Heinrich Hering

Minimal Moment Conditions in the Limit Theory for General Markov Branching Processes

Preprint October 1976

16

Institute of Mathematical Statistics
University of Copenhagen
Heinrich Hering*

Minimal Moment Conditions in the Limit Theory
for General Markov Branching Processes

Preprint 1976 No. 16

INSTITUTE OF MATHEMATICAL STATISTICS
UNIVERSITY OF COPENHAGEN

October 1976

*On leave from the University of Regensburg, Germany
**Supported by the Danish Natural Science Research Council
Abstract

Assuming positive regularity in a sense suggested by branching diffusions on bounded domains, some of the basic limit theorems for Markov branching processes are formulated with a general set of types and minimal moment conditions.

AMS 1975 subject classifications:
Primary 60J80, 60F99; Secondary 46N05, 47D05, 47H05.

Key words and phrases:
Markov branching process, general set of types, non-local branching law, positive regularity, minimal moment conditions, equivalence of moment conditions, limit theorems.
Several standard limit theorems for Markov branching processes have recently been proved with a general set of types in as sharp a form as they were known with a finite set of types. However, the theory as presented in [6, 7, 10] is somewhat inhomogeneous. The degree of generality varies from paper to paper. In particular, [6, 10] and the application to branching diffusions in [7] assume a local branching law, thus excluding for example multitype branching diffusions. Besides, the moment conditions in [6] are not quite minimal. In this note we formulate a coherent theory in a completely general setting and discuss its conditions for processes constructed from a transition function on the type space, a bounded termination density, and a not necessarily local branching kernel.

1.Set-up

Let \((\mathcal{X}, \mathcal{B})\) be a measurable space, \(\mathcal{B}\) the Banach algebra of all bounded, complex-valued, \(\mathcal{B}\)-measurable functions \(\xi\) on \(\mathcal{X}\) with supremum-norm \(\|\xi\|\), \(\mathcal{B}_+\) the non-negative cone in \(\mathcal{B}\), and \(\overline{\mathcal{B}} := \{\xi \in \mathcal{B} : \|\xi\| \leq 1\}\), \(\overline{\mathcal{B}_+} := \mathcal{B}_+ \cap \overline{\mathcal{B}}\). Define

\[
\hat{\mathcal{X}} := \bigcup_{n=0}^{\infty} \mathcal{X}^{(n)}
\]

where \(\mathcal{X}^{(n)}\), \(n \geq 1\), is the symmetrization of the direct product of \(n\) disjoint copies of \(\mathcal{X}\) and \(\mathcal{X}^{(0)} = \{\emptyset\}\) with some extra point \(\emptyset\). Let \(\hat{\mathcal{B}}\) be the \(\sigma\)-algebra on \(\hat{\mathcal{X}}\) induced by \(\mathcal{B}\).

By definition a transition function \(P_t(\hat{\mathcal{X}}, \hat{\mathcal{B}})\) on \((\hat{\mathcal{X}}, \hat{\mathcal{B}})\) with parameter set \(T = \mathbb{Z}_+\), or \(T = \mathbb{R}_+\), is a branching transition function if its generating functional,
\[ F_t(\xi, \theta) := P_t(\hat{X}, \{\theta\}) + \sum_{n=1}^{\infty} \int_{X^n} P_t(\hat{X}, d\langle x_1, \ldots, x_n \rangle) \prod_{v=1}^{n} \xi(x_v); \quad \xi \in \mathbb{S}, \]

satisfies

\[ F_t(\theta, \xi) = 1, \]

(1.1)

\[ F_t(\langle x_1, \ldots, x_n \rangle, \xi) = \prod_{v=1}^{n} F_t(\langle x_v \rangle, \xi) \]

for all \( t \in T, \xi \in \mathbb{S} \), and \( \langle x_1, \ldots, x_n \rangle \in X^n, n > 0 \). Correspondingly, a Markov process \( (\hat{X}_t, P_t) \) on \( (\hat{X}, \mathbb{S}) \) is a Markov branching process if it has a branching transition function.

In particular, we shall refer to the following more explicit setting:

Suppose \( T = \mathbb{R}_+ \), let \( X \) be a locally compact Hausdorff space with countable open base, and let \( \mathcal{U} \) be the topological Borel algebra on \( X \). If \( X \) is non-compact, let \( X \cup \{\partial\} \) be the one-point compactification of \( X \). Define \( C_0 \) as the subalgebra of all continuous \( \xi \in \mathbb{S} \) such that \( \lim_{X \to \partial} \xi(x) = 0 \) if \( X \) is non-compact. Suppose to be given

(A.1) a transition semigroup \( \{T_t\}_{t \geq 0} \) on \( \mathcal{B} \), which is strongly continuous on \( C_0 \) with \( T_t C_0 \subseteq C_0 \) for \( t \geq 0 \),

(A.2) a termination density \( k \in \mathcal{B}_+ \) and a branching kernel \( \pi \) on \( X \otimes \hat{\mathcal{U}} \).

As is wellknown, these data uniquely determine a right-continuous strong Markov branching process on \( (\hat{X}, \mathbb{S}) \), cf.\[3,4\]. If \( \{T_t\} \) is the transition semigroup of a diffusion, this process is called a branching diffusion.

For every \( \mathcal{U} \)-measurable function \( \xi \) on \( X \) define

\[ \mathcal{X}[\xi] := 0; \quad x = \theta, \]

\[ := \sum_{v=1}^{n} \xi(x_v); \quad x = \langle x_1, \ldots, x_n \rangle \in X, n > 0. \]
Let $O(x) := 0$ and $1(x) := 1 \ \forall x \in \mathbb{X}$. If $P_t$ is a branching transition function on $(\mathbb{X}, \mathcal{A})$ such that

$$M^t[\xi](\cdot) := \int_{\mathbb{X}} \mathbb{X}[\xi] P_t(\langle \cdot \rangle, d\mathbb{X}) \in \mathbb{F} ; \quad t \geq 0$$

for $\xi = 1$ and thus all $\xi \in \mathbb{F}$, then $(M^t)_{t \geq 0}$ is a semigroup of linear-bounded operators on $\mathbb{F}$. In the (A.1-2) framework the assumption

(A.3)

$$\int_{\mathbb{X}} \mathbb{X}[\cdot] \pi(\cdot, d\mathbb{X}) \in \mathbb{F}$$

assures that

$$m[\cdot](y) := \int_{\mathbb{X}} \mathbb{X}[\cdot] \pi(y, d\mathbb{X}) ; \quad y \in \mathbb{X},$$

defines a linear-bounded operator on $\mathbb{F}$, which in conjunction with $k \in \mathbb{F}_+$ implies $M^t : \mathbb{F} \to \mathbb{F}$ for all $t \geq 0$, cf.[3].

We assume throughout that the following condition is satisfied:

(M) The moment semigroup $(M^t)_{t \geq 0}$ can be represented in the form

$$M^t = \varphi^t \varphi^* + Q_t ; \quad t > 0,$$

where $\varphi \in ]0, \infty[ \quad \varphi \in \mathbb{F}_+ \quad \varphi^*$ is a non-negative, linear-bounded functional on $\mathbb{F}$, and $Q_t : \mathbb{F} \to \mathbb{F}$ such that

$$\varphi^*[\varphi] = 1 \quad \varphi^*[Q_t[\cdot]] = 0 \quad Q_t[\varphi] = 0 ; \quad t > 0,$$

$$|Q_t[\xi]| \leq \alpha_t \varphi^*[\xi] \varphi ; \quad \xi \in \mathbb{F}_+, \quad t > 0,$$

with some $\alpha_t : T \to [0, \infty[ \quad$ satisfying

$$\varphi^{-t} \alpha_t \to 0 ; \quad t \to \infty.$$

We propose to call a Markov branching process positively regular if it satisfies (M). For finite $\mathbb{X}$ this definition is equivalent to the historic one. Verifications of (M) for
large classes of branching diffusions and related processes are to be found in [6,7] and particularly in [9]. While admitting \( \inf_{x \in X} \varphi(x) = 0 \), we can assume w.l.o.g. that \( \varphi(x) > 0 \ \forall \ x \in X \). In case of a restricted branching diffusion this merely means that any totally absorbing barrier is by definition not included in \( X \). Note also that \( \varphi[\mathbf{1}_A] \) is automatically \( \sigma \)-additive in \( A \in \mathcal{B} \). Here \( \mathbf{1}_A \) is the indicator function of \( A \).

2. Limit theorems

Let us first recall two results on supercritical processes \( (\rho > 1) \).

**THEOREM 1 ([7]).** If \( \{\hat{X}_t, P^\hat{X}\} \) is a Markov branching process satisfying \( (M) \) with \( \rho > 1 \), then there exists a random variable \( W \) such that

\[
\lim_{n \to \infty} \rho^{-n} \hat{X}_n[\xi] = \varphi^*[\xi] W \quad \text{a.s. } [P^\hat{X}]
\]

for every \( \xi \) absolutely integrable with respect to \( \varphi^*[\mathbf{1}_s] \). If

\[
(2.1) \quad \varphi^*[E^{\langle \cdot \rangle} \hat{X}_t[\varphi] \log \hat{X}_t[\varphi]] < \infty
\]

for some \( t \in T \setminus \{0\} \), then this inequality holds for all \( t \in T \), and \( E^{\langle x \rangle} W = \varphi(x) \ \forall \ x \in X \). Otherwise \( W = 0 \) a.s. \( [P^\hat{X}] \).

In order to handle \( t \to \infty \), \( t \in \mathbb{R}_+ \), some additional structure is needed:

(C.1) There exists a set of non-negative random variables \( \{\hat{\Gamma}_t, t > 0\} \) such that \( \hat{\Gamma}_s[\mathbf{1}] \leq \hat{\Gamma}_t \ \forall \ s \in [0,t] \) and \( \|E^{\langle \cdot \rangle} \hat{\Gamma}_t\|_1 \) as \( t \downarrow 0 \).

If \( \{\hat{X}_t, P^\hat{X}\} \) can be constructed from a system \( [T_t, k, \pi] \).
satisfying (A.1-3), define \( y_t := \hat{x}_t^{[1]} + n_t \) with \( n_t := \{ \tau : \hat{x}_{\tau-0}^{[1]} > \hat{x}_t^{[1]} ; 0 < \tau \leq t \} \). Then \( \hat{x}_s^{[1]} \leq y_t \) \( \forall s \in [0, t] \) a.s. \([P^{\hat{x}}]\) and \( 1 \leq \|E^{x_t} y_t\| \leq \exp \{ \|k\| (\|m[1] + 1\| t) \} \), so that (C.1) is satisfied, cf. [7].

**THEOREM 1** ([7, 8]). Let \( X \) be a separable metric space, \( \mathcal{B} \) the topological Borel algebra, and \( \{\hat{x}_t, P^{\hat{x}}\} \) a right-continuous Markov branching process satisfying (M) with \( \rho > 1 \) and (C.1). Then

\[
\lim_{t \to \infty} \rho^{-t} \hat{x}_t^{[\eta]} = \varphi^{*}[\eta] \quad \forall \text{a.s.} \quad [P^{\hat{x}}]
\]

for all \( \eta \in \mathcal{B} \) which are continuous a.e. \([\varphi[1,]],\)]

Given (A.1-3), condition (2.1) can be expressed in terms of \( k \) and \( \pi \). For this we need the following property:

(B.1) There exists an \( \varphi^{*} \in \mathbb{R}^{+} \) such that \( \varphi^{*}[km[\xi]] \leq \varphi^{*}[\xi] \) for all \( \xi \in \mathcal{B}^{+} \).

Clearly, (B.1) has to be discussed. For finite \( X \), or a local \( \pi \), this condition is, of course, empty. However, let \( \{T_t\} \) be, for example, the transition semigroup of the restricted Brownian motion on the bounded interval \((\alpha, \beta) \subseteq \mathbb{R}\) with total absorption at \( \alpha \) and \( \beta \). Suppose

\[
k(x)m[\xi](x) = \int_{\mathcal{C}^{\beta}} k(x, y) \xi(y) \, dy,
\]

\( k \) continuous on \([\alpha, \beta] \otimes [\alpha, \beta] \). Then (B.1) cannot be satisfied unless \( k(\cdot, \alpha) = k(\cdot, \beta) = 0 \). Concerning the involvement of \( \varphi^{*} \) see Remark 1 at the end of this section.

**PROPOSITION 1.** Given (A.1-3), suppose (M) and (B.1) are satisfied. Let \( f : \mathbb{R}^{+} \to \mathbb{R}^{+} \) be concave with \( f(0) = 0 \). Then for any \( t > 0 \)
(2.2) \[ \phi^* \left[ E^{\langle \cdot \rangle} \mathcal{X}^\mathcal{X}_t \mathbf{f}(\mathcal{X}^\mathcal{X}_t) \right] < \infty \]

if and only if

(2.3) \[ \phi^* \left[ k \mathcal{X} \mathcal{X} \left( \mathcal{X}^\mathcal{X} \mathbf{f}(\mathcal{X}) \pi (\cdot, d\mathcal{X}) \right) \right] < \infty . \]

The proof is a routine extension of the proof given in a more special setting in [7], and there is no need to repeat the details. Note that, while \( f(x) = \log x \) does not satisfy the assumptions of Proposition 1, (2.2) and (2.3) with

\[ f(x) = \mathcal{X}[0,e](x) \frac{x}{e} + \mathcal{X}[e,\infty)(x) \log x \]

are equivalent to (2.1) and

(2.4) \[ \phi^* \left[ k \mathcal{X} \mathcal{X} \left( \mathcal{X}^\mathcal{X} \mathbf{f}(\mathcal{X}) \pi (\cdot, d\mathcal{X}) \right) \right] < \infty , \]

respectively. Concerning extensions of Proposition 1 see Remark 2 below.

Turning now to \( \rho < 1 \), we introduce the mappings \( F_t[\cdot] : \mathcal{X} \to \mathcal{Y} \), \( t \in T \), defined by \( F_t[\cdot](x) := F_t(\langle x \rangle, \cdot) \); \( x \in \mathcal{X} \). If \( M^t : \mathcal{X} \to \mathcal{Y} \), there exists a mapping \( R^t(\cdot, \cdot) : \mathcal{Y} \otimes \mathcal{Y} \to \mathcal{Y} \), sequentially continuous with respect to the product topology on bounded regions, non-increasing in the first and linear-bounded in the second variable, such that

(2.5) \[ 0 = R^t(1)[\eta] \leq R^t(\zeta)[\eta] \leq M^t[\eta] ; \quad (\zeta, \eta) \in \mathcal{Y} \otimes \mathcal{Y} . \]

(2.6) \[ 1 - F_t[\xi] = M^t[1 - \xi] - R^t(\xi)[1 - \xi] ; \quad \xi \in \mathcal{Y} . \]

cf. [2,6]. We shall need the following property:

(R) For every \( t \in T \setminus \{0\} \) there exists a mapping \( g_t : \mathcal{Y} \to \mathcal{Y} \) such that

\[ R^t(\xi)[1 - \xi] = g_t[\xi]^t \phi^*[1 - \xi] \phi ; \quad \xi \in \mathcal{Y}^+ , \]

\[ \lim_{\|1 - \xi\| \to 0} \| g_t[\xi] \| = 0 . \]
If $X$ is finite, $(R)$ is automatically satisfied. To prove $(R)$ in the $[T_t, k, \pi]$ setting for general $X$, we need another consistency condition:

(B.2) There exists a $c \in \mathbb{R}_+$ such that $km[\varphi] \leq c \varphi$.

As (B.1) this condition is empty if $X$ is finite, or if $\pi$ is local. Returning to the example given in connection with (B.1), note that (B.2) cannot hold unless $\kappa(\alpha, \cdot) = \kappa(\beta, \cdot) = 0$. Concerning the role of $\varphi$ we again refer to Remark 1.

PROPOSITION 2. Given (A.1-3), suppose $(M)$ and (B.1-2) are satisfied. Then $(R)$ holds.

A proof is to be found in section 3. It extends the argument given in [10].

In accordance with the remark at the end of section 1 we tacitly assume from now on that $\varphi(x) > 0$ for all $x \in X$.

In connection with the subcritical case $(\rho < 1)$ we shall need the following continuity property:

(C.2) The space $(X, \mathcal{U})$ is a topological measurable space, and there exists a compactification $\overline{X}$ of $X$ such that $(1 - F_t[\xi])/\varphi$ has a continuous extension on $\overline{X}$ for every $t \in T \setminus \{0\}$ and $\xi \in \mathcal{S}_+^+$. A verification of (C.2) for a large class of branching diffusions has been given in [10]. The proof does not depend on whether or not $\pi$ is local.

If $P_t$ is a branching transition function satisfying $(M)$ with $\rho < 1$, then by (2.4), (2.5), $\lim_{t \to \infty} P(\hat{x}, \{0\}) = 1$ uniformly in $\hat{x} \in X^{(n)}$ for every $n > 0$. 

THEOREM 2. Let $P_t$ be a branching transition function satisfying $(M)$ with $\rho < 1$ and $(R)$. Then there exists a $\gamma \in \mathbb{R}_+$ such that

$$\lim_{t \to \infty} \rho^{-t} P_t(\hat{x}, \{\hat{y} \neq \emptyset\}) = \gamma \hat{x}[\varphi]$$

uniformly in $\hat{x} \in X^{(n)}$ for every $n > 0$. Moreover, $\gamma > 0$ if and only if

$$\varphi^*[E(\cdot) \hat{x}_t[\varphi] \log \hat{x}_t[\varphi]] < \infty$$

for some (and thus all) $t \in T \setminus \{0\}$. If $\gamma = 0$, suppose $(C.1)$ is satisfied. Then for any $A_v = \emptyset$ and $n_v \in \mathbb{Z}_+$, $v = 1, \ldots, j$, with $\bigcup_{v=1}^j A_v = X$, $j > 0$, the limit

$$p(A_1, \ldots, A_j; n_1, \ldots, n_j) := \lim_{t \to \infty} \frac{P_t(\hat{x}, \{\hat{y}[1_{A_v}] = n_v; v=1, \ldots, j\} \cap \{\hat{y} \neq \emptyset\})}{P_t(\hat{x}, \{\hat{y} \neq \emptyset\})}$$

exists uniformly in $\hat{x} \in X^{(n)}$ for every $n > 0$ and is independent of $\hat{x}$. The limits form a consistent set of probabilities, and if $X$ is a locally compact Hausdorff space and $\mathcal{V}$ the topological Borel algebra, this set determines a probability measure $P$ on $(\hat{x}, \mathcal{V})$ such that

$$P(\hat{y}[1_{A_v}] = n_v; v=1, \ldots, j) = p(A_1, \ldots, A_j; n_1, \ldots, n_j).$$

If $\gamma > 0$, then

$$\int_{\hat{x}} \hat{x}[\cdot] P(d\hat{x}) = \frac{1}{\gamma} \varphi^*[\cdot],$$

but if $\gamma = 0$, then $P$ does not have a bounded first moment functional.

For the proof we refer to section 3. Details are given only where the argument deviates from the more special proof in [10].

We now turn to the critical case ($\rho = 1$).
LEMMA 3. Let $P_t$ be a branching transition function satisfying (M) with $p = 1$. Then the value of

$$
\mu := \frac{1}{2t} \varphi^* \int_{\hat{X}} \left[ \hat{\xi}^2 - \hat{\xi}^2 \right] P_t(\langle \cdot, \theta \rangle, d\hat{\xi}),
$$

which is non-negative, possibly infinite, does not depend on $t \in T \setminus \{\theta\}$.

PROPOSITION 3. Given (A.1-3) such that (M) with $p = 1$ and (B.1) are satisfied,

$$
\mu = \frac{1}{2} \varphi^* \left[ k \int_{\hat{X}} (\hat{\xi}^2 - \hat{\xi}^2) \pi(\cdot, d\hat{\xi}) \right].
$$

Again, the proofs are deferred to section 3.

Let $P_t$ be a branching transition function satisfying (M) with $p = 1$. Clearly, $\mu = 0$ if and only if $P_t(\langle \cdot, X(1) \rangle) = 1$ a.s. $[\varphi^*] \forall t \in T$. If $\mu > 0$, then $\varphi^*[P_t(\langle \cdot, \theta \rangle)] > 0 \forall t \in T \setminus \{\theta\}$.

Assuming $\mu > 0$, define

$$
N(t) := \{ x \in X : P_t(\langle \cdot, \theta \rangle) = 0 \}; \ t \in T \setminus \{\theta\},
$$

$$
q(x) := \lim_{t \to \infty} P_t(\langle x, \theta \rangle); \ x \in X.
$$

If $\varphi^*[N(t)] = 0$ for some $t > 0$, then $q = 1$ a.s. $[\varphi^*]$ as in [11, III, no. 11, 12]. If $\varphi^*[N(t)] > 0 \forall t > 0$, fix $s > 0$ such that $\varphi_s < 1$ and define

$$
N := \bigcap_{n \in \mathbb{N}} N(ns).
$$

A routine extension of [1; II, no. 6] shows that $P_{2s}(\langle x \rangle, \{ y[1] > 1 \}) > 0 \forall x \in X$ and, if

$$
(2.10) \quad \inf_{x \in X} P_{2s}(\langle x \rangle, \{ \varphi[1] \leq d \}, y[1] > 1) > 0,
$$

that $\{ 0 < \varphi[1] \leq d \}$, $d > 0$, is a transient event of the process $\{ \tilde{\xi}_{2ns}, P^\tilde{\xi}; n \in \mathbb{Z}_+ \}$ determined by $P_{2s}$, which implies again that $q = 1$. 
If $X$ is finite, (2.10) is automatic. If more generally $(X, \mathcal{U})$ is a topological measurable space and $N$ compact, then continuity of $P_{2s}(\langle x \rangle, \{\xi[1 \leq N] \leq 1\})$ in $x \in N$ is sufficient for (2.10). Given (A.1-2), this continuity is guaranteed, if $T_t \xi(x)$ is continuous in $x \in X$ for all $t > 0$ and $\xi \in \mathcal{B}$, and that is the case for many diffusions, cf. [9].

From (1.1) and the Chapman-Kolmogorov equation

$$(2.11) \quad P_{t+s}[\xi] = P_t[P_s[\xi]]; \quad t, s \in T, \xi \in \mathcal{F}.$$ 

By use of (2.11), (2.5), (2.6), and (M) it follows from $\varphi^*[1 - q] = 0$ that $\lim_{t \to -\infty} P_t(\hat{X}, \{\theta\}) = 1$ uniformly in $\hat{X} \in X^{(n)}$ for every $n > 0$.

We shall need the following continuity property:

(C.3) If $T = \mathbb{R}_+$, then for every $x \in X$ and every decomposition $\{A_1, \ldots, A_j\}$ of $X$ with $A_v \in \mathcal{U}, v \neq 1, \ldots, j, j > 0$, the function $P_t(\langle x \rangle, \{\hat{\gamma}[1 \leq A_v] = n_v; v = 1, \ldots, j\}$ is continuous in $t \in T$.

In the (A.1-3) setting (C.3) is automatic.

THEOREM 3. Let $P_t$ be a branching transition function satisfying (M) with $\rho = 1$ and (R). If $\varphi^*[1 - q] = 0$ and $\mu < \infty$, then

$$\lim_{t \to -\infty} P_t(\hat{X}, \hat{X} \setminus \{\theta\}) = \frac{1}{\mu} \hat{X} \varphi$$

uniformly in $\hat{X} \in X^{(n)}$ for every $n > 0$. If in addition (C.3) is satisfied, then for every decomposition $\{A_1, \ldots, A_j\}$ of $X$ with $A_v \in \mathcal{U}, v = 1, \ldots, j, j > 0$, and every $\xi \in \hat{X} \setminus \{\theta\}$

$$\lim_{t \to -\infty} \frac{P_t(\hat{X}, \{\bigcup_{v=1}^j \hat{\gamma}[1 \leq A_v] = \lambda_v; v = 1, \ldots, j\} \cap \{\hat{\gamma} \neq \theta\})}{P_t(\hat{X}, \{\hat{\gamma} \neq \theta\})} = \left\{ \begin{array}{ll} 0; \quad & \min \lambda_v \leq 0, \\
1 - \exp \left\{ -\min_v \left( (\mu \varphi^*[1 \leq A_v])^{-1} \lambda_v \right) \right\}; & \min \lambda_v > 0 \end{array} \right.$$
uniformly in \((\lambda_1, \ldots, \lambda_j) \in \mathbb{R}^j\).

REMARK 1. The conditions (2.1), (2.4), (B.1-2), and \(\mu < \infty\) are less implicit than they may appear to be. There is often enough general information about \(\varphi^*\) and \(\varphi\) to allow more explicit expressions. For example, if \((\mathcal{X}_t, P^\mathcal{X})\) is a branching diffusion on a bounded domain in \(\mathbb{R}^n\) with mixed boundary conditions, then with sufficient smoothness assumptions \(\varphi\) is the restriction to \(X\) of a smooth function on the closure \(\overline{X}\) which vanishes on \(\overline{X} \setminus X\) and has a strictly negative derivative in the direction of the exterior normal there, while \(\varphi^*\) has a Lebesgue density with the same properties. As one of the simplest cases consider again a Brownian motion on a bounded interval \((\alpha, \beta)\) with total absorption at both endpoints. We may then replace \(\varphi\) by \((x - \alpha)(\beta - x)\), and \(\varphi^*[\xi]\) by \(\int_\alpha^\beta (x - \alpha)(\beta - x)x(x)dx\) and arrive at conditions which are equivalent to the original ones.

REMARK 2. Although Proposition 1 is already more general than is needed here, the full scope of the method of proof in [7] is of interest:

(a) In order to prove that

\[
(2.12) \quad \varphi^*[k \int_{\mathcal{X}} \mathcal{X}[\varphi]^n f(\mathcal{X}[\varphi])\pi(d\mathcal{X})] < \infty
\]

is sufficient for

\[
\varphi^*[E(\cdot) \mathcal{X}_t[\varphi]^n f(\mathcal{X}_t[\varphi])] < \infty
\]

with \(f\) as in the proposition and \(n = 2, 3, 4, \ldots\), the corresponding higher order analogue of (A.3),

\[
\int_X \mathcal{X}[1]^n \pi(\cdot, d\mathcal{X}) = \theta
\]

is needed. For finite \(X\) this is, of course, already contained
in (2.12), but in general it is not. The necessity part of the proof goes through as before.

(b) When replacing $\varphi^*$, or $\varphi$, the sensitive details of the proof are the following. The sufficiency part relies on (B.2) and (3.11), the necessity part on (3.12) and the submartingale property of $\{\hat{X}_t[\varphi]/\rho^t\}$. In fact, (3.11), (3.12), and the submartingale property are needed only with some positive continuous function in place of $\rho^t$.

3. Proofs

PROOF OF PROPOSITION 2. Given (A.1-2), let \( \{x_t, P^x\} \) be the Markov process determined by \( \{T_t\} \) and \( E^x \) the expectation with respect to \( P^x \). Define

\[
T_0^x\xi(x) := E^x \left( \xi(x_t) \exp \left\{ -\int_0^t k(x_s) ds \right\} \right); \quad \xi \in \mathcal{F}, x \in X,
\]

and let \( f[\cdot](x) \) be the generating functional of \( \pi(x, \cdot) \), \( x \in X \). Then for every \( \xi \in \mathcal{F} \) the function \( F_t[\xi](x); t \geq 0, x \in X \), is the unique solution of

\[
u_t(x) = T_0^x\xi(x) + H_t(x) + \int_0^t T_0^s(kf[u_{t-s}](x))ds,
\]

\[
H_t(x) := 1 - \tau_0^t(x) - \int_0^t T_0^s(k(x))ds.
\]

If we also assume (A.3), then for every \( \xi \in \mathcal{F} \) the function \( M_t[\xi](x); t \geq 0, x \in X \), is the unique solution of

\[
v_t(x) = T_0^x\xi(x) + \int_0^t T_0^s(km[v_{t-s}](x))ds,
\]

cf. [3], [4]. It follows by use of (2.6) and the corresponding expansion for \( f \),

\[
1 - f[\xi] = m[1 - \xi] - r(\xi)[1 - \xi]; \quad \xi \in \mathcal{F},
\]

that for every \( \varepsilon > 0 \) and \( \xi \in \mathcal{F} \) the function \( R^t(\xi)[1 - \xi](x) \);
\[ t \geq \varepsilon, \ x \in X, \text{ solves} \]

\[ w_t(x) = A_t(x) + B^\xi_t(x) + \int_0^t T^O_s[km[w_{t-s}]](x)dx, \]

\[ A_t(x) := \int_0^t T^O_s[kr(F_{t-s}[^\xi])[1 - F_{t-s}[^\xi]]](x)ds, \]

\[ B^\xi_t(x) := \int_0^t T^O_s[km[R^g([^\xi)][1 - ^\xi]]](x)ds. \]

In fact, \( R^t([^\xi][1 - ^\xi])(x) \) is the only bounded solution in \([\varepsilon, \varepsilon + \tau]\) for any \( \tau > 0 \), and thus equals the limit of the (non-decreasing) iteration sequence \( \{ w^{(v)}_t(x) \}, v = Z^*_+, w_0^0 = 0 \). We estimate this sequence, modifying the argument given in [10].

Suppose \( 0 < \delta < \varepsilon/2 \) and \( ^\xi \in \overline{\varepsilon}^S_+ \). By (2.5) and (2.6) there exists a \( c_1 > 0 \) such that \( F_{t-s}[^\xi] \geq 1 - c_1 \| 1 - ^\xi \| 1 \) for \( \delta \leq s \leq t - \delta, t \leq \varepsilon + \tau \). Equation (3.1) implies \( T^O_t \leq M^t \) on \( \varepsilon^S_+ \). Finally, we have \( 0 = r(1[^\xi]) \leq r([^\xi]) \leq m(^\xi) \forall (^\xi, ^\xi) \in \overline{\varepsilon}^S_+ \otimes \varepsilon^S_+ \). Hence, making use of (M) and (B.1-2), for \( t \geq \varepsilon \)

\[ A_t(x) = \int_0^\delta \int_t^t M^S[km[M^{t-s}[1 - ^\xi]]](x)ds 
\]

\[ + \int_\delta^t M^S[kr(1 - c_1 \| 1 - ^\xi \| 1)[M^{t-s}[1 - ^\xi]]](x)ds \]

\[ \leq \delta (c + c^*) (1 + \rho^{-\varepsilon/2\alpha_{\varepsilon}/2}) \rho^t \varphi^* \varphi(x) \]

\[ + t (1 + \rho^{-\delta_{\varepsilon}/2} \| k \varphi \| \]

\[ \times \varphi^*[r(1 - c_1 \| 1 - ^\xi \| 1][\varphi]) \rho^t \varphi^*[1 - ^\xi] \varphi(x). \]

Since \( \varphi^*[r(1 - c_1 \| 1 - ^\xi \| 1][\varphi]) \to 0 \) as \( \| 1 - ^\xi \| \to 0 \), and since \( \delta \) can be chosen arbitrarily small, this shows that

\[ A_t \leq t \Delta_{\varepsilon, \tau}[^\xi] \rho^t \varphi^*[1 - ^\xi] \varphi; \quad \varepsilon \leq t \leq \varepsilon + \tau, \]

(3.3)

\[ \lim_{\| 1 - ^\xi \| \to 0} \Delta_{\varepsilon, \tau}[^\xi] = 0; \quad \varepsilon > 0, \tau > 0. \]

Using (2.5), (3.1), and the fact that \( T^O_{t-s} \leq M^{t-\varepsilon}M^S \) on \( \varepsilon^S_+ \),
Again by use of (M) and (R.1-2) it follows from (3.2-4) that

\[
\lim_{\nu \to \infty} \nu \leq \left\{ e^{ct} t \left[ \delta_{t} \right] + \varepsilon \cdot e^{t} \left( 1 + \rho^{-1} \right) \delta_{t} \right\} \rho^{t} \phi \left[ 1 - \xi \right] \phi; \quad \varepsilon < t \leq \varepsilon + \tau.
\]

Since \( \varepsilon, \tau > 0 \) were arbitrary, this implies (R). \( \square \)

The following lemma is used in the proofs of Theorems 2 and 3.

**Lemma 1.** If \( P_{t} \) is a branching transition function such that (M) and (R) are satisfied and \( \lim_{t \to \infty} P_{t} \langle \langle x \rangle, \theta \rangle \rightarrow 1 \) \( \forall x = x_{t} \), then there exists for every \( t \in T \setminus \{0\} \) a mapping\hfill 

\[ h_{t} : \overline{S} \rightarrow \mathbb{R} \]

such that

\[ 1 - F_{t} [\xi] = \left( 1 + h_{t} [\xi] \right) \phi \left[ 1 - F_{t} [\xi] \right] \phi; \quad \xi \in \overline{S}, \]

\[ \lim_{t \to \infty} \| h_{t} [\xi] \| = 0 \quad \text{uniformly on } \overline{S}, \]

where \( \phi \left[ 1 - F_{t} [\xi] \right] > 0 \forall t > 0, \xi \in \overline{S} \cap \{ \phi \left[ 1 - \xi \right] > 0 \} \).

The proof of this lemma is the same as in [10] except for the case when \( \delta > 1 \), which we verify as follows. Suppose \( \xi \in \overline{S} \cap \{ \phi \left[ 1 - \xi \right] > 0 \} \) and \( t > 0 \). If \( 0 < \delta < 1 \), then by (M) and (R)

\[ \phi \left[ 1 - F_{t} [\xi] \right] \geq \phi \left[ 1 - F_{t} [\xi - \delta (1 - \xi)] \right] \geq \rho^{t} \delta \phi \left[ 1 - \xi \right] \left( 1 - \| g_{t} [1 - \delta (1 - \xi)] \| \right) \],

and there is a \( \delta = \delta (t) \) such that \( \| g_{t} [1 - \delta (1 - \xi)] \| < 1 \).

**Proof of Theorem 2.** Given (M) with \( \rho < 1 \), there exists a \( \gamma \in \mathbb{R}^{+} \) such that

\[ \rho^{-t} \phi \left[ 1 - F_{t} [0] \right] \downarrow \gamma \quad \text{as } t \uparrow \infty. \]
Moreover, \( \gamma > 0 \) if and only if for some \( \epsilon < \| \varphi \|^{-1} \)

\[
(3.6) \quad \sum_{\nu=1}^{\infty} \varphi^* [ \mathcal{R}^t (1 - \epsilon \varphi \nu^t) \mathcal{J} ] < \infty,
\]

where \( t \in T \setminus \{ 0 \} \) is arbitrary. The proof of these two statements is the same as in \([10]\). It is a routine extension of the argument given in \([2]\).

Lemma 1 and (3.5) imply (2.7). The equivalence of (3.6) and (2.8) follows from the next lemma.

**Lemma 2.** Let \( \mathcal{P}_e^{\cdot \cdot} \) be a stochastic kernel on \( X \otimes \hat{\mathcal{U}} \) such that

\[
M[\cdot](x) := \int_X \delta[\cdot] P(x, d\bar{\gamma}); \quad x \in X,
\]

defines a bounded operator \( M \) on \( \mathfrak{B} \). Let \( F[\cdot](x) \) be the generating functional of \( \mathcal{P}(x, \cdot) \), and expand

\[
1 - F_t[\xi] = M[1 - \xi] - R(\xi)[1 - \xi]; \quad \xi \in \Re,
\]

as in (2.6). Finally, let \( \xi^* \) be a non-negative, linear-bounded functional on \( \mathfrak{B} \), sequentially continuous with respect to the product topology on bounded regions, let \( \xi \in \Re^+ \) such that \( \xi(x) > 0 \forall x \in X \), and let \( \lambda \in (0, 1) \). Then

\[
(3.7) \quad \sum_{\nu=1}^{\infty} \xi^*[ \mathcal{R} (1 - \lambda^\nu \xi)[\xi] ] < \infty
\]

if and only if

\[
(3.8) \quad \xi^*[\int_X \mathfrak{h}[\xi] \log \mathfrak{h}[\xi] P(\cdot, d\mathfrak{h})] < \infty.
\]

**Proof.** We extend the proof of \([10]: \text{Lemma } 4\). Notice the relation to the argument used in \([5]\). Clearly,

\[
\int_0^\infty \xi^*[ \mathcal{R}(1 - \lambda^\nu \xi)[\xi]] dt - \xi^*[M[\xi]] \leq
\]
\[ \sum_{n=1}^{\infty} \xi \left[ R(1 - T^n \xi) \right] \leq \int_{0}^{\infty} \xi \left[ R(1 - T^t \xi) \right] dt \cdot \]

With the substitution \( s = s(\hat{x}, t) = -\hat{x} \left[ \log (1 - T^t \xi) \right] / \hat{x}[\xi] \)

\[
\int_{0}^{\infty} \xi \left[ R(1 - T^t \xi) \right] dt = \\
= \xi \left[ \int_{0}^{\infty} (\exp(\hat{x}[\log (1 - T^t \xi)]) - 1 + T^t \hat{x}[\xi]) \lambda^{-t} dt \right] P(\cdot, d\hat{x}) \\
= \xi \left[ \right] s(\hat{x}, 0) \left\{ s^{-2} (\exp(-\hat{x}[\xi] s) - 1 - \hat{x}[\xi] s) + a(\hat{x}, s) \right\} b(\hat{x}, s) ds P(\cdot, d\hat{x}) \}

\[
a(\hat{x}, s(\hat{x}, t)) = s^{-2} (\lambda^t - s) \hat{x}[\xi] = \frac{\hat{x}[\lambda^t \xi] - \hat{x}[\log (1 - T^t \xi)]}{(\hat{x}[\log (1 - T^t \xi)] / \hat{x}[\xi])^2} ,
\]

\[
b(\hat{x}, s(\hat{x}, t)) = -\lambda^{-t} s^2 \left( \frac{\partial s}{\partial t} \right)^{-1} = \frac{1}{\lambda} \left[ \frac{\hat{x}[\log (1 - T^t \xi)]^2}{\hat{x}[\lambda^t \xi] \hat{x}[\lambda^t \xi] / (1 - T^t \xi)} \right] .
\]

Observing that \( a(\hat{x}, s(\hat{x}, t)) \) and \( b(\hat{x}, s(\hat{x}, t)) \) are bounded as functions of \( (\hat{x}, t) \) on \( \hat{x} \otimes R_+ \), even if \( \inf \xi = 0 \), and substituting \( \sigma = x[\xi] s \), we obtain the equivalence of (3.7) and

(3.9) \[ \xi \left[ \int_{0}^{\infty} \hat{x}[\xi] \int_{0}^{\infty} \hat{x}[\log (1 - \xi)] \sigma^{-2} (e^{-\sigma} - 1 + \sigma) d\sigma P(\cdot, d\hat{x}) \right] < \infty .
\]

Since there exist real constants \( C_1 \) and \( C_2 \) such that

\[ 0 < C_1 \leq [\log (1 + \omega)]^{-1} \int_{0}^{\infty} \sigma^{-2} (e^{-\sigma} - 1 + \sigma) d\sigma \leq C_2 < \infty \]

for all \( \omega > 0 \), (3.9) is equivalent to

\[ \xi \left[ \int_{0}^{\infty} \hat{x}[\xi] \log (1 + \hat{x}[\log (1 - \xi)]) P(\cdot, d\hat{x}) \right] < \infty ,
\]

which is clearly equivalent to (3.8). \( \square \)

The remaining parts of the proof of Theorem 2 are the same as in \([10]\).
PROOF OF LEMMA 3. Let $\mathcal{P}$ be the set of all non-negative, not necessarily finite-valued, $\mathcal{U}$-measurable functions on $X$. Extend $\hat{\varphi}[\cdot]^2 - \hat{\varphi}[(\cdot)^2]$ to $\mathcal{P}$ in the obvious way. Then

$$M_{(2)}^t[\xi](\cdot) := \int_X (\hat{\varphi}[\xi]^2 - \hat{\varphi}[\xi^2]) P_t(\cdot, d\hat{\varphi})$$

defines a quadratic mapping $M_{(2)}^t : \mathcal{P} \to \mathcal{P}$ for every $t \in T$. Extending also $M^t[\cdot](x)$ to $\mathcal{P}$, we deduce from (2.11) that

$$M_{(2)}^t[\xi] = M_{(2)}^t[M^S[\xi]] + M_{(2)}^t[M^S_{(2)}[\xi]] \forall s, t \in T, \xi \in \mathcal{P},$$

If we have (M) with $\rho = 1$ and extend $\varphi^*$ to $\mathcal{P}$, it follows that $\varphi^* M_{(2)}^t[\varphi]$ is non-decreasing in $t \in T$ and

$$\varphi^*[M_{(2)}^t[\varphi]] = t \varphi^*[M_{(2)}^t[\varphi]]$$

for all rational $t \in T \setminus \{0\}$. Consequently (3.10) holds for all $t \in T \setminus \{0\}$. □

PROOF OF PROPOSITION 3. Let $\xi \in \mathcal{B}_+$, define

$$m_{(2)}[\xi](x) := \int_X (\hat{\varphi}[\xi]^2 - \hat{\varphi}[\xi^2]) \pi(x, d\hat{\varphi}),$$

and extend $T_t^0[\cdot](x)$ and $m[\cdot](x)$ to $\mathcal{P}$. Then the function $M_{(2)}^t[\xi](x), t \geq 0, x \in X$, is the minimal non-negative solution of

$$z_t(x) = \int_0^t T_s^0[km[z_{t-s} + km_{(2)}[M_{(2)}^{t-s}[\xi]]]](x) ds,$$

cf.[3]. Given (M), it follows from (3.1) that

$$\varphi^*[T_t^0[\xi]] \leq \rho^t \varphi^*[\xi]; \quad t \geq 0, \xi \in \mathcal{P},$$

and, using (B.1), that

$$\varphi^*[T_t^0[\xi]] \geq (1-c^t) \rho^t \varphi^*[\xi]; \quad t \geq 0, \xi \in \mathcal{P},$$

Hence, if $\rho = 1$,

$$0 \leq \varphi^* \left[ \int_0^t T_s^0[km[M_{(2)}^{t-s}[\varphi]]] ds \right] \leq t c^t \sup_{s \in [0,t]} \varphi^*[M_{(2)}^s[\varphi]] = 2c^t t^2 \mu,$$
Letting \(0 < t \downarrow 0\), we have (2.9). □

PROOF OF THEOREM 3. Given Lemmata 1, 3, and 4, the proof is the same as in [6].

LEMMA 4. Assuming (M) with \(\rho = 1\), (R), and \(\mu < \infty\), we have

\[
\lim_{N \to \infty} \frac{1}{n^\delta} \left( \varphi^{*}[1 - F_{n^\delta}[\xi]]^{-1} - \varphi^{*}[1 - \xi]^{-1} \right) = 0
\]

uniformly on \(\mathbb{R}_+ \cap \{\varphi^{*}[1 - \xi] > 0\}\) for every \(\delta \in \mathbb{T} \setminus \{0\}\).

PROOF. Let \(\xi \in \mathbb{R}_+ \cap \{\varphi^{*}[1 - \xi] > 0\}\). Then by Lemma 1 also

\[
\varphi^{*}[1 - F_t[\xi]] > 0 \quad \forall t \in \mathbb{T}.
\]

Using (2.11),

\[
\frac{1}{n^\delta} \left( \varphi^{*}[1 - F_{n^\delta}[\xi]]^{-1} - \varphi^{*}[1 - \xi]^{-1} \right) = \frac{1}{n} \sum_{n=0}^{n-1} \frac{1}{\delta} \left( \varphi^{*}[1 - F_{\delta}[\xi]]^{-1} - \varphi^{*}[1 - F_{\xi}][\xi]^{-1} \right)
\]

\[
= \frac{1}{n} \sum_{n=0}^{n-1} \sum_{\nu=0}^{\delta} \left( 1 - \varphi^{*}[1 - F_{\nu}[\xi]] \Lambda_{\delta}[F_{\nu}[\xi]] \right)^{-1} \Lambda_{\delta}[F_{\nu}[\xi]],
\]

\[
\Lambda_{\delta}[\xi] := \varphi^{*}[1 - \xi]^{-2} \left( \varphi^{*}[1 - \xi] - \varphi^{*}[1 - F_{\delta}[\xi]] \right).
\]

Given \(\mu < \infty\), there exists for every \(t \in \mathbb{T} \setminus \{0\}\) a functional \(\varphi^{*}[R^t_{(2)}(\cdot)[\cdot]]\) on \(\mathbb{R}_+ \otimes \mathbb{B}_+\), sequentially continuous on bounded regions in \(\mathbb{R}_+ \otimes \{\xi = \eta \varphi : \eta \in \mathbb{B}_+\}\), such that

\[
0 = \varphi^{*}[R^t_{(2)}(1)[\eta \varphi]] \leq \varphi^{*}[R^t_{(2)}(\xi)[\eta \varphi]] \leq \varphi^{*}[M^t_{(2)}[\eta \varphi]] \leq 2t \mu \|
\eta \|^2
\]

for \(t \geq 0, (\xi, \eta) \in \mathbb{R}_+ \otimes \mathbb{B}_+\), and

\[
\varphi^{*}[1 - F_t[\xi]] = \varphi^{*}[M^t[1 - \xi]] - \frac{1}{2} \varphi^{*}[N^t_{(2)}[1 - \xi]] + \frac{1}{2} \varphi^{*}[R^t_{(2)}(\xi)[1 - \xi]]
\]

for \(t \geq 0, \xi = 1 - \eta \varphi \in \mathbb{R}_+, \eta \in \mathbb{B}_+\). In view of \(\rho = 1\) and Lemma 1,
\[ \Lambda_0 [F_t [\xi]] = \frac{1}{2} \varphi^* [M(2)_t [(1 + h_t [\xi]) \varphi]] - \frac{1}{2} \varphi^* [R(1-F_t [\xi])[1+h_t [\xi]) \varphi]]. \]

Since \( 1 > F_t [\xi](x) > F_t [0](x) = P_t (\langle x \rangle, \{ \emptyset \}) \), we have
\[
\lim_{t \to \infty} \Lambda_0 [F_t [\xi]] = 0 \mu
\]
uniformly in \( \xi \). This completes the proof. \( \square \)

ACKNOWLEDGEMENT. I would like to thank Søren Asmussen and his colleagues at the Institut for Matematisk Statistik for the invitation to Copenhagen, and I gratefully acknowledge the financial support by the Danish Natural Science Research Council.

REFERENCES


