$  and  $(i\tau, t]$  only. Then if

$$R_i = \int_{(i-1)\tau}^{i\tau} X_u du, \quad Y_i = X_{i\tau} - X_{(i-1)\tau},$$

$$\hat{\lambda}_i = \frac{B_t - Y_i}{S_t - R_i}.$$

Pseudovalues are now defined by

$$\tilde{\lambda}_i = n\hat{\lambda} - (n-1)\hat{\lambda}_i$$

and the jackknifed estimator is

$$\tilde{\lambda} = \frac{1}{n} \sum_{i=1}^n \tilde{\lambda}_i = n\hat{\lambda} - \frac{n-1}{n} \sum_{i=1}^n \hat{\lambda}_i.$$

One might then expect to get a perfect jackknife by letting  $n \rightarrow \infty$  as Gaver and Hoel (1970) did it in the case of the Poisson process.

It is easily seen that almost surely  $R_i \leq \tau X_t$  and  $S_t \geq \tau x_0$  so that

$$R_i/S_t \leq X_t/(nx_0)$$

and hence  $R_i/S_t \rightarrow 0$  as  $n^{-1}$  as  $n \rightarrow \infty$ . By expansion, therefore,

$$\hat{\lambda}_i = \frac{B_t - Y_i}{S_t - R_i} = \frac{B_t - Y_i}{S_t} \left(1 + \frac{R_i}{S_t} + O(n^{-2})\right)$$

that is, approximately

$$\begin{aligned} S_t \sum_{i=1}^n \hat{\lambda}_i &= nB_t - \sum Y_i + B_t \frac{\sum R_i}{S_t} - \frac{1}{S_t} \sum Y_i R_i \\ &= nB_t - \frac{1}{S_t} \sum Y_i R_i \end{aligned}$$

since  $B_t = \sum Y_i$  and  $S_t = \sum R_i$ .

The jackknifed estimator is

$$\begin{aligned} \tilde{\lambda} &= n \frac{B_t}{S_t} - \frac{n-1}{n} \sum \hat{\lambda}_i \\ &\approx n \frac{B_t}{S_t} - (n-1) \frac{B_t}{S_t} + \frac{n-1}{n} \frac{\sum Y_i R_i}{S_t^2} \\ &= \hat{\lambda} + \frac{n-1}{n} \frac{\sum Y_i R_i}{S_t^2} \end{aligned}$$

For large  $n$ ,  $Y_i$  is either 1 or 0 according as there is a birth in  $((i-1)\tau, i\tau]$  or not. If  $Y_i = 1$ , there is a  $k$  such that  $X_{(i-1)\tau} = x_0 + k - 1$  and  $X_{i\tau} = x_0 + k$ , so that  $R_i = \int_{(i-1)\tau}^{i\tau} X_u du$  is very nearly  $\tau(x_0 + k - 1/2)$  and it is seen that

$$\sum_{i=1}^n Y_i R_i \approx \tau \sum_{k=x_0+1}^{X_t} (k - \frac{1}{2})$$

which certainly goes to 0 as  $n \rightarrow \infty$ . So  $\tilde{\lambda} \rightarrow \hat{\lambda}$  a.s. as  $n \rightarrow \infty$ .

Consider then the variance of the pseudovalues

$$\frac{1}{n-1} \sum_{i=1}^n (n\hat{\lambda} - (n-1)\hat{\lambda}_i - n\hat{\lambda} + \frac{n-1}{n} \sum \hat{\lambda}_i)^2 = (n-1) \sum_{i=1}^n (\hat{\lambda}_i - \bar{\lambda})^2$$

with  $\bar{\lambda} = \frac{1}{n} \sum \hat{\lambda}_i$ .

Here we get

$$\hat{\lambda}_i - \bar{\lambda} = \frac{B_t}{S_t} - \frac{Y_i}{S_t} + \frac{B_t R_i}{S_t^2} - \frac{R_i Y_i}{S_t^2} - \frac{B_t}{S_t} + \frac{R_i Y_i}{n S_t^2} + o\left(\frac{1}{n^2}\right)$$

so that, approximately

$$S_t^4 \sum_{i=1}^n (\hat{\lambda}_i - \bar{\lambda})^2 = \sum_{i=1}^n (-Y_i S_t + B_t R_i - R_i Y_i + \frac{1}{n} \sum R_i Y_i)^2$$

This sum of squares contains the term

$$S_t^2 \sum_{i=1}^n Y_i^2 \rightarrow B_t S_t^2$$

as  $n \rightarrow \infty$ , and it may be seen that all other terms are of order at most  $n^{-1}$ .

Thus

$$B_t^2 \sum_{i=1}^n R_i^2 \approx B_t^2 \tau^2 \sum_{i=1}^n X_{(i-\frac{1}{2})}^2$$

$$-2B_t S_t \sum_{i=1}^n R_i Y_i \approx -2B_t S_t \tau \sum_{k=x_0+1}^{X_t} (k - \frac{1}{2})$$

and

$$-2S_t \frac{1}{n} \sum R_i Y_i \approx -2S_t \frac{1}{n} \tau \sum_{k=x_0+1}^{X_t} (k - \frac{1}{2}) B_t$$

and so on. Therefore, as  $n \rightarrow \infty$ , the variance of the pseudovalues is approximately

$$(n - 1)B_t/S_t^2 .$$

As estimate of the variance of the jackknife estimator one usually takes  $n^{-1}$  variance of the pseudovalues, in this case this then is approximately  $B_t/S_t^2$ .

From Keiding (1974), Theorem 3.4(a) it follows that  $(\lambda S_t)^{1/2} (\hat{\lambda}/\lambda - 1)$  is asymptotically normal,  $\hat{\lambda}$  being the maximum likelihood estimator. This suggests  $\lambda/S_t$  as an estimate of the variance of  $\hat{\lambda}$ . A natural estimator of  $\lambda/S_t$  is  $\hat{\lambda}/S_t = B_t/S_t^2$ , or the same estimator as obtained above.

### 3. COMMENT

Notice that the assumption of approximate normality of the pseudovalues (see Miller (1974), p.1) is violated for large  $n$ . In fact

$$\tilde{\lambda}_i = \begin{cases} \frac{B_t}{S_t} - \frac{t B_t X_{(k-\frac{1}{2})\tau}}{S_t^2} & \text{if } Y_i = 0 \\ \frac{B_t}{S_t} - \frac{t B_t X_{(k-\frac{1}{2})\tau}}{S_t^2} + \frac{n-1}{S_t} - \frac{t X_{(k-\frac{1}{2})\tau}}{S_t^2} & \text{if } Y_i = 1 . \end{cases}$$

An explanation of the asymptotic equivalence of the two methods may be found in the representation of the pure birth process as a time-transformed Poisson process, cf. Keiding (1975) and Aalen (1975). The maximum likelihood estimator of  $\lambda$  in the Poisson process is linear (given by  $X_t/t$ ), thus unbiased, and hence jackknife and maximum likelihood method identical in this case.

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