Søren Asmussen & Heinrich Hering

Some Modified Branching Diffusion Models

Preprint
October
1976
15

Institute of Mathematical Statistics
University of Copenhagen

Søren Asmussen and Heinrich Hering*

SOME MODIFIED BRANCHING DIFFUSION MODELS

Preprint 1976 No. 15

INSTITUTE OF MATHEMATICAL STATISTICS

UNIVERSITY OF COPENHAGEN

OCTOBER 1976

*Research supported in part by the Danish Natural Science Research Council.

ABSTRACT

The paper deals with some modifications of positively regular one-dimensional branching diffusions motivated from the applications to e.g. epidemics and neutron transport theory. It is shown that known results on positive regularity and limiting behaviour extend to processes with several types of motion, retarded branching and/or general lifelengths.

§ 1. INTRODUCTION

For stochastic branching processes with a finite set X of types satisfactory limit results have been known for some time, [1], [2]. Results of comparative strength for more general X are of more recent date, [3], [4], [5], [6]. Some of the main mathematical difficulties here concern the question of minimal conditions. From an application point of view this is of course largely irrelevant, but the admission of an infinite X is often essential. A probe into the scope and flexibility of the general theory in this respect is therefore of some interest.

A standard example with infinite X is a branching diffusion with local branching law $p = (p_n(x);\ n = 0, 2, 3, \ldots;\ x \in X)$. In fact, under quite general conditions any Markov branching process with continuous motion and local branching law is a branching diffusion, [7], [8] Ch. 12. Here, X is a domain in some Euclidian space, the motion of the particles is specified by the differential generator A of a diffusion process, and a particle at $x \in X$ at time t dies in $[t, t + dt]$ with infinitesimal probability $k(x)dt$ and is replaced (at the time and point of death) by n new particles with probability $p_n(x)$. Different particles move and reproduce independently.

Branching diffusions may serve as approximate descriptions of the growth of a bacterial colony, the spread of an epidemic or an uncontrolled neutron cascade in a reactor. However, such an approximation can reflect reality only in a rather crude and general way. Not only may the branching process model only be appropriate for moderate population sizes, but in concrete situations it is often of considerable interest to study the way in which some specific phenomena influence the behaviour of the simplest model. An illness, for example, can have several phases, such as incubation, latent and infectious periods. In a classical reactor, on the other hand, branching is in general not instantaneous but retarded. After a neutron is captured by a nucleus, an approximately exponentially distributed time elapses until fission occurs. Also the description by the termination density $k(x)$ of the events of branching may be realistic in case of a neutron cascade, but for example in epidemics it would be desirable to incorporate some sort of age-dependence.

It is the purpose of the present paper to illustrate in some simple settings how such modifications can be handled within the general framework.

§ 2. SIMPLE DIFFUSIONS

Our starting point is the branching diffusion model of § 1, and we recapitulate some of the main facts about it. For simplicity, the present paper is restricted to one-dimensional diffusions, though the basic theory has been developed on more general domains, [9], [4], [5], [6].

Suppose X is a bounded interval with endpoints u, v and let C^ν be the set of functions which are restrictions to X of ν times continuously differentiable functions on [u, v]. Unless otherwise stated, we work with

(D) A <u>is given in terms of a velocity a and a drift b</u>,

$$Af(x) = a(x)f''(x) + b(x)f'(x)$$

<u>together with separated endpoint conditions</u>

$$f \in D(A) = \{g \in C^2 : \alpha g(u+) - \alpha' g'(u+) = \beta g(v-) + \beta' g'(v-) = 0\}$$

<u>for some constants</u> $\alpha, \alpha', \beta, \beta' \geq 0$ <u>such that</u> $\alpha + \alpha' > 0$, $\beta + \beta' > 0$. <u>Furthermore</u> $a \in C^2$, $b \in C^1$, $\inf_{x \in X} a(x) > 0$.

Here $u \in X (v \in X)$ if and only if $\alpha \neq 0 (\beta \neq 0)$, the case $\alpha = 0 (\beta = 0)$ corresponds to reflection at u (v) and the case $\alpha' = 0 (\beta' = 0)$ to total absorption at u(v), cf. [10].

Let B be the set of bounded measurable functions on X and $B_+ = \{\eta \in B : \eta \geq 0\}$. For $\xi \in B$ we define $\hat{x}_t[\xi]$ as 0 when the population at time t is extinct and as $\xi(x_1) + \ldots + \xi(x_n)$ when it consists of n particles at sites $x_1, \ldots, x_n \in X$. For example when $A \subseteq X$ is measurable, $\hat{x}_t[1_A]$ is the number of particles in A at time t. Let P^x, E^x refer to the probability and expectation when the initial population consist of just one particle at $x \in X$, and define

$$M_t\xi(x) := E^x\hat{x}_t[\xi], \quad m(x) := \sum_{n=0}^{\infty} np_n(x).$$

With a, b, k, m sufficiently smooth, $\{M_t\}_{t \geq 0}$ is a semigroup with differential generator

$$Lf(x) = Af(x) + k(x)(m(x)-1)f(x), \quad f \in D(L) = D(A)$$

and the branching diffusion is <u>positively regular</u> in the sense that $\{M_t\}$ has the following property:

(M) <u>There exist a real number</u> λ, $-\infty<\lambda<\infty$, <u>a</u> $\phi \in B_+$ <u>and a bounded measure</u> $\phi^* \geq 0$ <u>such that</u> $\phi^*[\phi] = 1$,

$$M_t\phi = e^{\lambda t}\phi, \quad \phi^* M_t = e^{\lambda t}\phi^* \text{ [1)]},$$

$$M_t\eta(x) = e^{\lambda t}\phi(x)\phi^*[\eta]\{1 + \Delta_t\eta(x)\}, \quad \eta \in B_+,$$

---

1) $\phi^* M_t[\xi] := \phi^*[M_t\xi]$

$$\sup_{\eta,x} |\Delta_t \eta(x)| \to 0, t \to \infty.$$

The quantities λ, ϕ, ϕ^* have similar interpretations as what in demography is called the Malthusian parameter, the reproductive value and the stable distribution, cf. [11].

The proof of (M) in [4] (the underlying idea is the same as in [3]) assumes a, k, $m \in C^2$, $b \in C^1$. It contains some additional information, in particular

$$(\Phi) \quad \begin{aligned} &\phi \in D(A),\ \phi(x) > 0\ \forall x \in X, \\ &\phi(u+) + \phi'(u+) > 0,\ \phi(v-) - \phi'(v-) > 0 \end{aligned}$$

and the corresponding relations - referred to as (Φ^*) - for the Lebesgue density of ϕ^*. It is shown that for all $t > 0$, $\eta \in B$, $x \in X$

$$(2.1) \quad M_t \eta(x) = \sum_{n=1}^{\infty} e^{\lambda_n t} \phi_n(x) \phi_n^*[\eta],$$

$$(2.2) \quad \phi_n^*[\eta] = \int_X \phi_n^*(x) \eta(x) dx,$$

$$(2.3) \quad M_t \phi_n = e^{\lambda_n t} \phi_n,\ \phi_n^* M_t = e^{\lambda_n t} \phi_n^*,\ \phi_n^*[\phi_m] = \delta_{nm},$$

where $\{\lambda_n, \phi_n\}$ solves the eigenvalue problem $Lf = \lambda f$, $f \in D(L)$, and $\{\lambda_n, \phi_n^*\}$ the adjoint problem $L^*g = \lambda g$, $g \in D(L)$, the λ_n being real. Then asymptotic formulas for $\lambda_n, \phi_n, \phi_n^*$ are used to derive (M) with $\lambda = \lambda_1, \phi = \phi_1, \phi^* = \phi_1^*$. While in general this leads into the theory of differential equations, there are special cases where the argument reduces to elementary calculations.

Suppose, for example, that the diffusion is absorbing Brownian motion on $(0,\Pi)$ (i.e. $\alpha' = \beta' = 0$, $b(x) = 0$, $a(x) = a$) and that $k(x) = k$, $m(x) = m$. Looking for the solutions of $Lf = \lambda f$ which satisfy $f(0) = f(\Pi)$, we get simply

(2.4) $\quad \phi_n(x) = \sin nx, \; \lambda_n = k(m-1) - an^2, \; n = 1, 2, \ldots,$

and it follows from elementary Fourier series theory that for all $\eta \in D(A)$, $x \in X$,

(2.5) $\quad \eta(x) = \sum_{n=1}^{\infty} c_n[\eta] \sin nx,$

where $c_n[\eta] := \frac{2}{\Pi} \int_0^{\Pi} \eta(x) \sin nx \, dx$, $\sum_{n=1}^{\infty} |c_n[\eta]| < \infty$.

Also $L\phi_n = \lambda_n \phi_n$ implies $M_t \phi_n = e^{\lambda_n t} \phi_n$ and it is easily seen that $||M_t[\eta]|| \leq e^{k(m-1)t} ||\eta||$, $||.||$being the supremumnorm. Thus, applying M_t to (2.5), we obtain (2.1) with $\phi_n^*[\eta] = c_n[\eta]$for all $\eta \in D(L)$. Since $\lambda_n = O(-n^2)$, we have a bounded measure on each side of (2.1), which hence remains true for all $\eta \in B$. Using the inequality $|\sin nx| \leq n \sin x$, $x \in (0,\Pi)$, we obtain for all $\eta \in B_+$

$$|\phi_n^*[\eta]\phi_n(x)| \leq n^2\phi_1^*[\eta]\phi_1(x)$$

Setting $\lambda = \lambda_1$, $\phi_1 = \phi$, and taking $\phi_1^* = \frac{2}{\Pi}\phi_1$ as the Lebesque density of ϕ^*, (M) follows from

$$|\Delta_t\eta(x)| \leq \sum_{n=2}^{\infty} n^2 e^{(\lambda_n-\lambda_1)t} = \sum_{n=2}^{\infty} n^2 e^{a(1-n^2)t} \to 0,\ t \to \infty.$$

A similar direct argument goes through if we admit a constant drift $b \neq 0$, with otherwise unchanged assumptions. Then

$$\lambda_n = k(m-1) - \frac{b^2}{4a} - an^2,$$

$$\phi_n(x) = e^{-\frac{bx}{2a}} \sin nx,\ \phi_n^*(x) = \frac{2}{\Pi} e^{\frac{bx}{2a}} \sin nx,\ n = 1, 2, \ldots$$

Another simple case is that of reflecting Brownian motion on $[0,\Pi]$, with $a(x) = a$, $b(x) = 0$, $k(x) = k$, $m(x) = m$, where

$$\lambda_1 = k(m-1),\ \phi_1(x) = 1,\ \phi_1^*(x) = \frac{1}{\Pi}$$

$$\lambda_n = -a(n-1)^2 + k(m-1),\ \phi_n(x) = \cos(n-1)x,\ \phi_n^*(x) = \frac{2}{\Pi}\cos(n-1)x,$$
$n = 2, 3, \ldots$

Returning to the general model with $a \in C^2$, $b \in C^1$, $k, m \in C^0$, we quote three limit theorems from [3], [5], [6]. For more special topics, see [12], [13], [14].

(I) If $\lambda > 0$, <u>the martingale</u> $e^{-\lambda t}\hat{x}_t[\phi]$ <u>converges a.s. to a</u>

<u>random variable</u> W <u>and for any a.e. continuous</u> $\xi \in B$,

$$\lim_{t\to\infty} e^{-\lambda t}\hat{x}_t[\xi] = W\phi^*[\xi] \text{ a.s.}$$

<u>Here</u> $E^x W = \phi(x)$ $\forall x \in X$ <u>if and only if</u>

(X LOG X) $\phi^*[E^{\cdot}\hat{x}_t[\phi]\log \hat{x}_t[\phi]] < \infty$ <u>for some (all)</u> $t > 0$

<u>while</u> $P^x(W = 0) = 1$ $\forall x \in X$ <u>otherwise</u>.

Let $\hat{X}$ be the set of all finite populations $\hat{x}$ with sites in X (i.e. the state space of the process) and $\hat{A}$ the σ-algebra induced on $\hat{X}$ by the Borel σ-algebra on X.

(II) <u>If</u> $\lambda < 0$, <u>the</u> $\hat{x}_t$<u>-process becomes eventually extinct</u>,

$$\lim_{t\to\infty} P^x(\hat{x}_t[1] = 0) = 1 \text{ uniformly in } x \in X$$

<u>and there exists a constant</u> $\gamma \geq 0$ <u>such that</u>

$$\lim_{t\to\infty} e^{-\lambda t} P^x(\hat{x}_t[1] \neq 0) = \gamma\phi(x) \text{ uniformly in } x \in X.$$

<u>Here</u> $\gamma > 0$ <u>if and only if</u> (X LOG X) <u>is satisfied</u>. <u>Moreover</u>, <u>there exists a probability measure</u> P <u>on</u> $(\hat{X}, \hat{A})$ <u>such that</u>

$$\lim_{t\to\infty} P^x(\hat{x}_t[1_{A_\nu}] = n_\nu,\ \nu = 1, \ldots, j \,|\, \hat{x}_t[1] > 0) =$$

$$P(\hat{x}[1_{A_\nu}] = n_\nu,\ \nu = 1, \ldots, j)$$

for all finite measurable partitions $\{A_\nu\}$ of X. If $\gamma > 0$, then for all $\xi \in B$ $\int \hat{x}[\xi]P(d\hat{x}) = \gamma^{-1}\phi^*[\xi]$, while otherwise $\int \hat{x}[\xi]P(d\hat{x}) = \infty$ for any $\xi \in B_+$ such that $\phi^*[\xi] > 0$.

(III) If $\lambda = 0$, then either the $\hat{x}_t$-process is a motion on X, $P^x(\hat{x}_t[1] = 1) = 1 \forall x \in X \forall t \geq 0$, or it becomes eventually extinct,

$$\lim_{t\to\infty} P^x(\hat{x}_t[1] = 0) = 1 \text{ uniformly in } x \in X.$$

Furthermore

$$\mu := \frac{1}{2t}\, \phi^*[E^{\cdot}\{\hat{x}_t[\phi]^2 - \hat{x}_t[\phi^2]\}]$$

is constant as a function of $t > 0$, with $\mu > 0$ if and only if extinction occurs. If $0 < \mu < \infty$, then

$$\lim_{t\to\infty} tP^x(\hat{x}_t[1] > 0) = \frac{1}{\mu}\,\phi(x) \text{ uniformly in } x \in X,$$

and for all finite measurable partitions $\{A_\nu\}$ of X and all $x \in X$ the distribution of $t^{-1}(\hat{x}_t[1_{A_1}], \ldots, \hat{x}_t[1_{A_j}])$ with respect to $P^x(\cdot\,|\,\hat{x}_t[1] > 0)$ converges to that of a vector of the form $(\phi^*[1_{A_1}], \ldots, \phi^*[1_{A_j}])w$, where $P^x(w > y) = e^{-y/\mu}$ $\forall x \in X \forall y \geq 0$. Finally,

$$\lim_{t\to\infty} \frac{1}{t}\, E^x(\hat{x}_t[\xi]\,|\,\hat{x}_t[1] > 0) = \mu\phi^*[\xi] \quad x \in X \ \xi \in B.$$

(I), (II), (III) are in their form typical for any positively regular Markov branching process. In fact, they have been derived under more general assumptions than those of this section and one of the purposes of the present paper is to provide some further examples of these types of limiting behaviour. In concrete examples, it is frequently possible to reformulate in terms of natural model parameters. For example in the simple branching diffusions considered so far, it can be proved that, regardless the value of λ, (X LOG X) is equivalent to

$$(\text{x log x}) \quad \phi^*[k\kappa\phi] < \infty,$$

where $\kappa(x) := \Sigma p_n(x)\, n \log n$. By ($\Phi$), ($\Phi^*$) this amounts to

$$(\text{x log x})' \begin{cases} \int_u^v k(x)\kappa(x)dx < \infty & \alpha' \neq 0,\ \beta' \neq 0 \\ \int_u^v k(x)\kappa(x)(x-u)^2dx < \infty & \alpha' \neq 0,\ \beta' = 0 \\ \int_u^v k(x)\kappa(x)(v-x)^2dx < \infty & \alpha' = 0,\ \beta' \neq 0 \\ \int_u^v k(x)\kappa(x)(x-u)^2(v-x)^2dx < \infty & \alpha' = 0,\ \beta' = 0 \end{cases}$$

Similarly in (III), μ can be computed as $\mu = \frac{1}{2}\,\phi^*[k\chi\phi^2]$, where $\chi(x) := \Sigma p_n(x)n(n-1)$, and by ($\Phi$), ($\Phi^*$) the condition $\mu < \infty$ is equivalent to the one obtained from (x log x)' by replacing κ with χ and all squares with cubes.

§ 3. COUPLED DIFFUSIONS

It has been suggested to model the early stages in the spread of an epidemic by a branching process, see e.g. [15], [16], [17], [18], [19], [20], [21]. Two important factors whose influence should be accounted for are the spatial motion and the possible existence of several (sometimes overlapping) periods of the illness, for example an incubation, a latent, and an infectious phase. The spatial motion has been modelled either in discrete time, [16], or as a jump process, [15], [19], [20], [21], but a diffusion model seems at least as reasonable.

As a simple example, suppose we want to distinguish between the infectious and non-infectious phase. In addition to position we then have to specify phase. That is, if the motion is modelled as a diffusion on X_o, the set of types is $X = X_1 \cup X_2$, where X_1, X_2 are disjoint copies of X_o. A non-infectious particle moves on X_1. The moment it becomes infectious, say at x, it switches to $x \in X_2$ and continues the motion on X_2. When the motion here is stopped, say at y, the particle either disappears - through curement, isolation, or death - or it infects others, i.e. it creates new particles at $y \in X_1$ and continues itself its motion on X_2.

Precisely and slightly more generally, let X_o be a bounded real interval, let the motion on X_i be a diffusion with differential generator A_i and suppose a particle at $x \in X_i$ stops its motion with infinitesimal probability $k_i(x)dt$ and is replaced by $n = n_1 + n_2$ new particles, n_1 of them at $x \in X_1$ and n_2 at $x \in X_2$. Let $p_{n_1}^{i1}(x)$, $p_{n_2}^{i2}(x)$ be the associated marginal probabilities and $m_{ij}(x) = \Sigma n p_n^{ij}(x)$. Write functions on X as pairs (ξ,η) of functions on X_o. Then - with suitable smoothness assumptions - the mean semigroup has the differential generator

$$L(\xi,\eta) = \begin{cases} A_1\xi(x) + k_1(x)(m_{11}(x)-1)\xi(x) + k_1(x)m_{12}(x)\eta(x), \\ \hfill x \in X_1, \\ \\ A_2\eta(x) + k_2(x)m_{21}(x)\xi(x) + k_2(x)(m_{22}(x)-1)\eta(x), \\ \hfill x \in X_2, \end{cases}$$

$\xi \in D(A_1)$, $\eta \in D(A_2)$. As in § 2, let us first study an example allowing explicit calculations.

In the two-phase epidemic outlined above, $m_{11}(x) = 0$, $m_{12}(x) = 1$, $m_{22}(x) \leq 1$. Suppose in addition that both diffusions are absorbing Brownian motion on $(0,\Pi)$ with velocities $a_1 = a_2 = a > 0$ and that k_1, k_2, m_{21}, m_{22} all are constant. Then L takes the form

$$L(\xi,\eta)(x) = \begin{cases} a\xi''(x) - \alpha\xi(x) + \alpha\eta(x), & x \in X_1, \\ \\ a\eta''(x) + \gamma\xi(x) - \beta\eta(x), & x \in X_2, \end{cases}$$

$\xi,\eta \in \{g \in C^2 : g(0+) = g(\Pi-) = 0\}$

with positive α, β, γ. As suggested by § 2 we look for eigenfunctions of the form

$$(\xi(x),\eta(x)) = (\sin nx, \varepsilon \sin nx)$$

Inserting into $L(\xi,\eta)(x) = \lambda(\xi,\eta)(x)$ yields

$$(3.1) \quad -an^2 - \alpha + \alpha\varepsilon = \lambda, \; -a\varepsilon n^2 + \gamma - \beta\varepsilon = \lambda\varepsilon$$

Eliminating λ, we get $\alpha\varepsilon^2 - (\alpha - \beta)\varepsilon - \lambda = 0$, and since $\gamma > 0$, there are two different real roots, $\varepsilon_+ > \varepsilon_-$ (say). Let the corresponding eigenvalues obtained from (3.1) be λ_{n+}, λ_{n-} and write $\phi_{n\pm}(x) = (\sin nx, \varepsilon_\pm \sin nx)$. It is then easy to see that $\{\phi_{n+}, \phi_{n-}\}$ is complete, that the analogue of (2.1),

$$(3.2) \quad M_t(\xi,\eta) = \sum_{n=1}^{\infty} \{e^{\lambda_{n+}} \phi_{n+}\phi^*_{n+}[\xi,\eta] + e^{\lambda_{n-}} \phi_{n-}\phi^*_{n-}[\xi,\eta]\},$$

holds for all $\xi,\eta \in B(0,\Pi)$, and that (M), (Φ), (Φ*) are satisfied with $\lambda = \lambda_{1+}$, $\phi = \phi_{1+}$, $\phi^* = \phi^*_{1+}$. In fact, if

$$(3.3) \quad (\xi,\eta) = \sum_{n=1}^{\infty} \{\phi_{n+}\phi^*_{n+}[\xi,\eta] + \phi_{n-}\phi^*_{n-}[\xi,\eta]\}$$

is the formal expansion of (ξ,η) in terms of ϕ_{n+}, ϕ_{n-}, we must have

$$\phi^*_{n+}[\xi,\eta] + \phi^*_{n-}[\xi,\eta] = c_n[\xi], \; \varepsilon_+\phi^*_{n+}[\xi,\eta] + \varepsilon_-\phi^*_{n-}[\xi,\eta] = c_n[\eta],$$

where the $c_n[\cdot]$ are the sine series coefficients as in § 2. Since $\varepsilon_+ \neq \varepsilon_-$, these equations have solutions of the form

$$\phi^*_{n\pm}[\xi,\eta] = r_\pm c_n[\xi] + s_\pm c_n[\eta]$$

That is, $\phi^*_{n\pm}[\cdot]$ has the density

$$\phi^*_{n\pm}(x) = (r_\pm \frac{2}{\Pi} \sin nx,\ s_\pm \frac{2}{\Pi} \sin nx)$$

Using $\lambda_{n\pm} \simeq - an^2$, we can now repeat the argument of § 2.

The explicit expressions for ϕ, ϕ^* have simple interpretations relative to the epidemic model. Thus in the mean both the non-infectious and infectious particles are asymptotically distributed on $(0,\Pi)$ as in the one-dimensional model, that is, when neglecting the effect of the non-infectious period. Similarly, the reproductive value of a particle at $x \in (0,\Pi)$ is $\sin x$, the same as in one dimension, up to a factor depending only on whether it is infectious or not.

Let us return to the general L. Then:

PROPOSITION <u>Suppose that</u> A_1, A_2 <u>satisfy</u> (D), <u>that</u> k_i, $m_{ij} \in C^0$, <u>and that</u> $k_1 m_{12}$, $k_2 m_{21}$ <u>do not vanish everywhere</u>. <u>Then</u> (M), (Φ), (Φ^*) <u>are satisfied and the statements</u> (I), (II), (III) <u>remain valid</u>.

The proof of (M), (Φ), (Φ*) is contained in [9], (I) is a special case of [4], and (II), (III) are covered by [6].

Writing $\phi = (\phi(1,\cdot), \phi(2,\cdot))$, $\phi^* = (\phi^*(1,\cdot), \phi^*(2,\cdot))$, we have instead of (x log x)

$$\sum_{i,j=1}^{2} \int_{X_o} \phi^*(i,x)k_i(x)\Sigma p_n^{ij}(x)n \log n \, \phi(j,x)dx < \infty$$

Similarly

$$\mu = \frac{1}{2} \sum_{i,j=1}^{2} \int_{X_o} \phi^*(i,x)k_i(x)\Sigma\, p_n^{ij}(x)n(n-1)\phi(j,x)^2 dx$$

The analogues of (x log x)' and the corresponding equivalent of $\mu < \infty$ are easily written down.

§ 4. RETARDED BRANCHING

In a classical reactor rods or spheres of fissionable material are embedded in a moderating substance. A fast neutron produced through fission is slowed down to thermal velocities through collisions in the moderator. If a thermal neutron is captured by a fissionable nucleus, fission does not occur instantaneously, but after an approximately exponentially distributed time. Since a diffusion appears to be one of the reasonable approximations to the motion of neutrons in a reactor, [22], the process may be modelled as the two-phase epidemic in § 3, but with $A_2 = 0$. The particles in X_1 represent free neutrons and those in X_2 captured neutrons. Of course a phase in which particles are fixed may be of interest also in connection with epidemics.

The differential generator of the semigroup now takes the form

$$L(\xi,\eta)(x) = \begin{cases} A\xi(x)-k_1(x)\xi(x)+k_1(x)\eta(x), & x\in X_1, \\ k_2(x)m(x)\xi(x)-k_2(x)\eta(x), & x\in X_2, \end{cases}$$

$\xi,\eta\in D(A)$

Given the methods of the preceding section, this generator presents a serious problem. Suppose again $X_o = (0,\Pi)$, let A describe absorbing Brownian motion on X_o with constant velocity $a > 0$, $D(A) = \{g\in C^2 : g(0+) = g(\Pi-) = 0\}$, and let

k_1, k_2, m be constants. Proceeding as in § 3, the solutions of the eigenvalue problem $Lf = \lambda f$, $f \in D(A)$, turns out to be

$$\lambda_{n\pm} = -an^2-k_1+k_1\varepsilon_{n\pm},$$
$$\phi_{n\pm} = (\sin nx, \varepsilon_{n\pm}\sin nx),$$
$$\varepsilon_{n\pm} = \frac{1}{2k_1}\{k_1-k_2+an^2\pm[(k_1-k_2+an^2)^2+4k_1k_2m]^{\frac{1}{2}}\},$$

n = 1, 2, ... Notice that

$$\lambda_{n+} = -k_2 + O(n^{-2}),$$

i.e. the spectrum has a finite accumulation point. This means that the methods of § 2-3 are no longer applicable. In fact, it is easily seen that in this case (M) cannot be satisfied. For, suppose (M) were satisfied. Then $\phi = \phi_{1+}$ and

$$(|\sin nx|, \varepsilon_{n+}|\sin nx|) = e^{-\lambda_{n+}}|M_1\phi_{n+}| \leq e^{-\lambda_{n+}}M_1|\phi_{n+}|$$
$$\leq K\phi\phi^*[|\phi_{n+}|] = K(\sin x, \varepsilon_{1+}\sin x)\phi^*[|\phi_{n+}|]$$

with $K < \infty$ not depending on x or n. Since $\phi^*[|\phi_{n+}|] = O(\varepsilon_{n+})$, this would imply

$$|\sin nx| \leq K'\sin x, \; x \in (0,\Pi),$$

with $K' < \infty$ independent of x, n. Clearly, this is false, so that (M) cannot be satisfied. Of course, the analogue of (3.2) still holds for $\xi,\eta \in D(A)$, but the argument extending (3.2) to $\xi,\eta\in B$, in particular to $\xi = \eta = 1$, does not carry over.

These problems can be avoided by modifying the model. We replace the continous set X_2 with a finite set of <u>depots</u> and redefine our process as follows. For simplicity suppose X_2 consists of just one point, denoted by y. This corresponds to a reactor with just one rod of fissionable material or, in the context of epidemics, to one infirmary. If the motion on X_1 is terminated at $x\in X_1$, an exponentially distributed time with decay constant k_2 elapses until n particles are released with probability p_n, each distributed with density f(x) on X_1. Defining $m(x) = f(x)\Sigma_n np_n$, the differential generator of the mean semigroup now becomes

$$L(\xi,\eta)(x) = \begin{cases} A\xi(x)-k_1(x)\xi(x)+k_1(x)\eta, & x\in X_1, \\ k_2\int_{X_1} m(z)\xi(z)dz-k_2\eta, & x = y, \end{cases}$$

$\xi\in D(A)$, $\eta\in R$.

From the underlying picture it is not unreasonable to work with functions k_1, m which vanish outside a certain interval in the interior of X_1, interpreted as the vicinity of the depot.

PROPOSITION <u>Suppose that</u> A <u>satisfies</u> (D), <u>that</u> k_1, $m\in C^1$ <u>with</u> k_1, k_2m <u>not vanishing everywhere and that</u> $k_1(x) \to 0$, $f(x) \to 0$ <u>as</u> $x \to x_0\in\overline{X}_1\setminus X_1$. <u>Then</u> (M), ($\Phi$), ($\Phi$)* <u>are satisfied</u>

<u>and statements</u> (I), (II), (III) <u>remain valid</u>.

As before, the result follows from [9], [4], [6]. It can be shown that in the present case the analogue of (x log x) reduces to

$$\Sigma p_n n \log n < \infty$$

while

$$\mu = \frac{1}{2}\,\phi^*(y)k_2\Sigma\, p_n n(n-1)[\int_{X_1} f(x)\phi(x)dx]^2.$$

§ 5. COMBINATIONS

The considerations of the previous sections can be extended to more elaborate models, coupling any finite number of diffusions and depots. Explicitly, suppose the set of types is

$$X = X_1 \cup \ldots \cup X_r \cup \{y_1\} \cup \ldots \cup \{y_s\}$$

where the X_i are disjoint copies of some bounded interval X_o and the y_j additional disjoint points. Let the diffusion on X_i be given by A_i, let the motion on X_i be terminated with density $k_i(x)$ and let a decay in y_ν occur with decay constant l_ν. Suppose a stopping at $x \in X_i$ results in $n(1)+ \ldots +n(r) +m(1)+ \ldots +m(s)$ new particles, $n(j)$ of them at $x \in X_j$ and $m(\nu)$ of them at y_ν, and let $p^{ij}_{n(j)}(x)$, $q^{i\nu}_{m(\nu)}(x)$ be the associated marginal probabilities. Similarly, let a decay in y_ν yield $n(1)+ \ldots +n(r)+m(1)+ \ldots +m(s)$ new particles, $n(i)$ of them in X_i and $m(\mu)$ at y_μ, with marginal probabilities $\gamma^{\nu i}_{n(i)}$, $\delta^{\nu\mu}_{m(\mu)}$, where the positions of the particles in X_i have marginal density $f^{\nu i}(x)$.

The generality of this model permits to give a detailed description of even quite complex phenomena. We give two examples.

Consider first a neutron reactor with X_o as the moderator and the y_j as the rods of fissionable material. If a neutron is captured by a nucleus in y_ν, fission occurs after an exponential decay time with rate l_ν. The result is a number of new fast neutrons, which are slowed down to thermal energies by collisions with nuclei in the moderator. Usually this process is modelled as a chain of diffusions $A_1, \ldots, A_r$ on X_o with decreasing velocities, [22], and only the slow neutrons in X_r can be captured by a nucleus. Thus when a particle is stopped at $x \in X_i$, $i = 1, \ldots, r-1$, it switches to $x \in X_j$ for some $j = i+1, \ldots, r$, while a stop at $x \in X_r$ results in capture in one of the depots y_ν, normally close to x. Finally a decay in y_ν results in a collection of new particles scattered over X_1 according to $f^{\nu 1}(x)$, normally concentrating around y_ν.

Consider next a malaria epidemic. Let $s = 0$, write $r = r_1 + r_2$ and let the particles in X_i represent the i^{th} type of infected humans, $i = 1, \ldots, r_1$, and those in X_{r_1+j} the j^{th} type of infected mosquitos. We let $i = 1, \ldots, r_1 - 1$ ($j = 1, \ldots, r_2 - 1$) represent the phases in the non-infectious period of a human (mosquito), cf. [23], pg. 313, and $i = r$, ($j = r_2$) the infectious period. Thus a particle stopped at $x \in X_i$, $i = 1, \ldots, r_1 - 1$, switches to $x \in X_{i+1}$, while a stop at $x \in X_{r_1}$ results in either death or creation of one new particle at $x \in X_{r_1+1}$. Similarly a particle stopped at $x \in X_{r_1+j}$, $j = 1, \ldots, r_2 - 1$, switches to $x \in X_{r_1+j+1}$, while a stop at $x \in X_{r_1+r_2}$ results in either death or creation of one new particle at $x \in X_1$. Refinements of the epidemic pattern as discussed in [23] ch. 17 are easily incorporated in a similar manner.

Returning to the abstract model, define

$$K^{ij}(x) := \Sigma n\, p_n^{ij}(x),\quad L^{i\nu}(x) := \Sigma n\, q_n^{i\nu}(x),$$

$$\Omega^{\nu i}(x) := f^{\nu i}(x)\ \Sigma\, n\gamma_n^{\nu i},\quad \Pi^{\nu\mu} := \Sigma\, n\delta_n^{\nu\mu}.$$

The differential generator of the mean semigroup takes the form

$$L(\xi_1, \ldots, \xi_r, \eta_1, \ldots, \eta_s)(x) =$$

$$\begin{cases} A_i\xi_i(x)+k_i(x)\{-\xi_i(x)+\sum\limits_{j=1}^{r} K^{ij}(x)\xi_j(x)+\sum\limits_{\nu=1}^{s} L^{i\nu}(x)\eta_\nu\}, & x\in X_i, \\ l_\nu\{-\eta_\nu+\sum\limits_{j=1}^{r} \int\limits_{X_o} \Omega^{\nu i}(z)\xi_i(z)dz+\sum\limits_{\mu=1}^{s} \Pi^{\nu\mu}\eta_\mu\}, & x = y_\nu, \end{cases}$$

$\xi_i\in D(A_i)$, $i = 1, \ldots, r$, $\eta_\nu \in R$, $\nu = 1, \ldots, s$

and we have

PROPOSITION <u>Suppose that the</u> A_i <u>satisfy</u> (D), <u>that</u> k_i, $K^{ij}\in C^o$ <u>for</u> $i, j = 1, \ldots, r$ <u>and</u> $L^{i\nu}$, $\Omega^{i\nu}\in C^1$ <u>for</u> $i = 1, \ldots, r$, $\nu = 1, \ldots, s$, <u>that</u> $k_i(x)L^{i\nu}(x) \to 0$, $l_\nu\Omega^{\nu i}(x) \to 0$ <u>as</u> $x \to x_o\in \overline{X}_i \setminus X_i$, <u>and that all components of</u> X <u>communicate in the sense that the</u> $(r+s)\times(r+s)$ <u>matrix</u>

$$\begin{pmatrix} \overline{K} & \overline{L} \\ \overline{\Omega} & \overline{\Pi} \end{pmatrix}$$

defined by

$$\overline{K}^{ij} = \int_{X_o} k_i(x) K^{ij}(x)dx, \quad \overline{L}^{i\nu} = \int_{X_o} k_i(x) L^{i\nu}(x)dx,$$

$$\overline{\Omega}^{\nu i} = l_\nu \int_{X_o} \Omega^{\nu i}(x)dx = l_\nu \Sigma n \gamma_n^{\nu i}, \quad \overline{\Pi}^{\nu\mu} = l_\nu \Pi^{\nu\mu},$$

is irreducible. *Then* (M), (Φ), (Φ*) *are satisfied and statements* (I), (II), (III) *remain valid*.

As before, the result follows from [9], [4], [6]. The explicit expression for μ and analogues of ($x \log x$) and ($\mu < \infty$) are easily derived from the general formulas in [6].

§ 6. AGE-DEPENDENCE

In the simplest models considered so far, with constant termination density k and no absorption, the particles have all exponential lifetimes. If k(x) is space-dependent, the pattern is more complex since the lifetime depends on the path followed by the particle but still, the model is essentially Markovian. From the point of view of many applications it is, however, of interest to be able to deal with more general reproduction mechanisms. Some examples are already in the literature. For example, a Bellman-Harris process with the particles performing Brownian motion on the line has been studied, [24], [25]. The present section gives some results for models of similar simple structure, age-dependent branching processes with one or several types where the particles move according to diffusions of type (D). The results as well as the models should be considered as preliminary. For example, we do not treat the critical and subcritical case and it would also be of interest to study models with interaction of motion and branching mechanism. We hope to present such extensions in a future publication.

Consider first the case of one type. That is, we study a Bellman-Harris process, specified by its lifelength distribution G and offspring distribution F, where the particles move on X according to a diffusion, specified by its transition semigroup $\{T_t\}_{t\geq 0}$ and with differential generator satisfying (D). Let κ, V, A be the Malthusian parameter, resp. reproductive value function, resp. stable age distribution (cf. [11]) of the Bellman-Harris process and $T_t\eta(x) \simeq e^{-\mu t} \phi(x)\phi^*[\eta]$ the asymptotic expansion of § 2 (here $\mu \geq 0$, with $\mu > 0$ if and only if absorption may occur). Let I_t be the set of particles present at time t in the Bellman-Harris process and let for $c \in I_t$ x_c, a_c be the position resp. age of c. If c or some of its ancestors has been absorbed at the boundary, we adapt the convention $\eta(x_c) = 0$ for all $\eta \in B$.

PROPOSITION <u>Let</u> $\lambda = \kappa - \mu$. <u>Then</u>

$$W_t := e^{-\lambda t} \sum_{c \in I_t} V(a_c)\phi(x_c)$$

<u>is a non-negative martingale</u>. <u>Let</u> $W = \lim_t W_t$ <u>and consider the supercritical case</u> $\lambda > 0$. <u>Then</u> $EW = EW_0$ <u>for all non-random</u> I_0 <u>if and only if</u> $\int_0^\infty x \log x \, dF(x) < \infty$ <u>while</u> $W = 0$ <u>otherwise</u>. <u>Furthermore for all a.e. continuous</u> $\eta \in B(X)$, $\xi \in B[0,\infty)$

$$\text{(6.1)} \quad \lim_{t\to\infty} e^{-\lambda t} \sum_{c \in I_t} \xi(a_c)\eta(x_c) = WA[\xi]\phi^*[\eta] \ \underline{\text{a.s.}}$$

Letting $\xi(a) = 1$ we have $A[\xi] = 1$ and (6.1) specializes to a limit statement on the positions of the particles. The form is the same as in (I), only with W defined somewhat different. A similar remark applies, by letting $\eta(x) = 1$, to the ages.

We omit the proof, which is easily carried out along the lines of [25].

The generalization to a finite number of motions is, however, not immediate. We give the explicit calculations for a case of particular interest, a r-type Bellman-Harris process ($1 < r < \infty$), cf. [2], where the particles of type i move according to absorbing Brownian motion on $(0,\Pi)$ with velocity $a_i > 0$. In the Bellman-Harris process, let $G_1, \ldots, G_r$ be the lifetime distributions, (m_{ij}) the offspring mean matrix, I_t the set of particles alive at time t, $j_c = 1, \ldots, r$ the type of $c \in I_t$, a_c the age and x_c the position, where as before we define $\eta(x_c) = 0$ for any $\eta \in B(0,\Pi)$ if c or some of its ancestors have been absorbed at the boundary. For notational convenience, we do not include the ages in X, so that as in § 3-5 X is the disjoint union $X = X_1 \cup \ldots \cup X_r$ of r copies of $(0,\Pi)$. The main difficulty in treating the model turns out to be to determine the asymptotic behaviour of $M_t(\xi_1, \ldots, \xi_r)(x)$, $x \in X_i$, $\xi_1, \ldots, \xi_r \in B(0,\Pi)$ (it is always assumed that the ancestor is of age 0) and our approach is to describe the spatial distribution in terms of the semigroup $\{T_t\}_{t \geq 0}$ of absorbing Brownian motion on $(0,\Pi)$ with velocity one. Obviously,

$$(6.2) \quad M_t(\xi_1, \ldots, \xi_r)(x) = E^x \sum_{c \in I_t} \sum_{j=1}^{r} \xi_j(x_c) I(j_c = j)$$

Now consider some particular $c \in I(t)$. Then the path obtained by following all ancestors of c is Brownian motion, where the velocity at time s, $0 \leq s \leq t$, is a_i, i being the type of the particular ancestor of c alive at time s. Letting $\tau_i(c)$ be the total time up to t spent by c and its ancestor in type i, $\tau(c) = a_1\tau_1(c)+\ldots+a_r\tau_r(c)$, the position x_c of c (given $\tau(c)$) is then simply described by $T_{\tau(c)}$ in the sense that if the process is initiated by some particle at $x \in (0,\Pi)$ at time $t = 0$, then

$$(6.3) \quad E(\xi(x_c)\,|\,\tau(c)) = T_{\tau(c)}\xi(x) = e^{-\tau(c)}\psi(x)\psi^*[\xi]\{1+\Delta_{\tau(c)}\xi(x)\}$$

where from § 2

$$\psi(x) = \sin x, \; \psi^*[\xi] = \frac{2}{\Pi}\int_0^{\Pi} \sin x \; \xi(x)dx.$$

Let $a := \min\{a_1, \ldots, a_r\} > 0$. Then $\tau(c) \geq at$ so that as $t \to \infty$, $\Delta_{\tau(c)}\xi(x)$ tends to zero, uniformly in $c \in I(t)$, $\xi \in B_+$, $x \in (0,\Pi)$. Letting

$$K_i^k(t) = E \sum_{c \in I_t} e^{-\tau(c)} I(j_c = k),$$

the expectation being computed in the Bellman-Harris process with ancestor of type i, and inserting (6.3) into (6.2) yields

$$(6.4)\quad M_t(\xi_1,\ldots,\xi_r)(x)/\psi(x)\sum_{k=1}^{r}\psi^*[\xi_k]K_i^k(t) =$$

$$1+\Delta_t'(\xi_1,\ \ldots,\ \xi_r)(x),\ x\in X_i,$$

where $\Delta_t'(\xi_1,\ \ldots,\xi_r)(x) \to 0$, uniformly in $\xi_1,\ldots,\xi_r\in B_+$ and $x\in(0,\Pi)$. Now a standard renewal argument shows that

$$(6.5)\quad K_i^k(t) = \delta_{ik}e^{-a_it}(1-G_i(t))+\int_0^t e^{-a_iu}\{\sum_{j=1}^{r} m_{ij}K_j^k(t-u)\}dG_i(u)$$

To deal with these equations, we need

LEMMA Let for $i,\ j = 1,\ \ldots,\ r$ F_{ij} be a known bounded non-negative measure concentrated on $(0,\infty)$ such that $(F_{ij}(\infty))$ is irreducible and that (F_{ij}) is non-lattice in the sense that there is no $\lambda > 0$ and no c_{ij} such that each F_{ij} is concentrated on $\{c_{ij}+n\lambda;n = 0,\ 1,\ 2,\ \ldots\}$ and that $c_{i(o)i(1)}+\ldots+c_{i(n)i(n+1)}$ is a multiple of λ for each chain $i(0)\ldots i(n+1)$ such that $i(0) = i(n+1)$, $F_{i(k)i(k+1)} > 0$ for $k = 0,\ \ldots,\ n$. Define for some fixed λ $p_{ij} = \int_0^\infty e^{-\lambda u}dF_{ij}(u)$, and suppose $P = (p_{ij})$ has spectral radius one. Choose ν, h such that $\nu_i > 0$, $h_i > 0$, $\nu P = \nu$, $Ph = h$ (cf. [26]) and let

$$d = \sum_{i,j=1}^{r} h_i\nu_j\int_0^\infty ue^{-\lambda u}dF_{ij}(u)$$

Then the solutions of

$$(6.6)\quad Z_i(t) = z_i(t)+\sum_{j=1}^{r}\int_0^t Z_j(t-u)dF_{ij}(u)$$

satisfy

$$\lim_{t\to\infty} e^{-\lambda t} Z_i(t) = \frac{h_i}{d} \sum_{j=1}^{r} \nu_j \int_0^\infty e^{-\lambda u} z_j(u)\,du$$

whenever the $e^{-\lambda t} z_i(t)$ _are directly Riemann integrable_.

Results of similar type can be found in [27], [28], [29], [2]. The present version is essentially contained in [27] when P is a transition matrix (i.e. $h_i = h$) and the general case can be reduced to this because (6.6) holds with Z_i, z_i, F_{ij} replaced by $\tilde{Z}_i = z_i/h_i$, $\tilde{z}_i = z_i/h_i$, $\tilde{F}_{ij} = F_{ij}h_j/h_i$ and because $\tilde{P} = (p_{ij}h_j/h_i)$ is a transition matrix (with $\tilde{\nu}_i = \nu_i h_i$).

Now suppose that the $dF_{ij}(u) = m_{ij}e^{-a_i u}dG_i(u)$ satisfy the assumptions of the lemma and that the $e^{-(a_i+\lambda)t}(1-G_i(t))$ are directly Riemann integrable. Defining

$$\tilde{\nu}_i = \frac{\nu_i}{d} \int_0^\infty e^{-(a_i+\lambda)t}(1-G_i(t))\,dt$$

it follows by letting $Z_i(t) = K_i^k(t)$ with k fixed that $e^{-\lambda t}K_i^k(t) \to h_i\tilde{\nu}_k$ and combining with (6.4) yields

PROPOSITION

$$M_t(\xi_1,\ldots,\xi_r)(x) = e^{\lambda t}\phi(x)\phi^*[\xi_1,\ldots,\xi_r]\{1+\Delta_t''(\xi_1,\ldots,\xi_r)(x)\}$$

where $\Delta_t''(\xi_1,\ldots,\xi_r)(x) \to 0$, _uniformly in_ $\xi_1,\ldots,\xi_r \in B_+$ _and_ $x \in X$, _and where_ ϕ _and the Lebesgue density of_ ϕ^* _are given by_

$(h_1 \sin x, \ldots, h_r \sin x)$, resp. $(\frac{2\tilde{\nu}_1}{\Pi} \sin x, \ldots, \frac{2\tilde{\nu}_r}{\Pi} \sin x)$

PROPOSITION *Define*

$$V_i(a) := e^{(a_i+\lambda)a} \int_a^\infty e^{-(a_i+\lambda)t} dG_i(t) \sum_{j=1}^r m_{ij} h_j$$

[note that $V_i(0) = h_i$]. *Then*

$$W_t := e^{-\lambda t} \sum_{c \in I_t} V_{j_c}(a_c) \sin x_c$$

is a non-negative martingale. Let $W := \lim_t W_t$ *and consider the supercritical case* $\lambda > 0$. *Then* $EW = EW_0$ *for all non-random* I_0 *if and only if* $\int_0^\infty x \log x \, dH_{ij}(x) < \infty$ *for all* i, j, (H_{ij}) *being the offspring distributions of the Bellman-Harris process, while* $W = 0$ *otherwise. Furthermore whenever* $\eta_1, \ldots, \eta_r \in B(0,\Pi)$ *and* $\xi_1, \ldots, \xi_r \in B[0,\infty)$ *are a.e. continuous,*

$$\lim_{t\to\infty} e^{-\lambda t} \sum_{c \in I_t} \xi_{j_c}(a_c) \eta_{j_c}(x_c) = W\mu[\xi_1, \eta_1, \ldots, \xi_r, \eta_r] \text{ a.s} \tag{6.7}$$

where μ *is the measure given on the* i^{th} *component*, $i = 1, \ldots, r$, *by*

$$\frac{2}{\Pi d} \nu_i e^{-(a_i+\lambda)a} (1-G_i(a)) \cdot \sin x \, da dx$$

As in the case $r = 1$ treated earlier in this section, (6.7) can be specialized to statements on the limiting distribution according to position, type and ages. In general, the form of the results are, however, somewhat more complex than when $r = 1$. Thus the limiting age-distribution is not the same in the Bellman-Harris process and in the branching diffusion and also, λ can not be expressed as a function of the a_i and the Malthusian parameter of the Bellman-Harris process alone.

In applications, one would often be interested in allowing for reproduction during the lifespan and not only at the end. This could be handled approximately by dividing into a finite number of phases but in fact, there is no difficulty in incorporating more exact models as e.g. the age dependent birth - and death process, see [11] pg. 159-161. Consider as a simple example the two-phase epidemic of § 3 modified such that a phase i particle performs absorbing Brownian motion on $(0,\Pi)$ with velocity $a_i > 0$ and has lifelength distribution G_i, that a phase 1 particle switches to a phase 2 particle at the time of death and that a phase 2 particle of age u creates a new phase 1 particle with infinitesimal probability $k(u)dt$. The equations similar to (6.5) become

$$(6.8)\quad \begin{cases} K_i^k(t) = \delta_{1k} e^{-a_1 t}(1-G_1(t)) + \int_0^t K_2^k(t-u) e^{-a_1 u}\, dG_1(u) \\[2ex] K_2^k(t) = \delta_{2k} e^{-a_2 t}(1-G_2(t)) + \int_0^t K_1^k(t-u) e^{-a_2 u}\, k(u) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (1-G_2(u))\,du \end{cases}$$

which can be handled exactly as above. For example, the process is supercritical, critical or subcritical according to whether

$$\int_0^\infty e^{-a_1 u} dG_1(u) \cdot \int_0^\infty e^{-a_2 u} k(u)(1-G_2(u))du$$

is > 1, = 1 or < 1.

REFERENCES

[1] K.B. Athreya and P. Ney, Branching processes, Springer Verlag, Berlin (1972).

[2] C.J. Mode, Multitype branching processes, American Elsevier, New York (1971).

[3] H. Hering, Limit theorem for critical branching diffusion processes with absorbing barriers, Math.Biosci. 19, 355-370 (1974).

[4] S. Asmussen and H. Hering, Strong limit theorems for general supercritical branching processes with applications to branching diffusions, Z. Wahrscheinlichkeitstheorie verw. Geb. (1976).

[5] H. Hering, Subcritical branching diffusions, Comp. Math. (1976).

[6] H. Hering, Minimal moment conditions in the limit theory for Markov branching processes, to appear (1976).

[7] N. Ikeda, M. Nagasawa and S. Watanabe, Branching Markov processes I-III, J. Math. Kyoto Univ. 8, 233-278, 365-410 (1968); 9, 95-160 (1969).

[8] L. Breiman, Probability, Reading Mass. (1968).

[9] H. Hering, Primitive semigroups and positive regularity of branching diffusions, to appear (1976).

[10] E.B. Dynkin, Markov processes I-II, Springer Verlag, Berlin Göttingen Heidelberg (1965).

[11] T.E. Harris, The theory of branching processes, Springer-Verlag, Berlin Göttingen Heidelberg (1963).

[12] S. Asmussen, Almost sure behaviour of linear functionals of supercritical branching processes, to appear (1975).

[13] S.Asmussen and H.Hering, Strong limit theorems for supercritical immigration-branching processes, Math.Scand. (1976).

[14] H. Hering, Asymptotic behaviour of immigration-branching processes with general set of types. I: Critical branching part, Adv. Appl. Prob. 5, 391-416 (1973).

[15] M.S. Bartlett, Deterministic and stochastic models for recurrent epidemics, Proc. 3rd Berkeley Symp. Math. Statist. Prob. 4, 81-109 (1956).

[16] J. Neyman and E.L. Scott, A stochastic model of epidemics, Stochastic Models in Medicine and Biology (J. Gurland ed.) 45-85, University of Wisconsin Press (1964).

[17] R. Bartoszynski, Branching processes and the theory of epidemics. Proc. 5th Berkeley Symp. Math. Statist. and Prob. 4, 259-269 (1967).

[18] D.A. Griffiths, A bivariate birth-death process which approximates to the spread of a disease involving a vector, J. Appl. Prob. 10, 15-26 (1972).

[19] J. Radcliffe, The initial geographical spread of host-vector and carrierborne epidemics, J. Appl. Prob. 10, 703-717 (1973).

[20] J. Radcliffe, The effect of the length of incubation period on the velocity of propagation of an epidemic wave, Math. Biosci. 19, 257-262 (1974).

[21] J. Radcliffe, The convergence of a position-dependent branching process used as an approximation to a model describing the spread of an epidemic, J. Appl. Prob. 13, 338-344 (1976).

[22] A.M. Weinberg and E.P. Wigner, The physical theory of neutron chain reactors, University of Chicago Press (1958).

[23] N.T.J. Bailey, The mathematical theory of infectious diseases and its applications, 2nd Ed., Griffin, London (1975).

[24] A.W. Davis, Branching diffusion processes with no absorbing boundary I-II, J. Math. Anal. Appl. 18, 275-296; 19, 1-25 (1967).

[25] N. Kaplan and S. Asmussen, Branching random walks II, Stoch. Proc. Appl. 4, 15-31 (1976).

[26] F.R. Gantmacher, Matrizenrechnung II, VEB Deutscher Verlag der Wissenschaften, Berlin (1959).

[27] E. Cinlar, Markov renewal theory, Adv. Appl. Prob. 1, 123-187 (1969).

[28] K.S. Crump, On systems of renewal equations, J. Math. Anal. Appl. 30, 425-434 (1970).

[29] T.A. Ryan, Jr., A multidimensional renewal theorem, Ann. Prob. 4, 656-661 (1976).