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Minimal Moment Conditions in the Limit Theory for General Markov Branching Processes

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Abstract

Assuming positive regularity in a sense suggested by branching diffusions on bounded domains, some of the basic limit theorems for Markov branching processes are formulated with a general set of types and minimal moment conditions.

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Several standard limit theorems for Markov branching processes have recently been proved with a general set of types in as sharp a form as they were known with a finite set of types. However, the theory as presented in [6, 7, 10] is somewhat inhomogeneous. The degree of generality varies from paper to paper. In particular, [6, 10] and the application to branching diffusions in [7] assume a local branching law, thus excluding for example multitype branching diffusions. Besides, the moment conditions in [6] are not quite minimal. In this note we formulate a coherent theory in a completely general setting and discuss its conditions for processes constructed from a transition function on the type space, a bounded termination density, and a not necessarily local branching kernel.

1. Set-up

Let $(X,\mathfrak{A})$ be a measurable space, $\mathfrak{B}$ the Banach algebra of all bounded, complex-valued, $\mathfrak{A}$-measurable functions ξ on X with supremum-norm $\|\xi\|$, $\mathfrak{B}_+$ the non-negative cone in $\mathfrak{B}$, and $\bar{S} := \{\xi \in \mathfrak{B} : \|\xi\| \leq 1\}$, $\bar{S}_+ := \mathfrak{B}_+ \cap \bar{S}$. Define

$$\hat{X} := \bigcup_{n=0}^{\infty} X^{(n)}$$

where $X^{(n)}$, $n \geq 1$, is the symmetrization of the direct product of n disjoint copies of X and $X^{(o)} := \{\theta\}$ with some extra point θ. Let $\hat{\mathfrak{A}}$ be the σ-algebra on $\hat{X}$ induced by $\mathfrak{A}$.

By definition a transition function $P_t(\hat{x},\hat{A})$ on $(\hat{X},\hat{\mathfrak{A}})$ with parameter set $T = \mathbb{Z}_+$, or $T = \mathbb{R}_+$, is a branching transition function if its generating functional,

$$F_t(\hat{x},\xi) := P_t(\hat{x},\{\theta\}) + \sum_{n=1}^{\infty} \int_{X^{(n)}} P_t(\hat{x}, d\langle x_1 .. , x_n\rangle) \prod_{\nu=1}^{n} \xi(x_\nu); \quad \xi \in \overline{S},$$

satisfies

$$F_t(\theta,\xi) = 1,$$

(1.1)

$$F_t(\langle x_1, .. , x_n\rangle, \xi) = \prod_{\nu=1}^{n} F_t(\langle x_\nu\rangle, \xi)$$

for all $t \in T$, $\xi \in \overline{S}$, and $\langle x_1, .. x_n\rangle \in X^{(n)}$, $n > 0$. Correspondingly, a Markov process $\{\hat{x}_t, P^{\hat{x}}\}$ on $(\hat{X},\hat{\mathfrak{A}})$ is a Markov branching process if it has a branching transition function.

In particular, we shall refer to the following more explicit setting:

Suppose $T = \mathbb{R}_+$, let X be a locally compact Hausdorff space with countable open base, and let $\mathfrak{A}$ be the topological Borel algebra on X. If X is non-compact, let $X \cup \{\partial\}$ be the one-point compactification of X. Define $\mathcal{C}_o$ as the subalgebra of all continuous $\xi \in \mathcal{B}$ such that $\lim_{x\to\partial} \xi(x) = 0$ if X is non-compact. Suppose to be given

(A.1) a transition semigroup $\{T_t\}_{t\geq 0}$ on $\mathcal{B}$, which is strongly continuous on $\mathcal{C}_o$ with $T_t \mathcal{C}_o \subseteq \mathcal{C}_o$ for $t \geq 0$,

(A.2) a termination density $k \in \mathcal{B}_+$ and a branching kernel π on $X \otimes \hat{\mathfrak{A}}$.

As is wellknown, these data uniquely determine a right-continuous strong Markov branching process on $(\hat{X},\hat{\mathfrak{A}})$, cf.[3,4]. If $\{T_t\}$ is the transition semigroup of a diffusion, this process is called a branching diffusion.

For every $\mathfrak{A}$-measurable function ξ on X define

$$\hat{x}[\xi] := 0; \qquad x = \theta,$$

$$:= \sum_{\nu=1}^{n} \xi(x_\nu); \quad x = \langle x_1, .. , x_n\rangle \in X, \ n > 0.$$

Let $0(x) := 0$ and $1(x) := 1 \ \forall x \in X$. If P_t is a branching transition function on $(\hat{X}, \hat{\mathfrak{A}})$ such that

$$M^t[\xi](\cdot) := \int_{\hat{X}} \hat{x}[\xi] \, P_t(\langle\cdot\rangle, d\hat{x}) \in \mathcal{B} ; \quad t \geq 0$$

for $\xi = 1$ and thus all $\xi \in \mathcal{B}$, then $\{M^t\}_{t\geq 0}$ is a semigroup of linear-bounded operators on $\mathcal{B}$. In the (A.1-2) framework the assumption

$$\text{(A.3)} \qquad \int_{\hat{X}} \hat{x}[1] \, \pi(\cdot, d\hat{x}) \in \mathcal{B}$$

assures that

$$m[\cdot](y) := \int_{\hat{X}} \hat{x}[\cdot] \, \pi(y, d\hat{x}); \quad y \in X ,$$

defines a linear-bounded operator on $\mathcal{B}$, which in conjunction with $k \in \mathcal{B}_+$ implies $M^t : \mathcal{B} \to \mathcal{B}$ for all $t \geq 0$, cf. [3].

We assume throughout that the following condition is satisfied:

(M) The moment semigroup $\{M^t\}_{t \geq 0}$ can be represented in the form

$$M^t = \rho^t \varphi \, \varphi^* + Q_t ; \quad t > 0,$$

where $\rho \in]0, \infty[$, $\varphi \in \mathcal{B}_+$, φ^* is a non-negative, linear-bounded functional on $\mathcal{B}$, and $Q_t : \mathcal{B} \to \mathcal{B}$ such that

$$\varphi^*[\varphi] = 1 , \quad \varphi^*[Q_t[\cdot]] = 0 , \quad Q_t[\varphi] = 0 ; \quad t > 0 ,$$

$$|Q_t[\xi]| \leq \alpha_t \, \varphi^*[\xi] \, \varphi ; \quad \xi \in \mathcal{B}_+ , \ t > 0 ,$$

with some $\alpha_{\cdot} : T \to [0, \infty[$ satisfying

$$\rho^{-t} \alpha_t \to 0 ; \quad t \to \infty .$$

We propose to call a Markov branching process positively regular if it satisfies (M). For finite X this definition is equivalent to the historic one. Verifications of (M) for large classes of branching diffusions and related processes are to be found in [6,7] and particularly in [9]. While admitting $\inf_{x \in X} \varphi(x) = 0$, we can assume w.l.o.g. that $\varphi(x) > 0 \quad \forall\ x \in X$. In case of a restricted branching diffusion this merely means that any totally absorbing barrier is by definition not included in X. Note also that $\varphi^*[1_A]$ is automatically σ-additive in $A \in \mathfrak{A}$. Here 1_A is the indicator function of A.

2. Limit theorems

Let us first recall two results on supercritical processes ($\rho > 1$).

THEOREM 1 ([7]). If $\{\hat{x}_t, P^{\hat{x}}\}$ is a Markov branching process satisfying (M) with $\rho > 1$, then there exists a random variable W such that

$$\lim_{N \ni n \to \infty} \rho^{-n}\, \hat{x}_n[\xi] = \varphi^*[\xi]\, W \quad \text{a.s. } [P^{\hat{x}}]$$

for every ξ absolutely integrable with respect to $\varphi^*[1_\circ]$. If

$$(2.1) \qquad \varphi^*[E^{\langle\cdot\rangle}\hat{x}_t[\varphi]\log \hat{x}_t[\varphi]] < \infty$$

for some $t \in T \setminus \{0\}$, then this inequality holds for all $t \in T$, and $E^{\langle x \rangle} W = \varphi(x) \ \forall\ x \in X$. Otherwise $W = 0$ a.s. $[P^{\hat{x}}]$.

In order to handle $t \to \infty$, $t \in \mathbb{R}_+$, some additional structure is needed:

(C.1) There exists a set of non-negative random variables $\{\Gamma_t ; t > 0\}$ such that $\hat{x}_s[1] \leq \Gamma_t \ \forall\ s \in [0,t]$ and $\| E^{\langle\cdot\rangle}\Gamma_t \| \downarrow 1$ as $t \downarrow 0$.

If $\{\hat{x}_t, P^{\hat{x}}\}$ can be constructed from a system $[T_t, k, \pi]$

satisfying (A.1-3), define $y_t := \hat{x}_t[1] + n_t$ with $n_t := \{\tau : \hat{x}_{\tau-0}[1] > \hat{x}_\tau[1]\,;\, 0 < \tau \leq t\}$. Then $\hat{x}_s[1] \leq y_t \;\; \forall\; s \in [0,t]$ a.s. $[P^{\hat{x}}]$ and $1 \leq \| E^{\langle\cdot\rangle} y_t \| \leq \exp\{\|k\|(\|m[1]+1)t\}$, so that (C.1) is satisfied, cf.[7].

THEOREM 1* ([7,8]). let X be a separable metric space, $\mathfrak{A}$ the topological Borel algebra, and $\{\hat{x}_t, P^{\hat{x}}\}$ a right-continuous Markov branching process satisfying (M) with $\rho > 1$ and (C.1). Then

$$\lim_{t\to\infty} \rho^{-t}\,\hat{x}_t[\eta] = \varphi^*[\eta]\,W \quad \text{a.s. } [P^{\hat{x}}]$$

for all $\eta \in \mathfrak{B}$ which are continuous a.e. $[\varphi[1_\circ]]$.

Given (A.1-3), condition (2.1) can be expressed in terms of k and π. For this we need the following property:

(B.1) There exists a $c^* \in \mathbb{R}_+$ such that $\varphi^*[km[\xi]] \leq c^*\varphi^*[\xi]$ for all $\xi \in \mathfrak{B}_+$.

Clearly, (B.1) has to be discussed. For finite X, or a local π, this condition is, of course, empty. However, let $\{T_t\}$ be, for example, the transition semigroup of the restricted Brownian motion on the bounded interval $(\alpha,\beta) \subset \mathbb{R}$ with total absorption at α and β. Suppose

$$k(x)m[\xi](x) = \int_\alpha^\beta \varkappa(x,y)\,\xi(y)\,dy\,,$$

$\varkappa$ continuous on $[\alpha,\beta] \otimes [\alpha,\beta]$. Then (B.1) cannot be satisfied unless $\varkappa(\cdot,\alpha) = \varkappa(\cdot,\beta) = 0$. Concerning the involvement of φ^* see Remark 1 at the end of this section.

PROPOSITION 1. Given (A.1-3), suppose (M) and (B.1) are satisfied. Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be concave with $f(0) = 0$. Then for any $t > 0$

$$(2.2) \qquad \varphi^*[E^{\langle\cdot\rangle}\hat{x}_t[\varphi]\, f(\hat{x}_t[\varphi])] < \infty$$

if and only if

$$(2.3) \qquad \varphi^*[k\int_{\hat{X}} \hat{x}[\varphi]\, f(\hat{x}[\varphi])\, \pi(\cdot, d\hat{x})] < \infty .$$

The proof is a routine extension of the proof given in a more special setting in [7], and there is no need to repeat the details. Note that, while $f(x) = \log x$ does not satisfy the assumptions of Proposition 1, (2.2) and (2.3) with

$$f(x) = 1_{[0,e]}(x)\frac{x}{e} + 1_{[e,\infty)}(x)\log x$$

are equivalent to (2.1) and

$$(2.4) \qquad \varphi^*[\, k\int_{\hat{X}} \hat{x}[\varphi] \log \hat{x}[\varphi]\, \pi(\cdot, d\hat{x})] < \infty ,$$

respectively. Concerning extensions of Proposition 1 see Remark 2 below.

Turning now to $\rho \leq 1$, we introduce the mappings $F_t[\cdot]$: $\bar{\mathcal{S}} \to \bar{\mathcal{S}}$, $t \in T$, defined by $F_t[\cdot](x) := F_t(\langle x\rangle, \cdot)$; $x \in X$. If M^t: $\mathcal{B} \to \mathcal{B}$, there exists a mapping $R^t(\cdot)[\cdot] : \bar{\mathcal{S}} \otimes \mathcal{B} \to \mathcal{B}$, sequentially continuous with respect to the product topology on bounded regions, non-increasing in the first and linear-bounded in the second variable, such that

$$(2.5) \qquad 0 = R^t(1)[\eta] \leq R^t(\zeta)[\eta] \leq M^t[\eta] ; \qquad (\zeta, \eta) \in \bar{\mathcal{S}}_+ \otimes \mathcal{B}_+ ,$$

$$(2.6) \qquad 1 - F_t[\xi] = M^t[1-\xi] - R^t(\xi)[1-\xi] ; \qquad \xi \in \bar{\mathcal{S}} ,$$

cf. [2,6]. We shall need the following property:

(R) For every $t \in T \setminus \{0\}$ there exists a mapping $g_t : \bar{\mathcal{S}}_+ \to \mathcal{B}$ such that

$$R^t(\xi)[1-\xi] = g_t[\xi]\, \rho^t\, \varphi^*[1-\xi]\, \varphi ; \qquad \xi \in \bar{\mathcal{S}}_+ ,$$

$$\lim_{\|1-\xi\| \to 0} \|g_t[\xi]\| = 0 .$$

If X is finite, (R) is automatically satisfied. To prove (R) in the $[T_t, k, \pi]$ setting for general X, we need another consistency condition:

(B.2) There exists a $c \in \mathbb{R}_+$ such that $k\,m[\varphi] \leq c\,\varphi$.

As (B.1) this condition is empty if X is finite, or if π is local. Returning to the example given in connection with (B.1), note that (B.2) cannot hold unless $\varkappa(\alpha,\cdot) = \varkappa(\beta,\cdot) = 0$. Concerning the role of φ we again refer to Remark 1.

PROPOSITION 2. Given (A.1-3), suppose (M) and (B.1-2) are satisfied. Then (R) holds.

A proof is to be found in section 3. It extends the argument given in [10].

In accordance with the remark at the end of section 1 we tacidly assume from now on that $\varphi(x) > 0$ for all $x \in X$. In connection with the subcritical case ($\rho < 1$) we shall need the following continuity property:

(C.2) The space $(X,\mathfrak{A})$ is a topological measurable space, and there exists a compactification $\overline{X}$ of X such that $(1 - F_t[\xi])/\varphi$ has a continuous extension on $\overline{X}$ for every $t \in T\setminus\{0\}$ and $\xi \in \overline{\mathfrak{S}}_+$.

A verification of (C.2) for a large class of branching diffusions has been given in [10]. The proof does not depend on whether or not π is local.

If P_t is a branching transition function satisfying (M) with $\rho < 1$, then by (2.4), (2.5), $\lim_{t\to\infty} P(\hat{x}, \{\theta\}) = 1$ uniformly in $\hat{x} \in X^{(n)}$ for every $n > 0$.

THEOREM 2. Let P_t be a branching transition function satisfying (M) with $\rho < 1$ and (R). Then there exists a $\gamma \in \mathbb{R}_+$ such that

$$\lim_{t \to \infty} \rho^{-t} P_t(\hat{x}, \{\hat{y} \neq \theta\}) = \gamma \, \hat{x}[\varphi] \tag{2.7}$$

uniformly in $\hat{x} \in X^{(n)}$ for every $n > 0$. Moreover, $\gamma > 0$ if and only if

$$\varphi^*[E^{\langle\cdot\rangle} \hat{x}_t[\varphi] \log \hat{x}_t[\varphi]] < \infty \tag{2.8}$$

for some (and thus all) $t \in T \setminus \{0\}$. If $\gamma = 0$, suppose (C.1) is satisfied. Then for any $A_\nu \in \mathfrak{A}$ and $n_\nu \in \mathbb{Z}_+$, $\nu = 1,\ldots,j$, with $\bigcup_{\nu=1}^{j} A_\nu = X$, $j > 0$, the limit

$$p(A_1,\ldots,A_j; n_1,\ldots,n_j) := \lim_{t\to\infty} \frac{P_t(\hat{x}, \{\hat{y}[1_{A_\nu}] = n_\nu; \nu=1,\ldots,j\} \cap \{\hat{y} \neq \theta\})}{P_t(\hat{x}, \{\hat{y} \neq \theta\})}$$

exists uniformly in $\hat{x} \in X^{(n)}$ for every $n > 0$ and is independent of $\hat{x}$. The limits form a consistent set of probabilities, and if X is a locally compact Hausdorff space and $\mathfrak{A}$ the topological Borel algebra, this set determines a probability measure P on $(\hat{X}, \hat{\mathfrak{A}})$ such that

$$P(\hat{y}[1_{A_\nu}] = n_\nu;\ \nu=1,\ldots,j) = p(A_1,\ldots,A_j; n_1,\ldots,n_j).$$

If $\gamma > 0$, then

$$\int_{\hat{X}} \hat{x}[\cdot]\, P(d\hat{x}) = \frac{1}{\gamma} \varphi^*[\cdot],$$

but if $\gamma = 0$, then P does not have a bounded first moment functional.

For the proof we refer to section 3. Details are given only where the argument deviates from the more special proof in [10].

We now turn to the critical case ($\rho = 1$).

LEMMA 3. Let P_t be a branching transition function satisfying (M) with $\rho = 1$. Then the value of

$$\mu := \frac{1}{2t}\, \varphi^*\Big[\int_{\hat{X}} \{\, \hat{x}[\varphi]^2 - \hat{x}[\varphi^2]\}\, P_t(\langle\cdot\rangle, d\hat{x})\Big],$$

which is non-negative, possibly infinite, does not depend on $t \in T \setminus \{\theta\}$.

PROPOSITION 3. Given (A.1-3) such that (M) with $\rho = 1$ and (B.1) are satisfied,

$$\mu = \frac{1}{2}\, \varphi^*\Big[\, k \int_{\hat{X}} \{\hat{x}[\varphi]^2 - \hat{x}[\varphi^2]\}\, \pi(\cdot, d\hat{x})\Big].$$

Again, the proofs are deferred to section 3.

Let P_t be a branching transition function satisfying (M) with $\rho = 1$. Clearly, $\mu = 0$ if and only if $P_t(\langle\cdot\rangle, X^{(1)}) = 1$ a.s. $[\varphi^*]$ $\forall$ $t \in T$. If $\mu > 0$, then $\varphi^*[P_t(\langle\cdot\rangle, \{\theta\})] > 0$ $\forall$ $t \in T \setminus \{\theta\}$. Assuming $\mu > 0$, define

$$N(t) := \{\, x \in X : P_t(\langle\cdot\rangle, \{\theta\}) = 0 \,\}; \quad t \in T \setminus \{\theta\},$$

$$q(x) := \lim_{t \to \infty} P_t(\langle x\rangle, \{\theta\}) ; \qquad x \in X .$$

If $\varphi^*[N(t)] = 0$ for some $t > 0$, then $q = 1$ a.s. $[\varphi^*]$ as in [1; III, no. 11, 12]. If $\varphi^*[N(t)] > 0$ $\forall$ $t > 0$, fix $s > 0$ such that $\alpha_s < 1$ and define

$$N := \bigcap_{n \in \mathbb{N}} N(ns) .$$

A routine extension of [1 ; II, no. 6] shows that $P_{2s}(\langle x\rangle, \{\, y[1_N] > 1 \,\}) > 0$ $\forall$ $x \in X$ and, if

$$\inf_{x \in N} P_{2s}(\langle x\rangle, \{\, \hat{y}[1_N] > 1 \,\}) > 0, \tag{2.10}$$

that $\{\, 0 < \hat{y}[1] \leq d \,\}$, $d > 0$, is a transient event of the process $\{\, \hat{x}_{2ns}, P^{\hat{x}}; n \in \mathbb{Z}_+ \}$ determined by P_{2s}, which implies again that $q = 1$.

If X is finite, (2.10) is automatic. If more generally $(X,\mathfrak{A})$ is a topological measurable space and N compact, then continuity of $P_{2s}(\langle x\rangle,\{\hat{y}[1_N]\leq 1\})$ in $x\in N$ is sufficient for (2.10). Given (A.1-2), this continuity is guaranteed, if $T_t\xi(x)$ is continuous in $x\in X$ for all $t>0$ and $\xi\in\mathfrak{B}$, and that is the case for many diffusions, cf. [9].

From (1.1) and the Chapman-Kolmogorov equation

$$(2.11) \qquad F_{t+s}[\xi]=F_t[F_s[\xi]]; \quad t,s\in T, \quad \xi\in\overline{\mathfrak{S}}.$$

By use of (2.11), (2.5), (2.6), and (M) it follows from $\varphi^*[1-q]=0$ that $\lim_{t\to\infty}P_t(\hat{x},\{\theta\})=1$ uniformly in $\hat{x}\in X^{(n)}$ for every $n>0$.

We shall need the following continuity property:

(C.3) If $T=\mathbb{R}_+$, then for every $x\in X$ and every decomposition $\{A_1,\ldots,A_j\}$ of X with $A_\nu\in\mathfrak{A}$, $\nu=1,\ldots,j$, $j>0$, the function $P_t(\langle x\rangle,\{\hat{y}[1_{A_\nu}]=n_\nu;\ \nu=1,\ldots,j\}$ is continuous in $t\in T$.

In the (A.1-3) setting (C.3) is automatic.

THEOREM 3. Let P_t be a branching transition function satisfying (M) with $\rho=1$ and (R). If $\varphi^*[1-q]=0$ and $\mu<\infty$, then

$$\lim_{t\to\infty} t\,P_t(\hat{x},\hat{X}\setminus\{\theta\})=\frac{1}{\mu}\hat{x}[\varphi]$$

uniformly in $\hat{x}\in X^{(n)}$ for every $n>0$. If in addition (C.3) is satisfied, then for every decomposition $\{A_1,\ldots,A_j\}$ of X with $A_\nu\in\mathfrak{A}$, $\nu=1,\ldots,j$, $j>0$, and every $\hat{x}\in\hat{X}\setminus\{\theta\}$

$$\lim_{t\to\infty}\frac{P_t(\hat{x},\{\frac{1}{t}\hat{y}[1_{A_\nu}]\leq\lambda_\nu;\ \nu=1,\ldots,j\}\cap\{\hat{y}\neq\theta\})}{P_t(\hat{x},\{\hat{y}\neq\theta\})}$$

$$=\begin{cases}0; & \min\lambda_\nu\leq 0,\\ 1-\exp\{-\min_\nu[(\mu\varphi^*[1_{A_\nu}])^{-1}\lambda_\nu]\}; & \min\lambda_\nu>0\end{cases}$$

uniformly in $(\lambda_1,\ldots,\lambda_j) \in \mathbb{R}^j$.

REMARK 1. The conditions (2.1), (2.4), (B.1-2), and $\mu < \infty$ are less implicit than they may appear to be. There is often enough general information about φ^* and φ to allow more explicit expressions. For example, if $\{\hat{x}_t, P^{\hat{x}}\}$ is a branching diffusion on a bounded domain in $\mathbb{R}^n$ with mixed boundary conditions, then with sufficient smoothness assumptions φ is the restriction to X of a smooth function on the closure $\overline{X}$ which vanishes on $\overline{X}\backslash X$ and has a strictly negative derivative in the direction of the exterior normal there, while φ^* has a Lebesgue density with the same properties. As one of the simplest cases consider again a Brownian motion on a bounded interval (α,β) with total absorption at both endpoints. We may then replace φ by $(x-\alpha)(\beta-x)$ and $\varphi^*[\xi]$ by $\int_\alpha^\beta (x-\alpha)(\beta-x)\xi(x)dx$ and arrive at conditions which are equivalent to the original ones.

REMARK 2. Although Proposition 1 is already more general than is needed here, the full scope of the method of proof in [7] is of interest:

(a) In order to prove that

$$\varphi^*[k\int_{\hat{X}} \hat{x}[\varphi]^n f(\hat{x}[\varphi])\pi(\cdot.d\hat{x})] < \infty \tag{2.12}$$

is sufficient for

$$\varphi^*[E^{\langle\cdot\rangle}\hat{x}_t[\varphi]^n f(\hat{x}_t[\varphi])] < \infty$$

with f as in the proposition and $n=2,3,4,\ldots$, the corresponding higher order analogue of (A.3),

$$\int_{\hat{X}} \hat{x}[1]^n \pi(\cdot, d\hat{x}) \in \mathcal{B},$$

is needed. For finite X this is, of course, already contained in (2.12), but in general it is not. The necessity part of the proof goes through as before.

(b) When replacing φ^*, or φ, the sensitive details of the proof are the following. The sufficiency part relies on (B.2) and (3.11), the necessity part on (3.12) and the submartingale property of $\{\hat{x}_t[\varphi]/\rho^t\}$. In fact, (3.11),(3.12), and the submartingale property are needed only with some positive continuous function in place of ρ^t.

3. Proofs

PROOF OF PROPOSITION 2. Given (A.1-2), let $\{x_t, P^x\}$ be the Markov process determined by $\{T_t\}$ and E^x the expectation with respect to P^x. Define

$$T_t^o\xi(x) := E^x\Big(\xi(x_t)\exp\{-\int_0^t k(x_s)ds\}\Big); \qquad \xi \in \mathcal{B},\ x \in X,$$

and let $f[\cdot](x)$ be the generating functional of $\pi(x,\cdot)$, $x \in X$. Then for every $\xi \in \bar{\mathcal{S}}$ the function $F_t[\xi](x)$; $t \geq 0$, $x \in X$, is the unique solution of

$$u_t(x) = T_t^o\xi(x) + H_t(x) + \int_0^t T_s^o\{k\, f[u_{t-s}]\}(x)ds,$$

$$H_t(x) := 1 - T_t^o 1(x) - \int_0^t T_s^o k(x)ds.$$

If we also assume (A.3), then for every $\xi \in \mathcal{B}$ the function $M^t[\xi](x)$; $t \geq 0$, $x \in X$, is the unique solution of

$$(3.1) \qquad v_t(x) = T_t^o\xi(x) + \int_0^t T_s^o\{k\, m[v_{t-s}]\}(x)ds,$$

cf.[3],[4]. It follows by use of (2.6) and the corresponding expansion for f,

$$1 - f[\xi] = m[1-\xi] - r(\xi)[1-\xi]; \qquad \xi \in \bar{\mathcal{S}},$$

that for every $\varepsilon > 0$ and $\xi \in \bar{\mathcal{S}}_+$ the function $R^t(\xi)[1-\xi](x)$;

$t \geq \varepsilon$, $x \in X$, solves

$$w_t(x) = A_t(x) + B_t^{\varepsilon}(x) + \int_0^{t-\varepsilon} T_s^o\{k\,m[\,w_{t-s}]\}(x)\,dx, \tag{3.2}$$

$$A_t(x) := \int_0^t T_s^o\{k\,r(F_{t-s}[\xi])[1 - F_{t-s}[\xi]]\}(x)\,ds,$$

$$B_t^{\varepsilon}(x) := \int_0^{\varepsilon} T_{t-s}^o\{k\,m[\,R^s(\xi)[1-\xi]]\}(x)\,ds.$$

In fact, $R^t(\xi)[1-\xi](x)$ is the only bounded solution in $[\varepsilon\,,\,\varepsilon+\tau]$ for any $\tau > 0$, and thus equals the limit of the (non-decreasing) iteration sequence $\{w_t^{(\nu)}(x)\}_{\nu \in \mathbb{Z}_+}$, $w_t^o \equiv 0$. We estimate this sequence, modifying the argument given in [10].

Suppose $0 < \delta < \varepsilon/2$ and $\xi \in \overline{\mathcal{S}}_+$. By (2.5) and (2.6) there exists a $c_1 \geq 0$ such that $F_{t-s}[\xi] \geq 1 - c_1 \|1-\xi\|1$ for $\delta \leq s \leq t-\delta$, $t \leq \varepsilon+\tau$. Equation (3.1) implies $T_t^o \leq M^t$ on $\mathcal{B}_+$. Finally, we have $0 = r(1)[\xi] \leq r(\zeta)[\xi] \leq m[\xi]$ $\forall\, (\zeta,\xi) \in \overline{\mathcal{S}}_+ \otimes \mathcal{B}_+$. Hence, making use of (M) and (B.1-2), for $t \geq \varepsilon$

$$\begin{aligned} A_t(x) &= \int_0^{\delta} + \int_{t-\delta}^{t} M^s[k\,m[M^{t-s}[1-\xi]]](x)\,ds \\ &\quad + \int_{\delta}^{t-\delta} M^s[k\,r(1-c_1\|1-\xi\|1)[M^{t-s}[1-\xi]]](x)\,ds \\ &\leq \delta(c+c^*)(1+\rho^{-\varepsilon/2}\alpha_{\varepsilon/2})\,\rho^t\varphi^*[1-\xi]\,\varphi(x) \\ &\quad + t(1+\rho^{-\delta}\alpha_{\delta})(1+\rho^{-\varepsilon/2}\alpha_{\varepsilon/2})\,\|k\varphi\| \\ &\quad \times \varphi^*[r(1-c_1\|1-\xi\|1)[\varphi]]\,\rho^t\varphi^*[1-\xi]\,\varphi(x). \end{aligned}$$

Since $\varphi^*[r(1-c_1\|1-\xi\|1)[\varphi]] \to 0$ as $\|1-\xi\| \to 0$, and since δ can be chosen arbitrarily small, this shows that

$$\begin{aligned} &A_t \leq t\,\Delta_{\varepsilon,\tau}[\xi]\,\rho^t\varphi^*[1-\xi]\,\varphi; \qquad \varepsilon \leq t \leq \varepsilon+\tau, \\ &\lim_{\|1-\xi\| \to 0} \Delta_{\varepsilon,\tau}[\xi] = 0; \qquad \varepsilon > 0,\ \tau > 0. \end{aligned} \tag{3.3}$$

Using (2.5), (3.1), and the fact that $T_{t-s}^o \leq M^{t-\varepsilon}M^{\varepsilon-s}$ on $\mathcal{B}_+$,

$$B_t^\varepsilon(x) \leq M^{t-\varepsilon}[\int_0^\varepsilon M^{\varepsilon-s}[k\, m[M^s[1-\xi]]](x)ds =: \overline{B}_t^\varepsilon(x); \quad t \geq \varepsilon,$$

(3.4)

$$\int_0^{t-\varepsilon} T_s^o\{k\, m[\overline{B}_{t-s}^\varepsilon]\}(x)ds \leq \overline{B}_t^\varepsilon(x); \quad t \geq \varepsilon.$$

Again by use of (M) and (B.1-2) it follows from (3.2-4) that

$$\lim_{\nu\to\infty} w_t^\nu \leq \{e^{ct} t\, \Delta_{\varepsilon,\tau}[\xi] + \varepsilon c^*(1+\rho^{-(t-\varepsilon)}\alpha_{t-\varepsilon})\}\rho^t \varphi^*[1-\xi]\,\varphi;$$

$$\varepsilon < t \leq \varepsilon+\tau.$$

Since $\varepsilon, \tau > 0$ were arbitrary, this implies (R).□

The following lemma is used in the proofs of Theorems 2 and 3.

LEMMA 1. If P_t is a branching transition function such that (M) and (R) are satisfied and $\lim_{t\to\infty} P_t(\langle x\rangle, \Theta) = 1$ $\forall\, x \in X$, then there exists for every $t \in T\setminus\{0\}$ a mapping $h_t : \overline{\mathcal{S}}_+ \to \mathcal{B}$ such that

$$1 - F_t[\xi] = (1 + h_t[\xi])\,\varphi^*[1 - F_t[\xi]]\,\varphi; \qquad \xi \in \overline{\mathcal{S}}_+,$$

$$\lim_{t\to\infty} \|h_t[\xi]\| = 0 \text{ uniformly on } \overline{\mathcal{S}}_+,$$

where $\varphi^*[1 - F_t[\xi]] > 0 \ \forall\, t > 0, \ \xi \in \overline{\mathcal{S}}_+ \cap \{\varphi^*[1-\xi] > 0\}$.

The proof of this lemma is the same as in [10] except for the last statement, which we verify as follows. Suppose $\xi \in \overline{\mathcal{S}}_+ \cap \{\varphi^*[1-\xi] > 0\}$ and $t > 0$. If $0 < \delta < 1$, then by (M) and (R)

$$\varphi^*[1 - F_t[\xi]] \geq \varphi^*[1 - F_t[1-\delta(1-\xi)]]$$

$$\geq \rho^t \delta\, \varphi^*[1-\xi]\{1 - \|g_t[1-\delta(1-\xi)]\|\},$$

and there is a $\delta = \delta(t)$ such that $\|g_t[1-\delta(1-\xi)]\| < 1$.

PROOF OF THEOREM 2. Given (M) with $\rho < 1$, there exists a $\gamma \in \mathbb{R}_+$ such that

(3.5) $$\rho^{-t}\varphi^*[1 - F_t[0]] \downarrow \gamma \quad \text{as} \quad t \uparrow \infty.$$

Moreover, $\gamma > 0$ if and only if for some $\varepsilon < \|\varphi\|^{-1}$

(3.6) $$\sum_{\nu=1}^{\infty} \varphi^*[R^t(1-\varepsilon\varphi\rho^{\nu t})[\varphi]] < \infty ,$$

where $t \in T\setminus\{\theta\}$ is arbitrary. The proof of these two statements is the same as in [10]. It is a routine extension of the argument given in [2].

Lemma 1 and (3.5) imply (2.7). The equivalence of (3.6) and (2.8) follows from the next lemma.

LEMMA 2. Let $P(\cdot,\cdot)$ be a stochastic kernel on $X \otimes \hat{\mathfrak{A}}$ such that

$$M[\cdot](x) := \int_{\hat{X}} \hat{y}[\cdot]\, P(x,d\hat{y}); \qquad x \in X,$$

defines a bounded operator M on $\mathcal{B}$. Let $F[\cdot](x)$ be the generating functional of $P(x,\cdot)$, and expand

$$1 - F_t[\xi] = M[1-\xi] - R(\xi)[1-\xi] ; \qquad \xi \in \overline{\mathcal{S}},$$

as in (2.6). Finally, let ξ^* be a non-negative, linear-bounded functional on $\mathcal{B}$, sequentially continuous with respect to the product topology on bounded regions, let $\xi \in \overline{\mathcal{S}}_+$ such that $\xi(x) > 0\ \forall x \in X$, and let $\lambda \in (0,1)$. Then

(3.7) $$\sum_{\nu=1}^{\infty} \xi^*[R(1-\lambda^{\nu}\xi)[\xi]] < \infty$$

if and only if

(3.8) $$\xi^*[\int_X \hat{x}[\xi] \log \hat{x}[\xi]\, P(\cdot,d\hat{x})] < \infty .$$

PROOF. We extend the proof of [10: Lemma 4]. Notice the relation to the argument used in [5]. Clearly,

$$\int_0^{\infty} \xi^*[R(1-\lambda^t\xi)[\xi]]dt - \xi^*[M[\xi]] \leq$$

$$\leq \sum_{\nu=1}^{\infty} \xi^*[R(1-\lambda^{\nu}\xi)[\xi]] \leq \int_0^{\infty} \xi^*[R(1-\lambda^t\xi)[\xi]]dt .$$

With the substitution $s = s(\hat{x},t) := -\hat{x}[\log(1-\lambda^t\xi)]/\hat{x}[\xi]$

$$\int_0^{\infty} \xi^*[R(1-\lambda^t\xi)[\xi]dt =$$

$$= \xi^*[\int_{\hat{X}}\int_0^{\infty}\Big(\exp\{\hat{x}[\log(1-\lambda^t\xi)]\} - 1 + \lambda^t\hat{x}[\xi]\Big)\lambda^{-t}dt\, P(\cdot,d\hat{x})]$$

$$= \xi^*[\int_{\hat{X}}\int_0^{s(\hat{x},0)}\Big\{s^{-2}\Big(\exp\{-\hat{x}[\xi]s\} - 1 + \hat{x}[\xi]s\Big) + a(\hat{x},s)\Big\}\, b(\hat{x},s)\,ds\, P(\cdot,d\hat{x})],$$

$$a(\hat{x},s(\hat{x},t)) := s^{-2}(\lambda^t - s)\,\hat{x}[\xi] = \frac{\hat{x}[\lambda^t\xi] - \hat{x}[|\log(1-\lambda^t\xi)|]}{(\hat{x}[\log(1-\lambda^t\xi)]/\hat{x}[\xi])^2} ,$$

$$b(\hat{x},s(\hat{x},t)) := -\lambda^{-t}s^2\Big(\frac{\partial s}{\partial t}\Big)^{-1} = \frac{1}{|\log\lambda|}\,\frac{(\hat{x}[\log(1-\lambda^t\xi)])^2}{\hat{x}[\lambda^t\xi]\,\hat{x}[\lambda^t\xi/(1-\lambda^t\xi)]} .$$

Observing that $a(\hat{x},s(\hat{x},t))$ and $b(\hat{x},s(\hat{x},t))$ are bounded as functions of $(\hat{x},t)$ on $\hat{X}\otimes R_+$, even if $\inf\xi = 0$, and substituting $\sigma = x[\xi]s$, we obtain the equivalence of (3.7) and

(3.9) $$\xi^*[\int_{\hat{X}}\hat{x}[\xi]\int_0^{\hat{x}[|\log(1-\xi)|]}\sigma^{-2}(e^{-\sigma}-1+\sigma)\,d\sigma\, P(\cdot,d\hat{x})] < \infty .$$

Since there exist real constants C_1 and C_2 such that

$$0 < C_1 \leq [\log(1+\omega)]^{-1}\int_0^{\omega}\sigma^{-2}(e^{-\sigma}-1+\sigma)\,d\sigma \leq C_2 < \infty$$

for all $\omega > 0$, (3.9) is equivalent to

$$\xi^*[\int_{\hat{X}}\hat{x}[\xi]\log(1+\hat{x}[|\log(1-\xi)|])\,P(\cdot,d\hat{x})] < \infty ,$$

which is clearly equivalent to (3.8).□

The remaining parts of the proof of Theorem 2 are the same as in [10].

PROOF OF LEMMA 3. Let $\mathcal{P}$ be the set of all non-negative, not necessarily finite-valued, $\mathfrak{A}$-measurable functions on X. Extend $\hat{x}[\cdot]^2 - \hat{x}[(\cdot)^2]$ to $\mathcal{P}$ in the obvious way. Then

$$M_{(2)}^t[\xi](\cdot) := \int_{\hat{X}} \{\hat{y}[\xi]^2 - \hat{y}[\xi^2]\}\, P_t(\langle\cdot\rangle, d\hat{y})$$

defines a quadratic mapping $M_{(2)}^t : \mathcal{P} \to \mathcal{P}$ for very $t \in T$. Extending also $M^t[\cdot](x)$ to $\mathcal{P}$, we deduce from (2.11) that

$$M_{(2)}^{t+s}[\xi] = M_{(2)}^t[M^s[\xi]] + M^t[M_{(2)}^s[\xi]] \quad \forall\ s,t \in T,\ \xi \in \mathcal{P},$$

If we have (M) with $\rho = 1$ and extend φ^* to $\mathcal{P}$, it follows that $\varphi^*[M_{(2)}^t[\varphi]]$ is non-decreasing in $t \in T$ and

(3.10) $$\varphi^*[M_{(2)}^t[\varphi]] = t\,\varphi^*[M_{(2)}^1[\varphi]]$$

for all rational $t \in T \setminus \{0\}$. Consequently (3.10) holds for all $t \in T \setminus \{0\}$. □

PROOF OF PROPOSITION 3. Let $\xi \in \mathcal{B}_+$, define

$$m_{(2)}[\xi](x) := \int_X \{\hat{y}[\xi]^2 - \hat{y}[\xi^2]\}\, \pi(x, d\hat{y}),$$

and extend $T_t^o[\cdot](x)$ and $m[\cdot](x)$ to $\mathcal{P}$. Then the function $M_{(2)}^t[\xi](x)$, $t \geq 0$, $x \in X$, is the minimal non-negative solution of

$$z_t(x) = \int_o^t T_s^o\{k\,m[z_{t-s}] + k\,m_{(2)}[M^{t-s}[\xi]]\}(x)\,ds,$$

cf. [3]. Given (M), it follows from (3.1) that

(3.11) $$\varphi^*[T_t^o\xi] \leq \rho^t \varphi^*[\xi]; \qquad t \geq 0,\ \xi \in \mathcal{P},$$

and, using (B.1), that

(3.12) $$\varphi^*[T_t^o\xi] \geq (1 - c^*t)\rho^t \varphi^*[\xi]; \qquad t \geq 0,\ \xi \in \mathcal{P},$$

Hence, if $\rho = 1$,

$$0 \leq \varphi^*[\int_o^t T_s^o\{k\,m[M_{(2)}^{t-s}[\varphi]\}ds]$$

$$\leq t\,c^* \sup_{s \in [0,t]} \varphi^*[M_{(2)}^s[\varphi]] = 2c^*t^2\mu,$$

$$t(1-c^*t)\,\varphi^*[k\,m_{(2)}[\varphi]] \leq \varphi^*[\int_0^t T_s^o\{k\,m_{(2)}[\varphi]\}ds]$$
$$\leq t\,\varphi^*[k\,m_{(2)}[\varphi]]\,; \qquad t \in T.$$

Letting $0 < t \downarrow 0$, we have (2.9). □

PROOF OF THEOREM 3. Given Lemmata 1,3, and 4, the proof is the same as in [6].

LEMMA 4. Assuming (M) with $\rho = 1$, (R), and $\mu < \infty$, we have

$$\lim_{N \ni n \to \infty} \frac{1}{n\delta}\{\varphi^*[1-F_{n\delta}[\xi]]^{-1} - \varphi^*[1-\xi]^{-1}\} = \mu$$

uniformly on $\overline{S}_+ \cap \{\varphi^*[1-\xi] > 0\}$ for every $\delta \in T \setminus \{0\}$.

PROOF. Let $\xi \in \overline{S}_+ \cap \{\varphi^*[1-\xi] > 0\}$. Then by Lemma 1 also $\varphi^*[1-F_t[\xi]] > 0\ \forall t \in T$. Using (2.11),

$$\frac{1}{n\delta}\{\varphi^*[1-F_{n\delta}[\xi]]^{-1} - \varphi^*[1-\xi]^{-1}\}$$
$$= \frac{1}{n}\sum_{\nu=0}^{n-1}\frac{1}{\delta}\{\varphi^*[1-F_\delta[F_{\nu\delta}[\xi]]]^{-1} - \varphi^*[1-F_{\nu\delta}[\xi]]^{-1}\}$$
$$= \frac{1}{n}\sum_{\nu=0}^{n-1}\frac{1}{\delta}\Big(1-\varphi^*[1-F_{\nu\delta}[\xi]]\,\Lambda_\delta[F_{\nu\delta}[\xi]]\Big)^{-1}\Lambda_\delta[F_{\nu\delta}[\xi]],$$

$$\Lambda_\delta[\zeta] := \varphi^*[1-\zeta]^{-2}\{\varphi^*[1-\zeta] - \varphi^*[1-F_\delta[\zeta]]\}.$$

Given $\mu < \infty$, there exists for every $t \in T\setminus\{0\}$ a functional $\varphi^*[R^t_{(2)}(\cdot)[\cdot]]$ on $\overline{S}_+ \otimes B_+$, sequentially continuous on bounded regions in $\overline{S}_+ \otimes \{\xi = \eta\varphi : \eta \in B_+\}$, such that

$$0 = \varphi^*[R^t_{(2)}(1)[\eta\varphi]] \leq \varphi^*[R^t_{(2)}(\xi)[\eta\varphi]] \leq \varphi^*[M^t_{(2)}[\eta\varphi]] \leq 2t\mu\,\|\eta\|^2$$

for $t \geq 0$, $(\xi,\eta) \in \overline{S}_+ \otimes B_+$, and

$$\varphi^*[1-F_t[\zeta]] = \varphi^*[M^t[1-\zeta]] - \tfrac{1}{2}\varphi^*[M^t_{(2)}[1-\zeta]] + \tfrac{1}{2}\varphi^*[R^t_{(2)}(\zeta)[1-\zeta]]$$

for $t \geq 0$, $\zeta = 1-\eta\varphi \in \overline{S}_+$, $\eta \in B_+$. In view of $\rho = 1$ and Lemma 1,

$$\Lambda_\delta[F_t[\xi]] = \frac{1}{2}\varphi^*[M^\delta_{(2)}[(1+h_t[\xi])\varphi]] - \frac{1}{2}\varphi^*[R^\delta_{(2)}(1-F_t[\xi])[(1+h_t[\xi])\varphi]].$$

Since $1 \geq F_t[\xi](x) \geq F_t[0](x) = P_t(\langle x\rangle, \{\theta\})$, we have

$$\lim_{t\to\infty} \Lambda_\delta[F_t[\xi]] = \delta\mu$$

uniformly in ξ. This completes the proof. □

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