Søren Asmussen

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Institute of Mathematical Statistics
University of Copenhagen

Søren Asmussen

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INSTITUTE OF MATHEMATICAL STATISTICS
UNIVERSITY OF COPENHAGEN

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ABSTRACT. We prove the law of the iterated logarithm for the martingales associated with the eigenvectors of the mean matrix of a supercritical p-type $(1 < p < \infty)$ Galton-Watson process.

1. Introduction. Consider a p-dimensional Galton-Watson process $\{Z_n\} = \{Z_n(1) \cdots Z_n(p)\}$. We introduce as briefly as possible the basic parameters and refer to Athreya and Ney [6], Ch. V, for additional background material. Let I_n be the set of individuals of the n^{th} generation and, whenever $k \in I_n$, let U_k be the offspring of k so that

$$Z_{n+1} = \sum_{k \in I_n} U_k , \qquad n = 0, 1, 2, \ldots \quad .$$

Specific assumptions on I_0 are usually not relevant, but, whenever needed, we let P^i, E^i, Var^i etc. refer to the case where I_0 consists of one individual of type i. Letting $F_{n+1} = \sigma(U_k;\ k \in I_m;\ m \leq n)$ we see that Z_{n+1} is F_{n+1}-measurable and the basic branching property states, that for fixed n the U_k, $k \in I_n$, are independent conditioned upon F_n with $P(U_k \in A | F_n) = P^i(Z_1 \in A)$, $A \subseteq N^p$, where i is the type of k. Define $m_{i,j} = E^i Z_1(j)$ and assume $M = (m_{i,j})$ to be positively regular, i.e. all elements of M^t are strictly positive for some integer $t > 0$. Let ρ be the Frobenius-Perron root of M with associated left and right eigenvectors v, u. We consider throughout the supercritical case $\rho > 1$ and defining

$$ab' = a(1)b(1) + \cdots + a(p)b(p), \quad |a| = |a(1)| + \cdots + |a(p)|$$

for p-vectors a, b, we normalize by $vu' = 1$, $|v| = 1$. Since $E(Z_{n+1} | F_n) = Z_n M$, the relation $Mu' = \rho u'$ implies that $\rho^{-n} Z_n u'$ is a non-negative martingale. Defining $W = \lim_n \rho^{-n} Z_n u'$, it is well-known that [1)]

$$(1.1) \qquad \lim_n \rho^{-n} Z_n = Wv$$

and that $\{W > 0\}$ coincides with the set $\{Z_n \neq 0$ for all $n\}$ of non-extinction under mild moment conditions. In fact, our basic assumption

$$(1.2) \qquad E^i |Z_1|^2 < \infty, \qquad i = 1, \ldots, p \quad .$$

is more than sufficient for this.

The problem with which we are concerned is this. Given any p-vector a such that $va' = 0$, we want to describe the asymptotic behavior of the linear functional $Z_n a'$ in a manner more precise than the estimate $Z_n a' = o(\rho^n)$ provided by (1.1). This problem has received some attention in the literature. For results, see Kesten and Stigum [14] and Athreya [3], [4], [5]. Following Athreya [3] and Athreya and Ney [6], Ch. V, we restrict the problem somewhat by considering only a's which are eigenvectors of M, i.e. $Ma' = \rho_1 a'$ for some ρ_1. This

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1) all relations between random variables are understood to hold almost surely

case is somewhat simpler to deal with and of particular interest, since $W_n^* = \rho_1^{-n} Z_n a'$ is then a martingale in case $\rho_1 \neq 0$, as is easily seen. The first motivation for the results arises from the observation that $\mathrm{Var}\, W_n^*$ is $O(1)$, $O(n)$, or $O(\rho^n/\rho_1^{2n})$ according to whether $\rho_1^2 > \rho$, $\rho_1^2 = \rho$ or $\rho_1^2 < \rho$. If $\rho_1^2 > \rho$, the existence of $W^* = \lim_n W_n^*$ is immediate from the martingale convergence theorem, while otherwise the results of Kesten and Stigum and Athreya state (somewhat simplified) that if $\rho_1^2 = \rho$, $\sigma^2 = v(\mathrm{Var}\, W_1^*)' = \rho_1^{-2} \sum_{i=1}^{p} v(i) \mathrm{Var}^i Z_1 a'$, then

$$\lim_n P\left(\frac{Z_n a'}{(\sigma^2 Z_n u'\, n)^{1/2}} \leq y \mid W > 0\right) = \Phi(y) \tag{1.3}$$

while if $\rho_1^2 < \rho$, $\sigma^2 = (\rho - \rho_1^2)^{-1} \sum_{i=1}^{p} v(i) \mathrm{Var}^i Z_1 a'$, then

$$\lim_n P\left(\frac{Z_n a'}{(\sigma^2 Z_n u')^{1/2}} \leq y \mid W > 0\right) = \Phi(y) \quad . \tag{1.4}$$

Here as usual $\Phi(y) = \int_{-\infty}^{y} (2\pi)^{-1/2} e^{-x^2/2} dx$.

Though certainly useful and interesting in themselves, (1.3) and (1.4) are, however, only of limited value when studying the a.s. behavior of the process, that is, of one observed realization. The complete answer is given here by our main result,

THEOREM. Let $Ma' = \rho_1 a'$ for some real ρ_1 and suppose (1.2) holds. If $\rho_1^2 > \rho$, define

$$W^* = \lim_n \rho_1^{-n} Z_n a', \quad \sigma^2 = v(\mathrm{Var}\, W^*)' = (\rho_1^2 - \rho)^{-1} \sum_{i=1}^{p} v(i) \mathrm{Var}^i Z_1 a'$$

and suppose $\sigma^2 > 0$. *Then on* $\{W > 0\}$

$$(1.5) \qquad \overline{\lim}_n \frac{\rho_1^n W^* - Z_n a'}{(2\sigma^2 Z_n u' \log n)^{1/2}} = 1, \quad \underline{\lim}_n \frac{\rho_1^n W^* - Z_n a'}{(2\sigma^2 Z_n u' \log n)^{1/2}} = -1 .$$

If $\rho_1^2 \leq \rho$, *let* σ^2 *be defined as in* (1.3), (1.4) *and suppose* $\sigma^2 > 0$. *Then if* $\rho_1^2 = \rho$,

$$(1.6) \qquad \overline{\lim}_n \frac{Z_n a'}{(2\sigma^2 Z_n u' \, n \log_2 n)^{1/2}} = 1, \quad \underline{\lim}_n \frac{Z_n a'}{(2\sigma^2 Z_n u' \, n \log_2 n)^{1/2}} = -1$$

on $\{W > 0\}$, *letting* $\log_2 = \log\log$. *Finally if* $\rho_1^2 < \rho$, *it holds on* $\{W > 0\}$ *that*

$$(1.7) \qquad \overline{\lim}_n \frac{Z_n a'}{(2\sigma^2 Z_n u' \log n)^{1/2}} = 1, \quad \underline{\lim}_n \frac{Z_n a'}{(2\sigma^2 Z_n u' \log n)^{1/2}} = -1 .$$

In the case $a = u$, $\rho_1 = \rho$, $W^* = W$, (1.5) may be seen as the multitype analogue of a result of Heyde [11] (also see Heyde and Leslie [12] and Leslie [16]). (1.5) is also similar in form to a result of Chow and Teicher [8], Th. 3, for tail sums of independent random variables. Though (1.6), (1.7) may to some extent be motivated from (1.3), (1.4), it seems more natural to view the results within the framework of the general theory of the law of the iterated logarithm (LIL) for martingales, see Stout [19] and also for example Stout [20], Heyde [13]. We elaborate upon this point in full in §3, but briefly speaking the situation is this. The sequence

$$A_n = \sum_{k=1}^{n} \mathrm{Var}(W_k^* - W_{k-1}^* \mid F_{k-1}) \tag{1.8}$$

is known from the theory of square integrable martingales to be of fundamental importance. Thus W_n^* always converges on $\{\sup_n A_n < \infty\}$ while in contrast for a number of classes of martingales

$$\overline{\lim}_n W_n^* / (2A_n \log_2 A_n)^{1/2} = 1 \quad \text{on} \quad \{\sup_n A_n = \infty\} \ . \tag{1.9}$$

Explicit computation of A_n in our example shows that (1.9) is equivalent to (1.6) when $\rho_1^2 = \rho$ and to (1.7) when $0 < \rho_1^2 < \rho$ (in contrast, $\sup_n A_n < \infty$ if $\rho_1^2 > \rho$). No criterion in the literature seems, however, to yield (1.9) immediately and our proofs exploit a mixture of special properties of the process and results and methods developed for general martingales.

Though (1.5), (1.6), (1.7) form a complete trichotomy in the present setting, we feel it reasonable to point out that a number of new problems naturally arises. For example one would be interested in extending the results to continuous time and to arbitrary linear functionals rather than eigenvector functionals. Also it would be of considerable interest to prove similar results for branching diffusions, a large class of which have a complete set of eigenfunctions of the mean, see Hering [10] and Asmussen and Hering [2]. It would be tempting to think that here the case of an arbitrary linear functional could be treated by expanding in eigenfunctions, but, as was remarked by Harry Kesten, this is of course not immediate, since functionals corresponding to different eigenvalues can have normalizing factors of the same order, cf. (1.7).

We have in fact some results dealing with such generalizations. But they are not quite complete and also the proofs are in part more tedious than (and totally different from) the ones of this paper. This is why we have focused our attention on the present setting which is of particular importance anyway.

2. An auxiliary result. Proofs when $\rho_1^2 > \rho$ or $\rho_1 = 0$.

In order to avoid making trivial exceptions on the set of extinction, we assume from now on $P(W>0)=1$. Also, the proofs of the $\overline{\lim}$ and the $\underline{\lim}$ parts of the results are always similar and we treat only $\overline{\lim}$. In contrast, the proofs of $\overline{\lim} \leq 1$ and $\overline{\lim} \geq 1$ are certainly not the same. We shall need the following auxiliary result.

PROPOSITION. Let $Y = Y(Z_0, Z_1, \ldots)$ be some functional of the process such that

$$0 < \sigma^2 = v(\mathrm{Var}^{\cdot} Y)' < \infty, \quad E^i Y = 0, \quad i = 1, \ldots, p$$

and let Y_k be the corresponding functional of the line of descent initiated by $k \in I_n$. Then

$$(2.1) \qquad \overline{\lim}_n \frac{\sum_{k \in I_n} Y_k}{(2\sigma^2 Z_n u' \log n)^{1/2}} \leq 1$$

with the inequality replaced by equality if Y is F_m-measurable for some $m < \infty$.

The proof is based upon normal approximations and the elementary

LEMMA 1. Let $\{T_n\}$ be a sequence of random variables such that

$$(2.2) \qquad \sum_{n=0}^{\infty} \Delta_n < \infty$$

where $\Delta_n = \sup_{-\infty < y < \infty} |P(T_n \leq y | F_n) - \Phi(y)|$. Then

$$\overline{\lim}_n \; T_n/(2 \log n)^{1/2} \leq 1 \tag{2.3}$$

<u>with the inequality replaced by equality if</u> T_n <u>is</u> F_{n+m}<u>-measurable for some</u> $m < \infty$.

PROOF. It is well-known that $1 - \Phi(y) \cong e^{-y^2/2}/y$ as $y \to \infty$. Therefore for $\gamma > 1$

$$\sum_{n=0}^{\infty} P(T_n > (2\gamma \log n)^{1/2} | F_n) \leq \sum_{n=0}^{\infty} \{1-\Phi((2\gamma \log n)^{1/2}) + \Delta_n\} =$$

$$\sum_{n=0}^{\infty} \{O(\frac{1}{n^{\gamma}(\log n)^{1/2}}) + \Delta_n\} < \infty$$

and the conditional Borel-Cantelli lemma gives $\overline{\lim}\; T_n/(2 \log n)^{1/2} \leq \gamma^{1/2}$. The fact that one does not need to require T_n to be F_{n+1} - measurable is not in most standard textbooks and we refer to Meyer [18], pg.9. As $\gamma \to 1$, (2.3) follows. In the same way we get for $\gamma < 1$

$$\sum_{n=0}^{\infty} P(T_n > (2\gamma \log n)^{1/2} | F_n) \geq \sum_{n=0}^{\infty} \{1-\Phi((2\gamma \log n)^{1/2}) - \Delta_n\} =$$

$$\sum_{n=0}^{\infty} \{O(\frac{1}{n^{\gamma}(\log n)^{1/2}}) - \Delta_n\} = \infty .$$

If T_n is F_{n+m}-measurable, this implies $\overline{\lim}_n \; T_n/(2 \log n)^{1/2} \geq \gamma^{1/2}$. For this, it suffices to refer to the standard version of the conditional Borel-Cantelli lemma (Breiman [7], pg.96). Now let $\gamma \to 1$. $\square$

PROOF OF PROPOSITION. Define[1)]

$$Y_k' = Y_k I(|Y_k| \leq \rho^{n/2}) , \quad \tilde{Y}_k = Y_k' - EY_k' ,$$

$$S_n = \sum_{k \in I_n} \tilde{Y}_k , \quad \tau_n^2 = \mathrm{Var}(S_n | F_n), \quad T_n = S_n / \tau_n .$$

By a standard moment inequality,

$$E|\tilde{Y}_k|^3 \leq E|Y_k'|^3 + 6E|Y_k'|(E|Y_k'|)^2 + (E|Y_k'|)^3 \leq 8E|Y_k'|^3 = 8\int_0^{\rho^{n/2}} y^3 dF_{i(k)}(y)$$

where $i(k)$ is the type of k and $F_i(y) = P^i(|Y| \leq y)$. Letting C be the Berry-Esseen constant, we get

$$\Delta_n \leq 8C \tau_n^{-3} \sum_{i=1}^{p} Z_n(i) \int_0^{\rho^{n/2}} y^3 \, dF_i(y) .$$

It is readily checked that

$$(2.4) \qquad \lim_n \tau_n^2 / \sigma^2 Z_n u' = 1 .$$

Therefore $\tau_n^{-3} Z_n(i) = O(\rho^{-n/2})$, and (2.2) follows from

$$\sum_{n=0}^{\infty} \rho^{-n/2} \int_0^{\rho^{n/2}} y^3 dF_i(y) = \int_0^{\infty} y^3 \sum \rho^{-n/2} I(y \leq \rho^{n/2}) dF_i(y) =$$

$$\int_0^{\infty} y^3 O(y^{-1}) dF_i(y) = \int_0^{\infty} O(y^2) dF_i(y) < \infty .$$

Thus (2.3) holds and it only remains to prove

$$(2.5) \qquad \overline{\lim}_n \frac{\sum_{k \in I_n} Y_k}{(2\sigma^2 Z_n u' \log n)^{1/2}} = \overline{\lim}_n \frac{T_n}{(2 \log n)^{1/2}} .$$

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1) EY_k' etc. is a convenient notation, but strictly speaking we mean $E(Y_k' | F_n)$

Recalling (2.4) and the explicit definitions of Y_k', $\tilde{Y}_k$, T_n, it suffices that

$$\sum_{k\in I_n}\{Y_k - Y_k'\} = o(\rho^{n/2}(\log n)^{1/2}), \quad \sum_{k\in I_n} EY_k' = o(\rho^{n/2}(\log n)^{1/2})$$

or, appealing to Kronecker's lemma, that

$$(2.6) \qquad \sum_{n=0}^{\infty} \rho^{-n/2}(\log n)^{-1/2} \sum_{k\in I_n} |Y_k - Y_k'| < \infty \quad ,$$

$$\sum_{n=1}^{\infty} \rho^{-n/2}(\log n)^{-1/2} \sum_{k\in I_n} |EY_k'| < \infty \quad .$$

Noting that $|Z_n(i)| = O(\rho^n)$ and that

$$|EY_k'| = |E(Y_k' - Y_k)| \le E|Y_k' - Y_k| = E|Y_k| I(|Y_k| > \rho^{n/2}) = \int_{\rho^{n/2}}^{\infty} y \, dF_{i(k)}(y)$$

it suffices for both assertions of (2.6) that

$$(2.7) \qquad \sum_{n=0}^{\infty} \rho^{-n/2}(\log n)^{-1/2} \rho^n \int_{\rho^{n/2}}^{\infty} y \, dF_i(y) < \infty, \quad i = 1,\ldots,p$$

(for the first, take the mean). And (2.7) certainly holds since even

$$\sum_{n=0}^{\infty} \rho^{n/2} \int_{\rho^{n/2}}^{\infty} y \, dF_i(y) = \int_0^{\infty} y \sum \rho^{n/2} I(y > \rho^{n/2}) \, dF_i(y) = \int_0^{\infty} O(y^2) \, dF_i(y) < \infty. \ \square$$

PROOF OF (1.5). Letting $Y = W^* - W_0$ in (2.1) gives $\overline{\lim} \le 1$, $\underline{\lim} \ge -1$ in (1.5), since in that case

$$\sum_{k\in I_n} Y_k = \rho_1^n W^* - Z_n a' \quad .$$

Defining $\sigma_m^2 = v(\text{Var}' W_m^*)$ and letting $Y = W_m^* - W_0$ yields in a similar fashion

$$\overline{\lim}_n \frac{\rho_1^{-m} Z_{n+m} a' - Z_n a'}{(2\sigma_m^2 Z_n u' \log n)^{1/2}} = 1 \quad . \tag{2.8}$$

Combining these results, we obtain

$$\overline{\lim}_n \frac{\rho_1^n W^* - Z_n a'}{(2\sigma^2 Z_n u' \log n)^{1/2}} \geq$$

$$\underline{\lim}_n \rho_1^{-m} \frac{\rho_1^{n+m} W^* - Z_{n+m} a'}{(2\sigma^2 Z_n u' \log n)^{1/2}} + \overline{\lim}_n \frac{\rho_1^{-m} Z_{n+m} a' - Z_n a'}{(2\sigma^2 Z_n u' \log n)^{1/2}} \geq$$

$$-\rho_1^{-m} \lim_n \left(\frac{Z_{n+m} u' \log(n+m)}{Z_n u' \log n}\right)^{1/2} + \frac{\sigma_m}{\sigma} = -\left(\frac{\rho}{\rho_1^2}\right)^{m/2} + \frac{\sigma_m}{\sigma} \quad .$$

As $m \to \infty$, the right-hand side tends to $-0 + 1$. □

PROOF OF (1.7) WHEN $\rho_1 = 0$. Let $Y = Z_1 a'$. Then $\rho_1 = 0$ implies $E^i Y = 0$. It is thus immediate that (2.1) with equality applies and since

$$\sum_{k \in I_n} Y_k = Z_{n+1} a', \quad \lim_n Z_{n+1} u' / Z_n u' = \rho \ ,$$

(1.7) follows. □

3. <u>Outline of proofs</u> - $0 < \bar{\rho}_1^2 \leq \rho$

Let $\{G_n\}$ be an increasing sequence of σ-algebras and $\{X_n\}$ a square integrable martingale with respect to $\{G_n\}$. Define

$$Y_k = X_k - X_{k-1}, \quad s_n^2 = \sum_{k=1}^{n} E(Y_k^2 | G_{k-1}), \quad \varphi(a) = (2a \log_2 a)^{1/2} .$$

Stout [19] proved

LEMMA 2. *Suppose* $\sup_n s_n^2 = \infty$. *Then*

(3.1) $$\overline{\lim}_n X_n/\varphi(s_n^2) = 1$$

if there exists a sequence $\{K_n\}$ *with* K_n G_{n-1}*-measurable,* $\lim_n K_n = 0$, $K_n \geq 0$ *such that*

(3.2) $$|Y_k| \leq K_k s_k/(\log_2 s_k^2)^{1/2} .$$

This result extends Kolmogorov's LIL [15] in full to the martingale case. A classical counterexample (Marcinkiewicz and Zygmund [17]) shows that (3.1) cannot hold in general without some condition like (3.2). However, assuming special properties of the martingale it is frequently possible to eliminate (3.2). To this end, one truncates Y_n and proceeds in much the same manner as when deriving the Hartman-Wintner LIL [9] from the Kolmogorov LIL. For work in this direction, see Stout [20], Heyde [13], Tomkins [21].

In our specific example we see, recalling the definition (1.8) of A_n, that (1.9) and (3.1) are the same statement. Also we have

$$A_n = \sum_{i=1}^{n} \rho_1^{-2i} Z_{i-1}(\mathrm{Var}^{\cdot} Z_1 a')'$$

and combining with (1.1) one obtains

(3.4) $\quad \lim_n A_n/n = Wv(\mathrm{Var}^{\cdot} Z_1 a')'/\rho = W\sigma^2 \quad \text{if} \quad \rho_1^2 = \rho$

(3.5) $\quad \lim_n A_n/(\rho/\rho_1^2)^n = Wv(\mathrm{Var}^{\cdot} Z_1 a')'/(\rho-\rho_1^2) = W\sigma^2 \text{ if } \rho_1^2 < \rho$.

Inserting in (1.9) and noting that $Z_n u' \cong W\rho^n$, it is seen that (1.9) is equivalent to (1.6) if $\rho_1^2 = \rho$ and to (1.7) if $0 < \rho_1^2 < \rho$. Now the increments of our martingale are

(3.6) $\quad W_n^* - W_{n-1}^* = \rho_1^{-n} \sum_{k\in I_{n-1}} (U_k - EU_k)a'$.

Obviously the condition corresponding to (3.2) does not hold and also the method of truncating (3.6) directly does not seem very tractable. Instead we define an auxiliary martingale X_n, whose increments are the single $\rho_1^{-n}(U_k-EU_k)a'$ rather than those given by (3.6). The intuitive content of this device is, of course, that we let the individuals of the n^{th} generation reproduce one by one rather than all at the same time. One might note here that this new structure of the reproduction mechanism is automatic in continuous time and thus the method might be well suited to treat this case directly (see also in this connection the proof of Theorem 5 of Asmussen [1]). But even in continuous time, some extra work is still required and new problems arise, and returning to discrete time, we proceed as follows. A simple way to formalize the intuitive description of X_n given above is to assume a specific representation of the set I_n of individuals of the n^{th} generation as the set of all $k \in N$ with $\tau_{n-1} < k \leq \tau_n$. We then define $G_0 = F_0$, $X_0 = W_0^* = Z_0 a'$ and inductively

$$G_k = \sigma(G_{k-1}, U_k),\ Y_k = \rho_1^{-n-1}(U_k - EU_k)a' \text{ when } \tau_{n-1} < k \leq \tau_n.$$

Recalling that $|Z_n|$ is the total number of individuals in the n^{th} generation, we have

$$\tau_0 = |Z_0|,\ \tau_1 = \tau_0 + |Z_1|, \ldots, \tau_n = \tau_{n-1} + |Z_n|$$

and the martingale property of X_n, G_n is readily checked as well as

(3.7) $$W_{n+1}^* = X_{\tau_n},\ A_{n+1} = s_{\tau_n}^2,\ F_{n+1} = G_{\tau_n}$$

which permits us to relate a number of properties of X_n, G_n to those of Z_n, F_n. Thus we have immediately

(3.8) $$\overline{\lim}_n W_n^*/\varphi(A_n) \leq \overline{\lim}_n X_n/\varphi(s_n^2).$$

It is also frequently useful to keep in mind the following elementary observation.

LEMMA 3. <u>For any k, the random variable n defined by</u> $\tau_{n-1} < k \leq \tau_n$ (<u>i</u>.e., the generation of k) <u>is</u> G_{k-1}<u>-measurable.</u>

The plan is first to show (3.1) for the X_n-martingale and thus to obtain $\overline{\lim}_n \leq 1$ in (1.6), (1.7). To this end we introduce the truncation scheme

$$Y_k' = \rho_1^{-n-1}(U_k I(|U_k| \leq c_n) - EU_k)a' \quad \text{when} \quad \tau_{n-1} < k \leq \tau_n$$

$$\tilde{Y}_k = Y_k' - EY_k', \quad \tilde{X}_k = X_0 + \sum_{k=1}^{n} \tilde{Y}_k, \quad \tilde{s}_n^2 = \sum_{k=1}^{n} \mathrm{Var}(\tilde{Y}_k | G_{k-1}) .$$

By a suitable choice of c_n we ensure that

(3.9) $$\overline{\lim}_n \tilde{X}_n/\varphi(\tilde{s}_n^2) = \overline{\lim}_n X_n/\varphi(s_n^2),$$

that the condition corresponding to (3.2) holds for the $\tilde{X}_n$-martingale and that thus by Lemma 2 and (3.9)

(3.10) $$\overline{\lim}_n \tilde{X}_n/\varphi(\tilde{s}_n^2) = 1, \quad \overline{\lim}_n X_n/\varphi(s_n^2) = 1 .$$

This step uses techniques similar to those of Stout [20], Heyde [13], Chow and Teicher [8], Tomkins [21] and we defer the somewhat technical details to §4.

So far the proofs run parallel for the cases $\rho_1^2 = \rho$, $0 < \rho_1^2 < \rho$, though the details are somewhat more intricate in the latter case. On the contrary, the proofs of $\overline{\lim} \geq 1$ are not the same for (1.6) as for (1.7). In case $\rho_1^2 = \rho$, we show directly that the inequality in (3.8) can be reversed, while when $\rho_1^2 < \rho$, the method of §2 for proceeding from $\overline{\lim} \leq 1$ to $\overline{\lim} \geq 1$ works. The details follow in the next section.

4. Details of proofs - $0 < \rho_1^2 \leq \rho$

LEMMA 4. *Let* $\alpha_n = n$ *if* $\rho_1^2 = \rho$, $\alpha_n = (\rho/\rho_1^2)^n$ *if* $0 < \rho_1^2 < \rho$. *Then if* $k \to \infty$, $n \to \infty$ *such that* $k \in I_n$,

(4.1) $$\overline{\lim}\ s_k^2/\alpha_n < \infty$$

(4.2) $$\overline{\lim}\ \varphi(s_k^2)/\varphi(\alpha_n) < \infty .$$

PROOF. Use (3.4), (3.5) and $A_n \leq s_k^2 \leq A_{n+1}$. □

From (1.2), it is clear that there exists a distribution function F on $[0,\infty[$ with finite second moment such that

(4.3) $$E^i|Z_1 a'|I(|Z_1| > c) = O(\int_c^\infty x\ dF(x));\ i = 1,\ldots,p$$

LEMMA 5. *Let* $\{\alpha_n\}$ *be as above. Then the conditions*

(4.4) $$c_n \to \infty,\ c_n = d_n \rho_1^n (\alpha_n/\log_2 \alpha_n)^{1/2},\ d_n > 0,\ \lim_n d_n = 0$$

(4.5) $$\sum_{n=0}^{\infty} \rho_1^{-n} \rho^n / \varphi(\alpha_n) \int_{c_n}^{\infty} x\ dF(x) < \infty$$

are sufficient for (3.9), (3.10) *and thus (recalling the discussion of* §3) *for* $\overline{\lim} \leq 1$ *in* (1.6), (1.7).

PROOF. $c_n \to \infty$ is easily seen to imply

(4.6) $$\lim_n s_n^2/\tilde{s}_n^2 = 1,\quad \lim_n \varphi(s_n^2)/\varphi(\tilde{s}_n^2) = 1 .$$

Define

$$\tilde{A}_{n+1} = \tilde{s}_{\tau_n},\ \tilde{K}_k = d_n \left(\frac{\alpha_n/\log_2 \alpha_n}{\tilde{A}_n/\log_2 \tilde{A}_n}\right)^{1/2} \quad \text{when } k \in I_n .$$

Appealing to Lemma 3, $\tilde{K}_k$ is G_{k-1}-measurable and also $\lim_k \tilde{K}_k = 0$, since the square root occuring in the definition of $\tilde{K}_k$ is bounded by (4.6), (3.4), (3.5). For some C, $|Y_k'| \leq C\ \rho_1^{-n}\ c_n$ when

$k \in I_n$ and thus

$$|\tilde{Y}_k| \leq 2C\rho_1^{-n} c_n \leq 2C\tilde{K}_k(\tilde{A}_n/\log_2\tilde{A}_n)^{1/2} = \tilde{K}_k O(\tilde{s}_k/(\log_2\tilde{s}_k^2)^{1/2})$$

since $\tilde{A}_n \leq \tilde{s}_k^2$. Thus Lemma 2 applies to give the first half of (3.10). Recalling (4.6) and noting that $X_k - X_k' = \Sigma_1^k\{Y_i - Y_i' + EY_i'\}$, it suffices for (3.9) that

$$\sum_{i=1}^{k} |Y_i - Y_i'| = o(\varphi(s_k^2)), \quad \sum_{i=1}^{k} |EY_i'| = o(\varphi(s_k^2)) \quad .$$

By (4.2), we can replace s_k^2 by α_n when $\tau_{n-1} < k \leq \tau_n$ and it is then enough to take $k = \tau_n$, i.e.

$$\sum_{m=1}^{\infty} \rho_1^{-m} \sum_{k \in I_m} |U_k a' I(|U_k| > c_m)| = o(\varphi(\alpha_n)) \quad .$$

$$\sum_{m=1}^{\infty} \rho_1^{-m} \sum_{k \in I_m} |EU_k a' I(|U_k| > c_m)| = o(\varphi(\alpha_n)) \quad .$$

The sufficiency of (4.5) follows now in exactly the same manner as when deriving (2.7) by use of Kronecker's lemma.

PROOF OF (1.6). $\overline{\lim} \leq 1$ is immediate from Lemma 5 with $\alpha_n = n$, $c_n = \rho^{n/2}$. In fact, (4.4) is obvious and (4.5) also holds, since even

$$\sum_{n=0}^{\infty} \rho_1^{-n}\rho^n \int_{\rho^{n/2}}^{\infty} x\, dF(x) = \int_0^{\infty} x\, \Sigma\rho^{n/2} I(x > \rho^{n/2})\, dF(x) = \int_0^{\infty} O(x^2)\, dF(x) < \infty .$$

To prove $\overline{\lim} \geq 1$, we use the second part of (3.10). Let $\epsilon > 0$ and let $\varkappa(n)$ be a sequence of random times such that

$$(4.7) \qquad X_{\varkappa(n)} \geq (1-\epsilon)\varphi(s^2_{\varkappa(n)}) .$$

Define $\nu(n)$ by $\tau_{\nu(n)-1} < \varkappa(n) \leq \tau_{\nu(n)}$. We can assume that $\varkappa(n+1) > \tau_{\nu(n)}$. We shall show

$$(4.8) \qquad \sum_{n=0}^{\infty} P(X_{\tau_{\nu(n)}} - X_{\varkappa(n)} \geq 0 | G_{\varkappa(n)}) = \infty .$$

From the conditional Borel-Cantelli lemma (Breiman [7], pg. 96) we then have $X_{\tau_{\nu(n)}} \geq X_{\varkappa(n)}$ i.o. and thus from (4.7)

$$\overline{\lim}_n X_{\tau_n}/\varphi(A_{n+1}) \geq \overline{\lim}_n X_{\tau_{\nu(n)}}/\varphi(A_{\nu(n)+1}) \geq$$

$$\overline{\lim}_n X_{\varkappa(n)}/\varphi(s^2_{\varkappa(n)}) \cdot \varphi(s^2_{\varkappa(n)})/\varphi(A_{\nu(n)+1}) \geq 1-\epsilon$$

since $1 \geq \varphi(s^2_{\varkappa(n)})/\varphi(A_{\nu(n)+1}) \geq \varphi(A_{\nu(n)}) / \varphi(A_{\nu(n)+1})$ and the last expression tends to one by (3.4). Letting $\epsilon \to 0$ and using (3.7), $\overline{\lim} \geq 1$ in (1.6) follows.

For the proof of (4.8), we first remark that since $\tau_{\nu(n)-1}$ and $|Z_{\nu(n)}|$ both are $G_{\varkappa(n)}$-measurable, so is $N = \tau_{\nu(n)} - \varkappa(n)+1 = \tau_{\nu(n)-1} + |Z_{\nu(n)}| - \varkappa(n)+1$. Furthermore, up to the normalizing factor $\rho_1^{-\nu(n)}$ the distribution of $X_{\tau_{\nu(n)}} - X_{\varkappa(n)}$ conditioned upon $G_{\varkappa(n)}$ is that of $V_1 + \ldots + V_N$, where

$$P(V_j \leq y) = P^{i(j)}(Z_1a' - EZ_1a' \leq y)$$

for some $i(j) = 1,\dots,p$. No matter how the $i(j)$ are chosen, the limiting distribution of $V_1 + \dots + V_N$, properly normalized, is standard normal. Thus for some $\gamma > 0$, $P(V_1 + \dots + V_N \geq 0) \geq \gamma$ for all N, $i(1),\dots,i(N)$ and (4.8) is immediate. □

PROOF OF $\overline{\lim} \leq 1$ IN (1.7). We take $\alpha_n = (\rho/\rho_1^2)^n$ so that (4.4), (4.5) reduces to

(4.9) $c_n \to \infty$, $c_n = o(\rho^{n/2}/(\log n)^{1/2})$,

$$\sum_{n=0}^{\infty} \rho^{n/2}/(\log n)^{1/2} \int_{c_n}^{\infty} x \, dF(x) < \infty .$$

Assuming nothing more than the second moment of F, it is not quite easy to find $\{c_n\}$ directly satisfying (4.9). But fortunately, the techniques used in the literature in similar situations suffice for our application as well. Choose a sequence $K_n > 0$ such that

(4.10) $K_n \to 0$, $b_n = K_n(n/\log_2 n)^{1/2} \to \infty$, $\int_0^{\infty} x^2/K_{[x]} \, dF(x) < \infty$

and that $N(m) = \sup\{n: [b_n] \leq m\} = O(m^2 \log_2 m/K_m^2)$. For a proof of the existence of $\{K_n\}$, see Chow and Teicher [8], pg. 89, or Stout [20], pg. 2159. Obviously the choice $c_n = b_{[\rho^n]}$ is consistent with the two first requirements of (4.9). Defining $N^*(m) = \sup\{n: [c_n] \leq m\}$, we get by substituting $k = [\rho^n]$

$$N^*(m) \leq \sup\{\frac{\log(k+1)}{\log \rho}: k \in \{[\rho^n]\}, [b_k] \leq m\}$$

$$\leq \sup\{\frac{\log(k+1)}{\log \rho}: k \in N, [b_k] \leq m\} \leq \frac{\log(N(m)+1)}{\log \rho},$$

$$\frac{\rho^{N^*(m)}}{\log N^*(m)} = O(\frac{N(m)}{\log_2 N(m)}) = O(\frac{m^2 \log_2 m/K_m^2}{\log_2(m^2 \log_2 m)}) = O(\frac{m^2}{K_m^2})$$

and the last assertion of (4.9) follows from (4.10) since

$$\sum_{n=0}^{\infty} \rho^{n/2}/(\log n)^{1/2} I(x > c_n) = O(\rho^{n/2}/(\log n)^{1/2}|_{n=N^*([x])}) =$$

$$O(x/K_{[x]}). \quad \square$$

PROOF OF $\overline{\lim} \geq 1$ IN (1.7). We need the relation

$$(4.11) \qquad \lim_m v(\mathrm{Var}^{\cdot} W_m^*)/(\rho/\rho_1^2)^m = \sigma^2$$

see e.g. Athreya and Ney [6], expression (22), pg. 205. Using the $\underline{\lim} \geq -1$ part and (2.1) with equality for $Y = W_m^*$, we get

$$\overline{\lim}_n \frac{Z_n a'}{(2\sigma^2 Z_n u' \log n)^{1/2}} = \overline{\lim}_n \frac{Z_{n+m} a'}{(2\sigma^2 \rho^m Z_n u' \log n)^{1/2}} \geq$$

$$\overline{\lim}_n \rho_1^m \frac{\rho_1^{-m} Z_{n+m} a' - Z_n a'}{(2\sigma^2 \rho^m Z_n u' \log n)^{1/2}} + \underline{\lim}_n \rho_1^m \frac{Z_n a'}{(2\sigma^2 \rho^m Z_n u' \log n)^{1/2}} \geq$$

$$\rho_1^m \left(\frac{v(\mathrm{Var}^{\cdot} W_m^*)}{\sigma^2 \rho^m}\right)^{1/2} - \frac{\rho_1^m}{\rho^{m/2}} \to 1 - 0, \quad m \to \infty$$

and the proof is complete. $\square$

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INSTITUTE OF MATHEMATICAL STATISTICS,
UNIVERSITY OF COPENHAGEN, 5 UNIVERSITETSPARKEN, DK-2100 COPENHAGEN Ø, DENMARK