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An Investigation of the Asymptotic Properties of Geodetic Least Squares Estimates in Certain Types of Network

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AN INVESTIGATION OF THE ASYMPTOTIC PROPERTIES OF GEODETIC LEAST SQUARES ESTIMATES IN CERTAIN TYPES OF NETWORK

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Abstract. It is pointed out that the problem of asymptotic consistency of geodetic estimates changes when not only the number of observations but also the number of elements or parameters (such as coordinates etc.) tends to infinity. Some examples of asymptotic consistency and the reverse is given in this new framework.

A. Introduction

We suppose that we have a geodetic network with N points:

$$P_1, P_2, \ldots, P_N \, .$$

The coordinates of these points are, except for a common constant, completely characterized by the coordinate differences between the single points:

$$\Delta_{12}, \Delta_{23}, \ldots, \Delta_{ii+1}, \ldots, \Delta_{N-1N} \tag{1}$$

altogether N-1 differences. We simplify the notation by writing

$$\Delta_{ii+1} \equiv \Delta_i \tag{2}$$

so that our N-1 differences in (1) become

$$\Delta_1, \Delta_2, \ldots, \Delta_i, \ldots, \Delta_{N-1} \, . \tag{3}$$

In the following we shall assume that Δ_i is a scalar; however the extension to vectors is trivial, so that our results hold for both types of network although formally only one is treated.

We now suppose further that in this network our observations y_i consist of measurements of coordinate differences (that is a levelling network in the scalar case and a distance- and azimuth- network for horizontal control in the vector case). We denote the measurement between the i'th and the j'th point in the network

$$y_{ij} \qquad i \neq j \tag{4}$$

thus y_{21} is the measured coordinate difference between P_2 and P_1.

For these observations we shall make the following simplifying assumptions: The observations y_{ij} are stochastically independent and with a known variance, which we put to 1, and the mathematical expectation of y_{ij} exists and equals the sum of "elementary" coordinate differences from P_i to P_j:

$$E\{y_{ij}\} = \eta_{ij} = \sum_{k=i}^{j-1} \Delta_k \, . \tag{5}$$

When making Least Squares Adjustments of this network using the differences of (3) as elements we should get for each of these an estimate:

$$\hat{\Delta}_1, \hat{\Delta}_2, \ldots, \hat{\Delta}_i, \ldots, \hat{\Delta}_{N-1} \, . \tag{6}$$

What we want to investigate are the asymptotic properties of these estimates.

By this we mean specifically: Does an estimate $\hat{\Delta}_i$ converge in probability to the true value Δ_i as the number N of points tends to infinity, or is it possible to put down conditions under which this convergence takes place, respectively does not take place?

The problem of convergence in probability of our Least Squares Estimates is really a mathematical problem and in trying to solve this we may utilize known methods and theorems from mathematical statistics. This mathematical problem corresponds closely to the geodetic problem, which we may formulate more loosely: Given a geodetic network where the observations have a specified non-zero variance, is it then always possible to get a specified (low) variance for some coordinates or functions of coordinates or does there exist a lower limit for the variance of these quantities which no amount of measurements can make disappear? As we shall make clear in the following the answer to this question is by no means trivial.

We now return to our mathematical model. (The deductions which are made in the following are strictly speaking only valid in this model but we know of course that as long as our model gives a sensible picture of reality we are allowed to use the results in real geodetic networks. The interesting question of if and when model and reality parts company will not be treated here.)

In our subsequent investigations we shall have a closer look upon different types of network. A common feature for these investigations is of course that the number N of points tends to infinity. The difference between them therefore lies in the manner in which the number of observations tends to infinity that is the manner in which the observations are distributed across the network.

We first attack a network where the strategy for the distribution of observations is very simple indeed:

B. The Case of Observations between all Points

We now suppose that we have measured all possible coordinate differences in the network exactly once. We have in other words the following observations:

$$y_{12},\ \ldots,\ y_{1N},\ y_{23},\ \ldots,\ y_{2N},\ \ldots y_{N-1N} \tag{7}$$

altogether $\frac{N(N-1)}{2}$ observations.

We have for our sum of squares S:

$$S = \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} (y_{ij} - \eta_{ij})^2$$

and according to (5):

$$S = \sum_{i=1}^{N} \sum_{j=i+1}^{N} (y_{ij} - \sum_{k=1}^{j-1} \Delta_k)^2 \quad . \tag{8}$$

We now assume that the Least-Squares Estimates $\hat{\Delta}_i$ of Δ_i are the solutions to the following equations (consisting of N-1 equations, one for each Δ_i):

$$\frac{\partial S}{\partial \Delta_n} = \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} D_n(y_{ij} - \sum_{k=1}^{j-1} \Delta_k)^2 = 0 \qquad n=1,\ldots,N-1. \tag{9}$$

where D_n means differentiations with respect to Δ_n. (9) is N-1 equations with N-1 unknowns. Each equation contains n x(N-1) elements in the sums.

We have

$$D_n(y_{ij} - \sum_{k=i}^{j-1} \Delta_k)^2 = - (y_{ij} - \sum_{k=i}^{j-1} \Delta_k). \qquad \begin{matrix} \text{if } i \leqq n \leqq j-1 \\ 0 \text{ otherwise} \end{matrix} \tag{10}$$

Putting (10) into (9) we find

$$\sum_{i=1}^{n} \sum_{j=n+1}^{N} (y_{ij} - (\Delta_i+\Delta_{i+1}+ \ldots + \Delta_{j-1})) = 0. \tag{11}$$

That is for n=1

$$\sum_{j=2}^{N} (y_{1j}-(\Delta_1+ \ldots \Delta_{j-1})) = 0$$

or

$$\sum_{j=2}^{N} y_{1j}-\Delta_1-(\Delta_1+\Delta_2) \ldots - (\Delta_1+ \ldots + \Delta_{N-1}) = 0$$

or

$$\sum_{j=2}^{N} y_{1j}-(N-1)\Delta_1-(N-2)\Delta_2- \ldots -\Delta_{N-1} = 0. \tag{12}$$

For n = 2 we get

$$(12) - (y_{12} - \Delta_1) + \sum_{j=3}^{N} (y_{2j} - (\Delta_2 + \dots + \Delta_{j-1})) = 0$$

and since (12) = 0 we get

$$\sum_{j=3}^{N} y_{2j} - y_{12} + \Delta_1 - \Delta_2 - (\Delta_2 + \Delta_3) \dots - (\Delta_2 + \Delta_3 + \dots + \Delta_{N-1}) = 0 \quad \text{i.e.}$$

$$\sum_{j=3}^{N} y_{2j} - y_{12} + \Delta_1 - (N-2)\Delta_2 - \dots - \Delta_{N-1} = 0$$

that is

$$(N-2)\Delta_2 + (N-3)\Delta_3 + \dots + \Delta_{N-1} = \sum_{j=3}^{N} y_{2j} - y_{12} + \Delta_1 \, . \qquad (13)$$

Putting (13) into (12) we get:

$$\sum_{j=2}^{N} y_{1j} - (N-1)\Delta_1 - \sum_{j=3}^{N} y_{2j} + y_{12} - \Delta_1 = 0 \, .$$

That is our Least Squares Estimate of Δ_1, $\hat{\Delta}_1$, is given by

$$\hat{\Delta}_1 = \frac{2y_{12} + \sum_{j=3}^{N} (y_{1j} - y_{2j})}{N} \, . \qquad (14)$$

It is evident that (14) holds for all our coordinate differences Δ_i with the proper interpretation of the indices in (14).

We know from the Gauss-Markoff theorem that $E\{\hat{\Delta}_1\} = \Delta_1$; this we can also get by direct use of (14):

$$E\{\hat{\Delta}_1\} = \frac{2E\{y_{12}\} + \sum_{j=3}^{N} E\{y_{1j}-y_{2j}\}}{N} = \frac{2\Delta_1 + (N-2)\Delta_1}{N} = \Delta_1 .$$

Furthermore we find

$$\text{var}\{\hat{\Delta}_1\} = \frac{1}{N^2}(4 + 2(N-2)) = \frac{2}{N}$$

so that clearly

$$\lim_{N\to\infty} \text{var}\{\hat{\Delta}_1\} = 0. \qquad (15)$$

(15) shows: The Least Squares Estimates of the coordinate differences are, when all coordinate differences are measured once, asymptotically consistent.

We have thus established the asymptotic consistency in this case; now we remember that the number of observations is $\frac{(N-1)N}{2}$. We might therefore hope that asymptotic consistency is assured if only the number of observations grows, say, as $\sim N^2$. That this is false we perceive with the aid of the following counter-example:

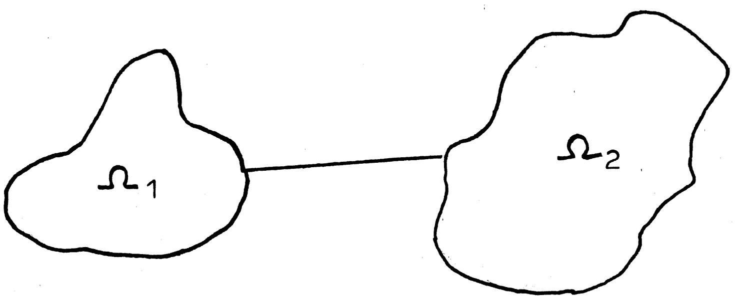

Fig. 1

Here we have a network consisting of two networks Ω_1 and Ω_2 connected with one observation. In Ω_1 and Ω_2 we have that all observations between the points have been made; therefore the number of observations will grow again as $\sim N^2$, but we see clearly that any coordinate difference involving coordinates both from Ω_1 and Ω_2 will not be asymptotically consistent.

In our next example we shall show that it is possible to get asymptotic consistency in a case where the number of observations grows only linearly with N provided that these observations are placed correctly.

C. A Square Network with Measurements only between Neighbouring Points

We take the example from [1] p. 28-29 i.e. a square network with measurements only between neighbouring points (see Fig. 2).

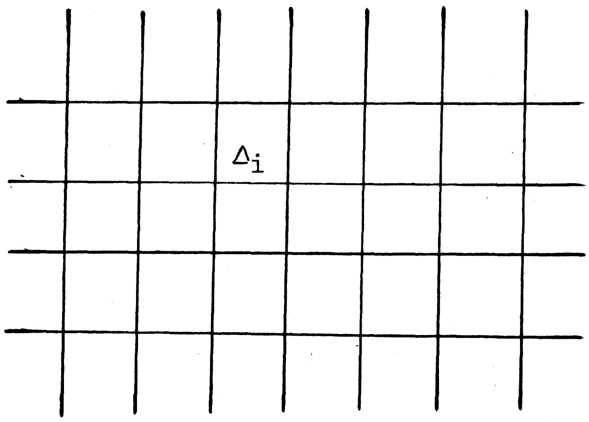

Fig. 2

The number N of points tends to infinity by covering the whole plane with a network as in Fig. 2. In [1] it is proved that for the Least Squares Estimate $\hat{\Delta}_i$ of Δ_i we get through this

$$\lim_{N \to \infty} \mathrm{var}\{\hat{\Delta}_i\} = \frac{1}{2}$$

i e. we have clearly not asymptotic consistency in such a network.

We may however get asymptotic consistency by making the number of points tend to infinity in the following way (see Fig. 3):

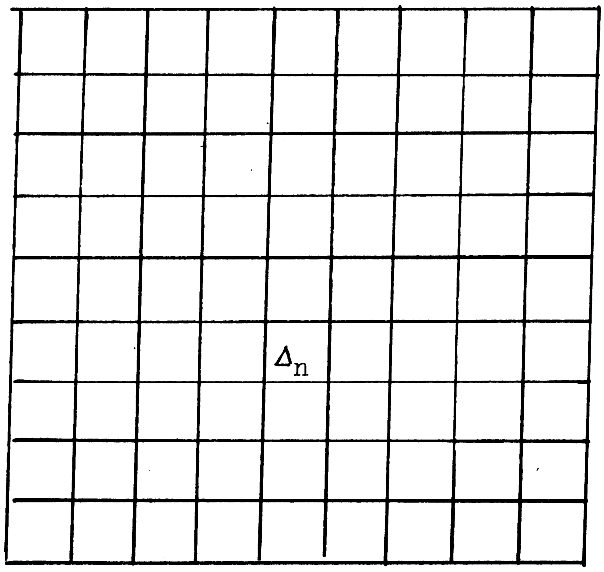

Fig. 3

Here we take only <u>one</u> of the squares from Fig. 2 and subdivides it into 4 squares then subdivides these again into 16 squares etc. During this process we still only measure between neighbouring points. From [1] p. 55 we conclude that

$$\sigma^2\{\hat{\Delta}\} \approx \frac{1}{\pi} \left(\frac{\Delta_n}{\Delta}\right)^2 \log\left(\frac{\Delta}{\Delta_n}\right)$$

where Δ_n is the coordinate difference of the n'th subdivision. Since $\Delta_n \longrightarrow 0$ while Δ remains constant we have

$$\lim_{N \longrightarrow \infty} \operatorname{var}\{\hat{\Delta}\} = 0$$

i.e. we have <u>asymptotic consistency</u> for this type of network if it grows "inward" instead of "outward".

D. Concluding Remarks

The examples show the complexity of the problem of asymptotic consistency for geodetic networks, when not only the number of observations, but also the number of geodetic parameters to be estimated, tend to infinity.

We remark that if we assume our observations to be normally distributed then we might everywhere write Maximum Likelihood Estimates instead of Least Squares Estimates.

References

1. Borre, K. and P. Meissl: Strength Analysis of Leveling-Type Networks. An Application of Random Walk Theory. - Geodætisk Institut, Meddelelse No. 50, 1974.