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Abstract

The paper is a survey of various martingale techniques useful when studying the supercritical Galton-Watson process $Z_0, Z_1, \ldots$ and its generalizations. Suppose $Z_0 = 1$, $Z_{n+1} = \sum_1^{Z_n} X_{n,i}$ with the $X_{n,i}$ i.i.d. and let $m = EX_{n,i}$ $(1 < m < \infty)$, $W_n = Z_n/m^n$, $W = \lim_n W_n$. Exploiting the similarity of

$$\sum_{n=0}^{\infty} \{W_{n+1} - W_n\} = \sum_{n=0}^{\infty} m^{-n-1} \sum_{i=1}^{Z_n} \{X_{n,i} - m\}$$

with a sum of independent r.v. with mean zero, a class of martingale series approximating $\Sigma \{W_{n+1} - W_n\}$ is used to give a new and short proof of the necessity and sufficiency of the condition $EX_{n,i} \log X_{n,i} < \infty$ for non-degeneracy of W and to study convergence rates (i.e.a.s. estimates of $W - W_n$) under related moment conditions. E.g. if $1 < p < 2$, $1/p + 1/q = 1$, then $W - W_n = o(m^{-n/q})$ if and only if $EX_{n,i}^p < \infty$. It is shown how this technique can be extended to the Bellman-Harris process, where (with some additional material) a full and self-contained treatment of the basic limit theory is given. Also a simple approach to the study of the moments of W is presented. It yields explicit inequalities like $EW^p \leqq 1 + EX_{n,i}^p/(m^p - m)$, $1 < p < 2$, and is based upon moment inequalities of the form

$$ESf(S) \leqq ESf(ES) + \sum_{i=1}^{n} EX_i f(X_i)$$

valid whenever $f: [0,\infty) \to [0,\infty)$ is concave and $S = X_1 + \ldots + X_n$ is a sum of independent r.v. $X_i \geqq 0$.

§ 1. Introduction.

Consider a Galton-Watson process $\{Z_n\}$ with offspring distribution F and offspring mean $m = \int_0^\infty x\,dF(x)$. We think of the process as constructed from a double array $\{X_{n,i}\}$ of independent random variables (r.v.) distributed according to F, such that

$$Z_0 = 1, \quad Z_{n+1} = \sum_{i=1}^{Z_n} X_{n,i} \,.$$

When $m \leq 1$, the only interesting a.s. statement on $\{Z_n\}$ seems to be the certainty of extinction and all limit results as well as their proofs are essentially analytic in nature. The flavour of the supercritical case $1 < m < \infty$, which we consider throughout, is quite different. Here in general growth to infinity occurs with positive probability, and the limit results are of strong type, describing for example the growth more precisely and more specific phenomena such as spatial distribution and age structure in the various generalizations of the model. Despite this fact, extensive use is made of analytic techniques, often successsfully and in a natural way, but often also in situations where the problems would suggest a different approach.

In the present paper, we present some probabilistic methods useful when dealing with certain aspects of the limit theory in the supercritical case. As is well-known, $W_n = Z_n/m^n$ is a non-negative martingale w.v.t. $\mathcal{F}_n = \sigma(X_{m,i};i=0,1,2,\ldots;m<n)$ and thus $W = \lim_n W_n$ exists [1]. Roughly speaking, our approach is to undertake a more refined study of $\{W_n\}$ and W in terms of the infinite series

$$\sum_{n=0}^{\infty} \{W_{n+1} - W_n\} = \sum_{n=0}^{\infty} m^{-n-1} \sum_{i=1}^{Z_n} \{X_{n,i} - m\} \tag{1.1}$$

and to exploit the structure of (1.1) as something in between a general martingale series and a sum of independent r.v. with mean zero. The core of the paper

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1) In such statements is frequently understood a.s.

is § 2-3, where we introduce some new martingale series approximating (1.1). The applications are in § 2 to give a new and short proof of the classical result of [13],

1.1 THEOREM $EW = EW_0 = 1$ if and only if

$$(x \log x) \qquad \int_0^\infty x \log x \, dF(x) < \infty$$

while $W = 0$ otherwise.

and in § 3 to study convergence rates, i.e. a.s. estimates of $W - W_n$,

1.2 THEOREM Suppose $(x \log x)$ holds. Then

(i) Let $1 < p < 2$, $1/p + 1/q = 1$. Then $W - W_n = o(m^{-n/q})$ if and only if $\int_0^\infty x^p \, dF(x) < \infty$

(ii) Let $\alpha > 0$. Then $W - W_n = o(n^{-\alpha})$ if and only if

$$(1.2) \qquad \int_y^\infty x[\log x - \log y] \, dF(x) = o([\log y]^{-\alpha})$$

(iii) Let $\alpha > 0$. Then $\Sigma_{n=0}^\infty n^{\alpha-1} \{W - W_n\}$ converges if and only if $\mu_{\alpha+1} < \infty$, where

$$\mu_\beta = \int_0^\infty x[\log^+ x]^\beta \, dF(x)$$

[to get a feeling for (1.2), note that

$$(1.3) \quad \mu_{\alpha+1} < \infty \Rightarrow \int_y^\infty x \log x \, dF(x) = o([\log y]^{-\alpha}) \Rightarrow (1.2) \Rightarrow \mu_{\alpha+1-\varepsilon} < \infty \ \forall \varepsilon > 0$$

as is easily seen upon integration by parts] 1.2 is a slight sharpening and extension of [1]. Results of similar form can be found in the theory of sums of i.i.d.r.v. $U_1, U_2, \ldots$ For example, letting $\mu = EU_1$, $\bar{U}_n = (U_1 + \ldots + U_n)/n - \mu$, it holds that

$$(1.4) \qquad \bar{U}_n = o(n^{-1/q}) \Leftrightarrow E|U_1|^p < \infty \quad (1 < p < 2,\ 1/p + 1/q = 1)$$

$$(1.5)\qquad \bar{U}_n = o([\log n]^{-\alpha}) \Leftrightarrow E|U_1|\ [\log^+|U_1|]^\alpha < \infty \quad (\alpha \geqq 0)$$

see [14], pg. 152-155, slightly extended. Together with

$$(1.6)\qquad \sup_n W_n < \infty,\quad P(\inf_n W_n > 0) = P(W > 0) > 0$$

$$(1.7)\qquad EW^p < \infty \Leftrightarrow \int_0^\infty x^p\ dF(x) < \infty \quad (p > 1)$$

$$(1.8)\qquad EW[\log^+ W]^\alpha < \infty \Leftrightarrow \mu_{\alpha+1} < \infty$$

which holds assuming (x log x), see 1.1 and [6], [9], (1.4) and (1.5) also provide a first motivation for 1.2, since conditioned upon F_n $W - W_n$ is distributed as $W_n \bar{U}_{Z_n}$ if we let $P(U \leqq u) = P(W \leqq u)$. To see this, let $W^{n,i}$ be the W-variable corresponding to the Galton-Watson process initiated by the i^{th} individual alive at time n and note that

$$(1.9)\qquad W - W_n = \frac{1}{m^n} \sum_{i=1}^{Z_n} \{W^{n,i} - 1\} = W_n \frac{1}{Z_n} \sum_{i=1}^{Z_n} \{W^{n,i} - 1\} .$$

Noting that $Z_n \simeq m^n$ and combining (1.4), (1.7), (1.9) leads precisely to part (i) of 1.2, while using instead (1.5), (1.8), (1.9) one is lead to expect the condition for $W - W_n = o(n^{-\alpha})$ to be $\mu_{\alpha+1} < \infty$, which is only slightly stronger than (1.2), cf. (1.3). However, in § 2 we sketch a different point of view on 1.2.

Also the technique in our proofs of 1.1, 1.2 relates to sums of independent r.v. As relevant background, we suggest to keep in mind Kolmogorov's three series criterion and the somewhat related standard proof of (1.4), (1.5), see [14], pg. 152-155. A common feature is here an approximation argument, which for example for the law of large numbers consists in studying

$$(1.10)\qquad \sum_{n=0}^\infty \{\tilde{U}_n - E\tilde{U}_n\}/n = \sum_{n=0}^\infty \{\tilde{U}_n - \mu + EU_n I(|U_n| > n)\}/n,$$

where $\tilde{U}_n = U_n\ I(|U_n| \leqq n)$, rather than the (non necessarily convergent) series $\Sigma\{U_n - \mu\}/n$. Adapting this idea to the Galton-Watson process, we study not the series (1.1) but instead a series

$$(1.11) \qquad \sum_{n=0}^{\infty} \{\tilde{W}_{n+1} - E(\tilde{W}_{n+1} \mid \mathcal{F}_n)\} = \sum_{n=0}^{\infty} \{\tilde{W}_{n+1} - W_n + R_n\}$$

defined in analogy by (1.10), that is, by

$$\tilde{W}_{n+1} = m^{-n-1} \sum_{i=1}^{Z_n} X_{n,i}\, I(X_{n,i} \leq c_n) ,$$

$$R_n = E(W_n - \tilde{W}_{n+1} \mid \mathcal{F}_n) = E(W_{n+1} - \tilde{W}_{n+1} \mid \mathcal{F}_n) = m^{-1} W_n \int_{c_n}^{\infty} x \, dF(x) .$$

By definition, (1.11) is again a martingale series. As a common feature in the proofs enters a routine calculation of three series similar to those of Kolmogorov,

$$(1.12) \qquad \sum_{n=0}^{\infty} P(\tilde{W}_{n+1} \neq W_{n+1}), \quad \sum_{n=0}^{\infty} ER_n, \quad \sum_{n=0}^{\infty} \mathrm{Var}\{\tilde{W}_{n+1} - W_n + R_n\}$$

It is here where the moment conditions on F come in, but the calculations in (1.12) alone does not prove the results. Additional ideas varying from case to case are required to complete the proofs.

There are numerous ways of varying the basic model and when developing techniques for dealing with the Galton-Watson process, it is important that these can be used in more general branching processes. The adaption to processes with several (even infinitely many) types has already been presented as part of [2] and we treat here in § 5 age-dependent processes. As example we have chosen the Bellman-Harris process ([12], Ch. 6) and give a full treatment of the limit theory. It turnes out that our proof of the analogue of 1.1 with some minor modifications provide one of the basic lemmas needed when treating the further a.s. convergence results on the distribution of the population according to ages, see e.g. [12], [7][1]. We have added sufficient material to make the exposition totally self-contained and borrow here some ideas from [4], [7] as well as [2].

Finnaly, § 4 is devoted to a remark on results of type (1.7), (1.8). The proofs in the literature, see e.g. [6], [9], are in part both deep and laborious and we sketch a different approach based on moment inequalities for sums of independent r.v. rather than expansions of Laplace transforms.

§ 2. The x log x condition.

Our first example on the use of the martingales $\Sigma\{\tilde{W}_{n+1} - W_n + R_n\}$ defined in § 1 is to give the proof of 1.1. We let $c_n = m^n$ and the series in (1.12) are then computed the following way:

$$(2.1) \qquad \sum_{n=0}^{\infty} P(W_{n+1} \neq \tilde{W}_{n+1}) =$$

$$\sum_{n=0}^{\infty} EP(X_{n,i} > m^n \text{ for some } i = 1,\dots,Z_n \mid F_n) \leqq$$

$$\sum_{n=0}^{\infty} EZ_n \int_{m^n}^{\infty} dF(x) = \int_0^{\infty} \sum_{n=0}^{\infty} m^n I(x > m^n)\, dF(x) = \int_0^{\infty} 0(x)\, dF(x)$$

$$(2.2) \qquad \sum_{n=0}^{\infty} ER_n = m^{-1} \sum_{n=0}^{\infty} \int_{m^n}^{\infty} x\, dF(x) =$$

$$m^{-1} \int_0^{\infty} x\Big(\sum_{n=0}^{\infty} I(x > m^n)\Big)\, dF(x) = \int_0^{\infty} x\, 0(\log^+ x)\, dF(x)$$

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1) In fact, the results of § 5 are slightly stronger than those of [7], since we need only finite mean and not (x log x), which is used in a technical way in [7]. However, a short direct treatment of the case when (x log x) fails can be found in [3].

(2.3)
$$\sum_{n=0}^{\infty} \mathrm{Var}\{\widetilde{W}_{n+1} - W_n + R_n\} = \sum_{n=0}^{\infty} E\ \mathrm{Var}(\widetilde{W}_{n+1} \mid F_n) =$$

$$\sum_{n=0}^{\infty} E\, m^{-2n-2} Z_n \mathrm{Var}[X_{1,1}\, I(X_{1,1} \leq m^n)] \leq$$

$$m^{-2} \sum_{n=0}^{\infty} m^{-n} \int_0^{m^n} x^2\, dF(x) = m^{-2} \int_0^{\infty} x^2 \left(\sum_{n=0}^{\infty} m^{-n} I(x \leq m^n)\right) dF(X) =$$

$$\int_0^{\infty} x^2\, 0(x^{-1})\, dF(x) = \int_0^{\infty} 0(x)\, dF(x)$$

From (2.3) and the convergence theorem for L^2-bounded martingales, we obtain

2.1 LEMMA <u>Without any moment conditions on</u> F <u>beyond</u> $1 < m < \infty$, $\Sigma\{\widetilde{W}_{n+1} - W_n + R_n\}$ <u>converges a.s. and in</u> L^1

$EW \leq 1$ is immediate form Fatous lemma. To prove the converse, assuming (x log x) we let $N \to \infty$ in the inequality

$$EW = E(W_0 + \sum_{n=0}^{N} \{W_{n+1} - W_n\} + \sum_{n=N+1}^{\infty} \{W_{n+1} - W_n\})$$

$$\geq 1 + 0 + E(\sum_{n=N+1}^{\infty} \{\widetilde{W}_{n+1} - W_n\})$$

which is obvious from $W_{n+1} \geq \widetilde{W}_{n+1}$. We then only have to prove the L^1-convergence of $\Sigma\{\widetilde{W}_{n+1} - W_n\}$ which in view of the lemma is equivalent to that of ΣR_n. Since $R_n \geq 0$, it suffices that $\Sigma ER_n < \infty$, which follows from (2.2).

To prove that $W = 0$ if (x log x) fails, we first note that the existence of $W = \lim_n W_n$ implies the a.s. convergence of the telescoping series $\Sigma\{W_{n+1} - W_n\}$ and therefore that of $\Sigma\{\widetilde{W}_{n+1} - W_n\}$, since $\widetilde{W}_{n+1} = W_{n+1}$ for n large by (2.1) and the Borel-Cantelli lemma. Combining this with Lemma 2.1, we have a.s. convergence of ΣR_n. Now let $\underline{W} = \inf_n W_n$ and note that $\{W > 0\} = \{\underline{W} > 0\}$. If (x log x) fails, $\Sigma \int_{m^n}^{\infty} x\, dF(x) = \infty$ as in (2.2) and $P(W > 0) = 0$ follows from

$$\infty > \sum_{n=0}^{\infty} R_n \geq n^{-1} \underline{W} \int_{m^n}^{\infty} x\, dF(x) = \infty \quad \text{on} \quad \{\underline{W} > 0\}$$

§ 3. Convergence rates.

We next consider the proof of 1.2. To estimate $W-W_N$ we write $W-W_N = \Sigma_N^\infty \alpha_n$ where $\alpha_n = W_{n+1} - W_n$ and use

3.1 LEMMA *Let* $\{\alpha_n\},\{\beta_n\}$ *be series of real numbers such that* $0 < \beta_n \uparrow \infty$. *Then*

$$(3.1) \qquad \sum_{n=0}^{\infty} \alpha_n \beta_n \text{ converges} \Rightarrow \sum_{n=N}^{\infty} \alpha_n = o(1/\beta_N)$$

Obviously (3.1) is analogous to Kronecker's lemma, which is used in the proof of (1.4), (1.5) and states that under the assumptions of (3.1) it holds that

$$(3.2) \qquad \sum_{n=0}^{\infty} \alpha_n/\beta_n \text{ converges} \Rightarrow \sum_{n=0}^{N} \alpha_n = o(\beta_N)$$

In fact, both (3.1) and (3.2) are immediate from Abel's lemma, [10] pg. 54.

We first consider part (ii) of (1.2), the proof of which is particularly well suited to demonstrate the ideas. We let $\beta_n = n^\alpha$ and instead of studying $\Sigma\, \alpha_n \beta_n = \Sigma\, n^\alpha\{W_{n+1} - W_n\}$, we approximate by $\Sigma\, n^\alpha\{\tilde{W}_{n+1} - W_n + R_n\}$ defined as in § 1 with $c_n = m^n/n^\alpha$. Calculations similar to (2.1), (2.3) yields

$$\sum_{n=0}^{\infty} P(\tilde{W}_{n+1} \neq W_{n+1}) \simeq \sum_{n=0}^{\infty} \mathrm{Var}[n^\alpha\{\tilde{W}_{n+1} - W_n + R_n\}] \simeq \mu_\alpha$$

and as in § 2, we have immediately

3.2 LEMMA *Let* $\alpha > 0$, $c_n = m^n/n^\alpha$ *and suppose* $\mu_\alpha < \infty$. *Then* $\Sigma\, n^\alpha\{\tilde{W}_{n+1} - W_n + R_n\}$, $\Sigma\, n^\alpha\{W_{n+1} - W_n + R_n\}$ *converges a.s.*

PROOF OF (ii). Suppose first $\mu_\alpha < \infty$ (which is substantially weaker than (1.2), cf. (1.3)). Combining 3.1 and 3.2 yields

$$o(N^{-\alpha}) = \sum_{n=N}^{\infty} \{W_{n+1} - W_n + R_n\} = W - W_N + \sum_{n=N}^{\infty} R_n .$$

Therefore $W - W_N = o(N^{-\alpha})$ is equivalent to

$$o(N^{-\alpha}) = \sum_{n=N}^{\infty} R_n = \sum_{n=N}^{\infty} m^{-1} W_n \int_{m^n/n^\alpha}^{\infty} x\, dF(x)$$

or, appealing to (1.6), to

(3.3) $$o(N^{-\alpha}) = \sum_{n=N}^{\infty} \int_{m^n/n^\alpha}^{\infty} x\,dF(x) .$$

Define $y_n = m^n/n^\alpha$, $N(x) = \sup\{n : y_n \leqq x\}$. Then (3.3) can be rewritten as

(3.4) $$o([\log y_N]^{-\alpha}) = \int_{y_N}^{\infty} x(N(x) - N)\,dF(x) , \quad N \to \infty.$$

Apparently (3.4) is weaker than

(3.5) $$o([\log y]^{-\alpha}) = \int_{y}^{\infty} x(N(x) - N(y))\,dF(x) , \quad y \to \infty$$

but if (3.4) holds, so does (3.5) since for $y_N \leqq y < y_{N+1}$ then

$$\int_{y}^{\infty} x(N(x) - N(y))\,dF(x) \leqq \int_{y_N}^{\infty} x(N(x) - N)\,dF(x) = o([\log y_N]^{-\alpha}) = o([\log y]^{-\alpha})$$

Now from the definition of N(x) it can be verified that

$$N(x) = \frac{\log x}{\log m} + \frac{\alpha}{\log m} \log\log x + O(1) .$$

As $x, y \to \infty$, the meanvalue theorem for the log yields $\log\log x - \log\log y = o(\log x - \log y)$ so that the right-hand side of (3.5) is

$$\int_{y}^{\infty} x(\log x - \log y)\left(\frac{1}{\log m} + o(1)\right) dF(x) + \int_{y}^{\infty} x\,O(1)\,dF(x) .$$

Since $\mu_\alpha < \infty$, the last term is $o([\log y]^{-\alpha})$ and therefore conditions (3.5) and (1.2) are equivalent, completing the proof when $\mu_\alpha < \infty$.

Suppose next $\mu_\alpha = \infty$. Then by (1.3), certainly (1.2) fails and we have to prove that $W - W_n = o(n^{-\alpha})$ must fail too. Since we assume $(x \log x)$, we can find β such that $1 \leqq \beta < \alpha$ and that $\mu_\beta < \infty$, $\mu_{\beta+1/2} = \infty$. Then from (1.3) and the first part of this proof it follows that $W - W_n = o(n^{-\beta})$ fails and the proof is complete since $\beta < \alpha$.□

PROOF OF (iii). Let $\beta_n = \Sigma_1^n k^{\alpha-1}$. Then

$$\sum_{n=1}^{N} \alpha_n \beta_n = \sum_{k=1}^{N} k^{\alpha-1} \sum_{n=k}^{\infty} \alpha_n - \beta_N \sum_{n=N+1}^{\infty} \alpha_n$$

and from (3.1) it follows by letting $N \to \infty$ that

$$(3.6) \qquad \sum_{n=1}^{\infty} \alpha_n \beta_n \text{ converges} \Rightarrow \sum_{k=1}^{\infty} k^{\alpha-1} \sum_{n=k}^{\infty} \alpha_n \text{ converges} .$$

Let $\tilde{W}_{n+1}$, R_n be defined as above with $c_n = m^n/n^\alpha$. Using $\beta_n \simeq n^\alpha$ one obtains

$$\sum_{n=0}^{\infty} P(\tilde{W}_{n+1} \neq W_{n+1}) \simeq \sum_{n=0}^{\infty} \mathrm{Var}\{\beta_n\{W_{n+1} - W_n + R_n\}\} \simeq \mu_\alpha .$$

Thus if $\mu_\alpha < \infty$, $\Sigma\, \beta_n\{W_{n+1} - W_n + R_n\}$ converges a.s. and from (3.6), we get the a.s. convergence of

$$\sum_{k=0}^{\infty} k^{\alpha-1} \sum_{n=k}^{\infty} \{W_{n+1} - W_n + R_n\} = \sum_{k=0}^{\infty} k^{\alpha-1}(W - W_k + \sum_{n=k}^{\infty} R_n) .$$

Thus the convergence of $\Sigma\, k^{\alpha-1}\{W - W_k\}$ is equivalent to that of

$$\sum_{k=0}^{\infty} k^{\alpha-1} \sum_{n=k}^{\infty} R_n = m^{-1} \sum_{k=0}^{\infty} k^{\alpha-1} \sum_{n=k}^{\infty} W_n \int_{m^n/n^\alpha}^{\infty} x\, dF(x)$$

or, appealing to (1.6), to that of

$$\sum_{k=0}^{\infty} k^{\alpha-1} \sum_{n=k}^{\infty} \int_{m^n/n^\alpha}^{\infty} x\, dF(x) = \sum_{n=0}^{\infty} \beta_n \int_{m^n/n^\alpha}^{\infty} x\, dF(x) .$$

Using $\beta_n \simeq n^\alpha$, this precisely reduces to $\mu_{\alpha+1} < \infty$ and the proof is complete when $\mu_\alpha < \infty$.

If $\mu_\alpha = \infty$, then of course $\mu_{\alpha+1} = \infty$. Assuming $(x \log x)$ we can choose β, $1 \leq \beta < \alpha$, such that $\mu_\beta < \infty$, $\mu_{\beta+1} = \infty$. Then the first part of the proof excludes the convergence of $\Sigma\, n^{\beta-1}\{W - W_n\}$ and Abel's criterion ([10], pg. 48) that of $\Sigma\, n^{\alpha-1}\{W - W_n\}$. □

PROOF OF (i). We let $\beta_n = m^{n/q}$, $c_n = m^{n/p}$. Then

$$(3.7)\quad \sum_{n=0}^{\infty} P(\widetilde{W}_{n+1} \neq W_{n+1}) \simeq \sum_{n=0}^{\infty} \mathrm{Var}[m^{n/q}\{\widetilde{W}_{n+1} - W_n + R_n\}] \simeq \int_0^{\infty} x^p \, dF(x) \ .$$

Assuming the right-hand side to be finite, we have a.s. convergence of $\Sigma\, m^{n/q}\{\widetilde{W}_{n+1} - W_n + R_n\}$, $\Sigma\, m^{n/q}\{W_{n+1} - W_n + R_n\}$ and (3.1) gives

$$o(m^{-N/q}) = W - W_N + \sum_{n=N}^{\infty} R_n = W - W_N + m^{-1} \sum_{n=N}^{\infty} W_n \int_{m^{n/p}}^{\infty} x\, dF(x) \ .$$

But the last term is $o(m^{-N/q})$, since

$$\sum_{n=0}^{\infty} m^{n/q} \int_{m^{n/p}}^{\infty} x\, dF(x) = \int_0^{\infty} O(x^p)\, dF(x) < \infty$$

and it follows that $W - W_N = o(m^{-N/q})$, proving one way of the result.

For the converse, the method in the proofs at part (ii), (iii) and in § 2 does not apply, because the condition for convergence in (3.7) is not weaker than that for the result. Our proof is here totally different and we proceed by reducing the necessity problem for the Galton-Watson process to that of sums of i.i.d.r.v., cf. (1.4).

Suppose $W - W_N = o(m^{-N/q})$. In particular, $W_{n+1} - W_n = o(m^{-n/q})$ so that on $\{W > 0\}$

$$(3.8)\qquad Z_n^{-1/p} \sum_{i=1}^{Z_n} \{X_{n,i} - m\} = W_n^{-1/p}\, m^{n/q}\, \{W_{n+1} - W_n\} \to 0 \ .$$

Let the r.v. U_n in (1.4) be distributed as $X_{n,i} - m$, let $q(n,\varepsilon) = P(n^{1/q}|\bar{U}_n| > \varepsilon)$ and let $U_1^c, U_2^c, \ldots$ be independent and follow the symmetrized distribution of U_n, that is, the distribution of $X_{n,1} - X_{n,2}$. Define $\bar{U}_n^c$, $q^c(n,\varepsilon)$ the obvious way. It is then well-known that U_n^c has p^{th} moment if and only if F has so, so by (1.4) it suffices to prove that $\bar{U}_n^c = o(n^{-1/q})$. By the conditional Borel-Cantelli lemma and (3.8), we have on $\{W > 0\}$

$$\sum_{n=0}^{\infty} q(Z_n, \varepsilon) = \sum_{n=0}^{\infty} P(|Z_n^{-1/p} \sum_{i=1}^{Z_n} \{X_{n,i} - m\}| > \varepsilon \mid \mathcal{F}_n) < \infty$$

and therefore also by a standard inequality $\Sigma\, q^c(Z_n, 2\varepsilon) \leq 2\,\Sigma\, q(Z_n, \varepsilon) < \infty$. Now pick a numerical sequence $\{k(n)\}$ of integers of the form $k(n) = Z_n(\omega)$, where ω belongs to the set of positive probability where $W > 0$, $\Sigma\, q^c(Z_n, \varepsilon) < \infty$ for all rational (and therefor all) $\varepsilon > 0$. Then

$$(3.9) \qquad \sum_{n=0}^{\infty} P(k(n)^{1/q} \, |\bar{U}^c_{k(n)}| > \varepsilon) = \sum_{n=0}^{\infty} q^c(k(n), \varepsilon) < \infty$$

implying $\bar{U}^c_{k(n)} = o(k(n)^{-1/q})$. Define

$$M_n = k(n)^{-1/p} \max_{1 \leq i \leq k(n)} |U^c_1 + \ldots + U^c_i|$$

By Levy's inequality and (3.7)

$$\sum_{n=0}^{\infty} P(M_n > \infty) \leq 2 \sum_{n=0}^{\infty} P(k(n)^{-1/p} |U^c_1 + \ldots + U^c_{k(n)}| > \infty) = 2 \sum_{n=0}^{\infty} q^c(k(n), \varepsilon) < \infty$$

so that $M_n \to 0$. One checks readily that when $k(n) \leq i \leq k(n+1)$,

$$i^{1/q} \, |\bar{U}^c_i| \leq k(n)^{1/q} \, |\bar{U}^c_{k(n)}| + 2 \left(\frac{k(n+1)}{k(n)} \right)^{1/p} M_{n+1}$$

and $\bar{U}^c_i = o(i^{-1/q})$, $i \to \infty$, follows since $k(n+1)/k(n) \to m$ (in particular, the sequence $k(n)$ is ultimately increasing).□

We conclude by some remarks on the relation of 1.2 to sums of independent r.v. It is possible to exploit the motivation for 1.2 given in § 1 somewhat further by using (1.7), (1.9) to prove

$$\sum_{n=0}^{\infty} P(m^{n/q} \, |W - W_n| > \varepsilon \mid \mathcal{F}_n) < \infty \quad \text{if} \quad \int_0^{\infty} x^p \, dF(x) < \infty$$

and thus one half of (i). Similarly, (1.8) and (1.9) combine to give $W - W_N = o(N^{-\alpha})$ if $\mu_{\alpha+1} < \infty$. However, the full strenght of 1.2 does not seem to follow this way and as is apparent from the proofs, we exploit the structure of $W - W_n$ as the tail sum of (1.1) rather than (1.9). Also, as remarked earlier not all results are the perfect analogous of (1.4), (1.5) to be expected from (1.7), (1.8), (1.9). Instead we state the following result on sums of independent r.v., whose form and proof is more similar.

3.3 THEOREM Let $U_1, U_2, \ldots$ be i.i.d. with common distribution G. Then (from Kolmogorov's three series criterion) $\Sigma\, U_n/n$ converges if and only if

$$(3.10) \qquad \int_{-\infty}^{\infty} |x|\, dG(x) < \infty, \qquad \sum_{n=0}^{\infty} \int x\, I(|x| > n)\, dF(x) \quad \text{converges}$$

If this is satisfied, then

(i) $\Sigma_N^{\infty}\, U_n/n = o(N^{-1/q})$ if and only if $\int_{-\infty}^{\infty} |x|^p\, dG(x) < \infty \quad (1 < p < 2,\ 1/p + 1/q)$

(ii) For $\alpha > 0$, $\Sigma_N^{\infty}\, U_n/n = o([\log N]^{-\alpha})$ if and only if

$$(3.11) \qquad \begin{aligned} \mu_\alpha &= \int_{-\infty}^{\infty} |x|\, |\log^+ x|^\alpha\, dG(x) < \infty, \\ &\sum_{n=N}^{\infty} \int x\, I(|x| > n/(\log n)^\alpha)\, dG(x) = o([\log N]^{-\alpha}) \end{aligned}$$

There are, of course, similar results for other weights than n^{-1} and also part (iii) of 1.2 has a counterpart. The conditions (3.10), (3.11) can not be expressed in terms of the μ_α in the same way as in (1.3). For example, if G is symmetric, (3.11) reduces to $\mu_\alpha < \infty$, while if G is concentrated on $[a, \infty[$ for some $a > -\infty$, then (3.11) reduces to (1.2) (with F replaced with G) and (1.3) holds.

§ 4. A remark on the moments of W.

We recall the results (1.7), (1.8) concerning the relation between the moments in the offspring distribution F and those of W. The aim of the present section is to sketch an approach different from that of [6], [9] to results of this type.

As set-up, we choose to consider moments of the form $EW^\nu f(W)$, where ν is an integer and f a suitable function satisfying $f(x) = o(x)$, $x \to \infty$, for example $f(x) = x^\alpha$, $0 < \alpha < 1$. A detailed treatment is given only for the case $\nu = 1$, which is of particular importance and suffices to demonstrate the ideas.

4.1 LEMMA <u>Let</u> $f: [0,\infty[\to [0,\infty[$ <u>be concave and let</u> $S = X_1 + \ldots + X_N$ <u>be a sum of</u> N <u>independent r.v.</u> $X_i \geqq 0$. <u>Then</u>

$$ESf(S) \leqq ESf(ES) + \sum_{i=1}^{N} E X_i f(X_i) \tag{4.1}$$

PROOF: The assumptions on f imply subadditivity, $f(a+b) \leqq f(a) + f(b)$, $a,b \geqq 0$. Thus

$$ESf(S) = \sum_{i=1}^{n} EX_i f(S) \leqq \sum_{i=1}^{n} \{EX_i f(\sum_{j\neq i} X_j) + EX_i f(X_i)\}$$

$$\leqq ESf(ES) + \sum_{i=1}^{n} E X_i f(X_i) ,$$

since by Jensen's inequality

$$EX_i f(\sum_{j\neq i} X_j) = EX_i Ef(\sum_{j\neq i} X_j) \leqq EX_i f(E \sum_{j\neq i} X_j) \leqq EX_i f(ES). \square$$

Letting $N = Z_n$, $X_i = X_{n,i}/m^{n+1}$, $S = m^{-n-1} \Sigma_{i=1}^{Z_n} X_{n,i} = W_{n+1}$ yields

$$E(W_{n+1}\, f(W_{n+1}) \mid \mathcal{F}_n) \leqq$$

$$E(W_{n+1} \mid \mathcal{F}_n)\, f(E(W_{n+1} \mid \mathcal{F}_n)) + \sum_{i=1}^{Z_n} E\left(\frac{X_{n,i}}{m^{n+1}}\, f\left(\frac{X_{n,i}}{m^{n+1}}\right) \mid \mathcal{F}_n\right) =$$

$$W_n f(W_n) + m^{-1}\, W_n \int_0^\infty x\, f\left(\frac{x}{m^{n+1}}\right) dF(x)$$

and it follows that

$$E\,W\,f(W) \leqq \underline{\lim}\, E\, W_{N+1} f(W_{N+1}) =$$

$$\underline{\lim}\, \{f(1) + \sum_{n=0}^{N} \{EW_{n+1} f(W_{n+1}) - EW_n f(W_n)\}\} \leqq$$

$$f(1) + \sum_{n=0}^{\infty} m^{-1}\, EW_n \int_0^\infty x\, f\left(\frac{x}{m^{n+1}}\right) dF(x) = f(1) + m^{-1} \int_0^\infty x \sum_{n=0}^{\infty} f\left(\frac{x}{m^{n+1}}\right) dF(x)$$

4.2 EXAMPLE Let $1 < p < 2$, $f(x) = x^{p-1}$. Computation of $\Sigma f(x/m^{n+1})$ and inserting yields

(4.2) $$EW^p \leqq 1 + \frac{\int_0^\infty x^p \, dF(x)}{m^p - m}, \qquad 1 < p < 2$$

In particular, $EW^p < \infty$ if the p^{th} moment in the offspring distribution is finite. The converse is immediate assuming $(x \log x)$, since then by convexity

$$\int_0^\infty x^p \, dF(x) = m^p EW_1^p = m^p E(E(W \mid F_1))^p \leqq m^p EW^p$$

4.3 EXAMPLE In (1.8), $[\log^+ x]^\alpha$ does not satisfy the assumption on $f(x)$, but so does

$$f(x) = \begin{cases} c_1 x & 0 \leqq x \leqq x_0 \\ [\log^+ x]^\alpha + c_2 & x_0 \leqq x < \infty \end{cases}$$

if we chose first $x_0 > 1$ such that $d^2/dx^2 \, (\log x)^\alpha < 0$ when $x \geqq x_0$ and let

$$c_1 = \frac{d}{dx} (\log x)^\alpha \Big|_{x=x_0}, \quad c_2 = c_1 x_0 - (\log x_0)^\alpha$$

$\Leftarrow$ in (1.8) follows at once, since one easily checks $\Sigma f(x/m^{n+1}) = 0([\log^+ x]^{\alpha+1})$.

We shall not here further work out the approach. Some problems, in particular to prove $\Rightarrow$ in (1.8) seems to require additional ideas, while others are immediate. For example, the method works in the multitype or age-dependent case with a mere change of notation by studying the one-dimensional martingale functionals of the process. Also moments of order higher than the second can be treated. We state here the following inequality, which is valid for $\nu = 1,2,\ldots$ under the hypothesis of 4.1:

(4.3) $$ES^\nu f(S) \leqq ES^\nu f(ES) + \sum_{\mu=1}^{\nu} \binom{\nu}{\mu} ES^{\nu-\mu} \sum_{i=1}^{N} EX_i^\mu f(X_i)$$

§ 5. The limit theory of the supercritical Bellman-Harris process.

The model is the following. All individuals have lifelengths governed by a distribution G on $]0,\infty[$. At the time of death of the parent a random number of children are born according to the offspring distribution F. The lifelength and number of children of any particular individual are independent, and all individuals evolve independently of each other.

For questions of existence and construction, we refer to [12]. As remarked at a number at occasions in the literature (going back at least to [12]), the process is most naturally considered as a Markovian multitype process identifying types with ages. Accordingly, we define the state Z_t of the process at time t not as the number n of individuals alive, but as the collection $Z_t = \langle x_1,\dots,x_n\rangle$ of their ages. By averaging Z_t with various η belonging to the set B of bounded measurable functions on $[0,\infty[$, we obtain a number of functionals useful in the study of the process, defined by $Z_t[\eta] = 0$ if the population is extinct at time t and by

$$Z_t[\eta] = \eta(x_1) + \dots + \eta(x_n) \quad \text{if} \quad Z_t = \langle x_1,\dots,x_n\rangle \quad .$$

For example, $|Z_t| = Z_t[1]$ is the total population size. Also, if we think of Z_t as a (random) measure on $[0,\infty[$, then simply

$$Z_t[\eta] = \int_0^\infty \eta(x)\ Z_t[dx].$$

Specific assumptions on Z_0 are usually not relevant but, whenever needed, P^x, E^x etc. refer to the case $Z_0 = \langle x\rangle$. We throughout consider the supercritical case

$$1 < m = \int_0^\infty x\,dF(x) < \infty$$

and assume as usual that G is non-lattice with $G(0) = 0$. Define $\alpha > 0$ as the (unique) root of

$$m \int_0^\infty e^{-\alpha x}\, dG(x) = 1$$

and let

$$A[dx] = e^{-\alpha x}\,(1-G(x))\,dx \,/ \int_0^\infty e^{-\alpha y}\,(1-G(y))\,dy$$

$$G^x(t) = (G(x+t) - G(x)) \,/\, (1-G(x))$$

$$n_1 = \int_0^\infty y\, e^{-\alpha y}\, dG(y) \,/ \int_0^\infty e^{-\alpha y}\,(1-G(y))\,dy$$

$$V(x) = n_1^{-1} \int_0^\infty e^{-\alpha y}\, dG^x(y) \qquad ^{1)}$$

$$M_t\eta(x) = E^x Z_t[\eta] \,, \qquad \mu M_t[\eta] = \int_0^\infty M_t\eta(x)\,\mu[dx]$$

It is then readily checked that $\{M_t\}_{t\geq 0}$ is a semigroup acting to the right on the set B of bounded Borel-measurable functions η on $[0,\infty[$ and to the left on the set of bounded measures μ on $[0,\infty[$. Furthermore:

5.1 LEMMA A,V *are eigenfunctions of* M_t *corresponding to the eigenvalue* $e^{\alpha t}$, *i.e.*

(5.1) $$A M_t = e^{\alpha t} A \,, \quad M_t V = e^{\alpha t} V .$$

Furthermore for any $\eta \in B$ *such that* $e^{-\alpha x}\,(1-G(x))\eta(x)$ *is directly Riemann integrable* (cf. [11], pg. 361-362)

(5.2) $$\sup_{0\leq x<\infty} \left| e^{-\alpha t} M_t\eta(x) - V(x)\,A[\eta] \right| \to 0 \,, \quad t \to \infty$$

The class of η's satisfying the assumptions for (5.2) is rather extensive and contains e.g. for all $0 \leq a \leq b \leq \infty$ $\eta(x) = I(a \leq x < b)$. Thus (5.2) states that in the mean the population at time t is asymptotically composed like the measure

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1) There is some ambiguity in the literature concerning the normalization of V. The present choice ensures $A[1] = A[V] = 1$, $V(0) = (m\, n_1)^{-1}$.

$e^{\alpha t}$ V(x) A , where x is the age of the ancestor, and for this reason and (5.1), A is usually called the stable age-distribution, V the reproductive value and α the Malthusian parameter, cf. [12].

5.2 REMARK Suppose the ancestor is of age x and let λ be the time of his death. Then from time λ on the process evolves like the sum of N independent processes with ancestors of age zero, N chosen at random according to F. In particular, P^x depends only on x through G^x. This explains somewhat further the role of λ and V, since $V(x) = n_1^{-1} E^x e^{-\alpha\lambda}$.

PROOF OF 5.1. We first prove (5.2). Let for some fixed η satisfying the assumptions $K^x(t) = E^x Z_t[\eta]$, $\tilde{K}(t) = e^{-\alpha t} K^0(t)$, $d\tilde{G}(x) = m e^{-\alpha x} dG(x)$. Appealing to 5.2,

$$K^x(t) = E^x Z_t[\eta]\, I(\lambda > t) + E^x(|Z_\lambda|\, K^0(t-\lambda)\, I(\lambda \leq t)) \tag{5.3}$$

$$= \eta(x+t)\frac{1-G(x+t)}{1-G(x)} + \int_0^t m K^0(t-u)\frac{dG(x+u)}{1-G(x)} .$$

Letting $x = 0$ and multiplying by $e^{-\alpha t}$ gives

$$\tilde{K}(t) = e^{-\alpha t}\eta(t)(1-G(t)) + \int_0^t \tilde{K}(t-u)\, d\tilde{G}(u) .$$

The choice of α ensures that $\tilde{G}$ is a probability measure so that by the renewal theorem

$$\lim_{t\to\infty} \tilde{K}(t) = \frac{\int_0^\infty e^{-\alpha t}\eta(t)(1-G(t))\, dt}{\int_0^\infty t\, d\tilde{G}(t)} = \frac{A[\eta]}{m\, n_1} = V(0)\, A[\eta]$$

Inserting in (5.3), (5.2) follows after some elementary estimates.

(5.1) is an easy consequence of (5.2). For example integrating (5.2) w.r.t. A yields $e^{-\alpha t} A M_t \eta \to A[\eta]$ for all η satisfying the assumptions for (5.2) and therefore by weak continuity for all a.e. [1] continuous η, cf. [8]. It is not

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1) Since A has a density, continuity a.e. on the essential span of F w.r.t. A or w.r.t. Lebesgue measure are the same concept.

difficult to see, that if $\eta \in B$ is continuous, then $M_s\eta$ is a.e. continuous. Therefore

$$A M_s[\eta] = A[M_s\eta] = \lim_{t\to\infty} e^{-\alpha t} A M_t M_s \eta$$

$$= \lim_{t\to\infty} e^{\alpha s} e^{-\alpha(t+s)} A M_{t+s}\eta = e^{\alpha s} A[\eta]$$

and $A M_s = e^{\alpha s} A$ follows. $M_s V = e^{\alpha s} V$ is proved in a similar manner. □

Let $\mathcal{F}_t$ be the σ-algebra containing all relevant information on the process up to time t. From (5.2), we get

$$E(Z_{t+s}[V] \mid \mathcal{F}_t) = Z_t M_s [V] = e^{\alpha s} Z_t [V]$$

and it follows that $\{W_t\}_{t\geq 0}$, where $W_t = e^{-\alpha t} Z_t[V]$, is a non-negative martingale w.r.t. $\{\mathcal{F}_t\}_{t\geq 0}$. Thus $W = \lim_t W_t$ exists a.s. and the main result on the limiting behaviour of the process is the following, the proof of which occupies the rest of this section:

5.3 THEOREM $E^x W = V(x)$, $x \geq 0$, <u>if and only if</u>

(x log x) $$\int_0^\infty x \log x \, dF(x) < \infty$$

<u>while</u> $P^x(W = 0) = 1, \forall x \geq 0$, <u>otherwise. Furthermore, for any</u> $\eta \in B$ <u>continuous a.e.</u>,

(5.4) $$\lim_{t\to\infty} e^{-\alpha t} Z_t[\eta] = W A[\eta]$$

Compared with the Galton-Watson process, the complications occur from the fact that the different lines of descent still evolve independently, but no longer according to the same law. That is, if $Y \geq 0$ is some functional of the process, $P^x(Y > y)$ depends on x. We work here as in [7] with the assumption

(5.5) $$P^x(Y > y) \leq 1 - H(y)$$

where H is some distribution on $[0,\infty[$ independent of x. The reduction to (5.5) follows essentially from 5.4 below. In the proof, we adapt as everywhere in the following without further explanation the convention, that $Y^{t,i}$ denotes the corresponding functional of the line of descent initiated by the i^{th} individual alive at time t.

5.4 LEMMA Let $t > 0$ and let $Y = Y_t$ be the total number of individuals which ever lived up to time t. Then (5.5) holds, where H may be taken with finite mean and satisfying

$$(5.6) \qquad (x \log x) \Rightarrow \int_0^\infty x \log x \, dH(x) < \infty .$$

In the proof, we need

5.5 LEMMA Let $N, U_1, U_2, \ldots$ be independent and non-negative with N integer-valued and $U_1, U_2, \ldots$ i.i.d. and let $S = 1 + U_1 + \ldots + U_N$. Define

$$\log^* x = \begin{cases} x/e & 0 \leq x \leq e \\ \log x & x \geq e \end{cases} ,$$

$\mu = E\, U_1 \log^* U_1$. Then there exist constants $c(\nu) < \infty$, $\nu \geq 0$ (dependent on the distribution of N) such that if $\mu < \infty$, $E\, U_1 \leq \nu$, $E\, N \log^* N < \infty$ then $E\, S \log^* S \leq c(\nu) + \mu\, E\, N$.

PROOF Since log* satisfies the assumptions of 4.1, we have

$$E(S \log^* S \mid N) \leq E(S \mid N) \log^*(E(S \mid N)) + 1 \log^* 1 + N\mu$$

so we have only to let $c(\nu) = E(1 + N\nu) \log^*(1 + N\nu) + e^{-1}$. □

PROOF OF 5.4 Let $N, U_1, U_2, \ldots$ be independent with $P(N \leq x) = F(x)$, $P(U_i \leq u) = P^0(Y \leq u)$ and let $S = 1 + \Sigma_1^N U_i$, $H(y) = P(S \leq y)$. Letting N be the number of children born at time λ we have, appealing to 5.2,

$$P^x(Y_t > y) \leqq P^x(Y_{\lambda+t} > y) \leqq P(S > y) = 1 - H(y)$$

and we have to prove $\int_0^\infty x\,dH(x) < \infty$ and (5.6). We treat only the latter and more complicated case, which obviously is equivalent to $E\,S\log^* S < \infty$, or appealing to 5.5, to $\infty > E\,U_1 \log^* U_1 = E^0\ Y_t \log^* Y_t = \mu(t)$ (say). Let $A_n(t)$ be the event that at most n deaths occur before time t. Obviously,

$$Y_t\ I(A_{n+1}(t)) \leqq 1 + \sum_{i=1}^{|Z_\lambda|} Y_{t-\lambda}^{\lambda,i}\ I(A_n(t)) \tag{5.7}$$

where for convenience $Y_s = 0$, $s < 0$. Define

$$\mu_n(t,\lambda) = E^0(Y_t \log^* Y_t\ I(A_n) \mid \lambda), \quad \mu_n(t) = E^0\ \mu_n(t,\lambda)$$

Letting $\nu = E^0\ Y_T$ where $T > t$ is fixed in 5.5 and using (5.7) gives

$$\mu_{n+1}(t,\lambda) \leqq c(\nu) + m\,\mu_n(t-\lambda)\ I(\lambda \leqq t),$$

$$\mu_{n+1}(t) \leqq c(\nu) + m \int_0^t \mu_n(t-\lambda)\ dG(\lambda) \leqq c(\nu) + mG(t)\ \mu_n(t)$$

If t is so small that $mG(t) < 1$, it therefore follows by iteration that $\mu(t) = \lim \mu_n(t) < \infty$. But if $\mu(t) < \infty$, then the 4.1 applied to the inequality

$$Y_{2t} \leqq Y_t + \sum_{i=1}^{|Z_t|} Y_t^{t,i}$$

shows easily that $\mu(2t) < \infty$ and therefore $\mu(s) < \infty\ \forall s$. □

The following lemma is rather standard and easily proven for example upon integration by parts:

5.6 LEMMA (5.5) <u>implies that for any</u> $x, y \geqq 0$

$$E^x\,Y\,I(Y > y) \leqq \int_y^\infty x\ dH(x) \tag{5.8}$$

$$E^x\ Y^2\ I(Y \leqq y) \leqq \int_0^y x^2\ dH(x) + y(1 - H(y)) \tag{5.9}$$

5.7 LEMMA <u>Define for some functional</u> $Y > 0$ <u>of the process and some fixed</u> $\delta > 0$

$$S_n = e^{-\alpha n\delta} \sum_{i=1}^{|Z_{n\delta}|} Y^{n\delta,i}, \quad \tilde{S}_n = e^{-\alpha n\delta} \sum_{i=1}^{|Z_{n\delta}|} Y^{n\delta,i}\, I(Y^{n\delta,i} \leq e^{\alpha n\delta}),$$

$$\eta(x) = E^x Y, \quad \varepsilon_n(x) = E^x(Y\, I(Y > e^{\alpha n\delta})), \quad T_n = E(S_n \mid \mathcal{F}_n) = e^{-\alpha n\delta}\, Z_n[\eta],$$

$$R_n = e^{-\alpha n\delta}\, Z_{n\delta}[\varepsilon_n] = E(S_n - \tilde{S}_n \mid \mathcal{F}_{n\delta}) = T_n - E(\tilde{S}_n \mid \mathcal{F}_{n\delta})$$

<u>Then</u> (5.5) <u>and</u> $\int_0^\infty x\, dH(x) < \infty$ implies <u>that</u>

$$\sum_{n=0}^{\infty} P(S_n \neq \tilde{S}_n) < \infty, \quad \sum_{n=0}^{\infty} \mathrm{Var}\{\tilde{S}_n - T_n + R_n\} < \infty. \tag{5.10}$$

<u>If furthermore</u> $\int_0^\infty x \log x\, dH(x) < \infty$, <u>then also</u> $\sum_{n=0}^\infty E R_n < \infty$

PROOF One just has to insert (5.5), (5.8), (5.9) in (2.1), (2.2), (2.3). For example,

$$\sum_{n=0}^{\infty} P(S_n \neq \tilde{S}_n) \leq \sum_{n=0}^{\infty} E\Big(\sum_{i=1}^{|Z_{n\delta}|} P(Y^{n\delta,i} > e^{\alpha n\delta} \mid \mathcal{F}_n) \Big) \leq$$

$$\sum_{n=0}^{\infty} E\, |Z_{n\delta}|(1 - H(e^{\alpha n\delta})) = \sum_{n=0}^{\infty} O(e^{\alpha n\delta}) \int_{e^{\alpha n\delta}}^{\infty} dH(x) = \int_0^\infty O(x)\, dH(x). \ \square$$

PROOF OF THE SUFFIENCY OF ($x \log x$). We study the $\{W_t\}_{t\geq 0}$ - martingale along the discrete subsequence $\{W_n\}_{n=0,1,2,...}$. Let $Y = W_1 = e^{-\alpha} Z_1[V]$ and $\delta = 1$ in 5.7. Then $\eta(x) = V(x)$, $S_n = W_{n+1}$, $T_n = W_n$. Writing $\tilde{W}_{n+1} = \tilde{S}_n$, 5.4,5.7 implies

$$\sum_{n=0}^{\infty} \mathrm{Var}\{\tilde{W}_{n+1} - W_n + R_n\} < \infty, \quad \sum_{n=0}^{\infty} E R_n < \infty$$

and thus the L^1-convergence of $\Sigma\{\tilde{W}_{n+1} - W_n + R_n\}$, ΣR_n, $\Sigma\{\tilde{W}_{n+1} - W_n\}$. From this $EW = EW_0$ follows exactly as in § 2. $\square$

Before discussing the problem of the necessity of ($x \log x$), we give the proof of (5.4).

5.8 LEMMA <u>Let</u> $M = \sup_{t \geq 0} e^{-\alpha t} |Z_t|$. <u>Then</u> $M < \infty$.

PROOF (H. Kesten, private communication. See also [3]). Since W exists, it is clear that $\tilde{M} = \sup_{t \geq 0} W_t < \infty$ a.s. If $\inf_{x \geq 0} V(x) = c > 0$, then $|Z_t| \leq c^{-1} Z_t[V]$ and thus $M \leq c^{-1} \tilde{M} < \infty$. In the general case, we always have $V(x) \geq \gamma > 0$ when $0 \leq x \leq 1$. Any individual alive at time t, $n \leq t \leq n+1$, was alive and of age at most 1 at one of the times $0,1,\ldots,n,t$. Thus

$$|Z_t| \leq \sum_{k=0}^{n} Z_k[I_{[0,1]}] + Z_t[I_{[0,1]}] \leq$$

$$\gamma^{-1}\Big(\sum_{k=0}^{n} Z_k[V] + Z_t[V]\Big) \leq \gamma^{-1} \tilde{M}\Big(\sum_{k=0}^{n} e^{\alpha k} + e^{\alpha t}\Big) = \tilde{M}O(e^{\alpha t})$$

and the assertion follows. □

5.9 LEMMA <u>In the notation of</u> 5.7, $\int_0^\infty x\,dH(x) < \infty$ <u>implies that</u> $S_n - T_n \to 0$.

PROOF (5.10) implies that $S_n = \tilde{S}_n$ for n large and that $\tilde{S}_n - T_n + R_n \to 0$ so we only have to prove $R_n \to 0$. But from (5.8)

$$0 \leq R_n \leq M \int_{e^{\alpha n \delta}}^{\infty} y\,dH(y) \to 0. \ \square$$

5.10 LEMMA <u>If</u> $\xi \in B$ <u>satisfies</u> (5.2), <u>then for any</u> $\delta > 0$ a.s.

$$e^{-\alpha n\delta} Z_n[\xi] \to WA[\xi] \tag{5.11}$$

PROOF Let

$$Y_m = e^{-\alpha m\delta} Z_{m\delta}[\eta], \quad \eta_m(x) = E^x Y_m, \quad c_m = \sup_{0 \leq x < \infty} |\eta_m(x) - V(x)A[\xi]|.$$

In the notation of 5.7 we get, using 5.9,

$$\overline{\lim}\, e^{-\alpha n\delta} Z_n[\xi] = \overline{\lim}\, S_n = \overline{\lim}\, T_n \leq$$

$$\lim e^{-\alpha n\delta} \{Z_{n\delta}[V]A[\xi] + Z_{n\delta}[1]\,c_m\} \leq WA[\xi] + Mc_m .$$

As $m \to \infty$, $\overline{\lim} \leqq$ in (5.10) follows. $\overline{\lim} \geqq$ is similar. □

PROOF OF (5.4) We first remark, that (5.11) holds whenever $\eta \in B$ is continuous a.e. This follows since we have weak convergence of the (random) measure $e^{-\alpha t} Z_t$ to WA for all realizations of the process such that (5.11) holds for $\eta(x) = I(0 \leqq x < a)$, $a = \infty$ or a rational, cf. [8]. The same argument shows that it suffices to establish (5.4) for an η of this specific form. Let then for $\delta > 0$

$$\underline{Y}_\delta = \inf_{0 \leqq t \leqq \delta} Z_t[\eta], \quad \overline{Y}_\delta = \sup_{0 \leqq t \leqq \delta} Z_t[\eta], \quad \underline{\xi}_\delta(x) = E^x \underline{Y}_\delta, \quad \overline{\xi}_\delta(x) = E^x \overline{Y}_\delta .$$

Obviously

$$\underline{\xi}_\delta(x) \geqq I(x + \delta < a) \; P^x(\lambda > \delta) = \underline{\xi}^*_\delta(x)$$

$$\overline{\xi}_\delta(x) \leqq I(x < a) \; P^x(\lambda > \delta) + c \; P^x(\lambda \leqq t) = \overline{\xi}^*_\delta(x)$$

(say), using 5.2, 5.4 for the last estimate. Since $\underline{\xi}^*_\delta$, $\overline{\xi}^*_\delta$ are a.e. continuous, we get from 5.9, 5.10

$$\overline{\lim_{t\to\infty}} \; e^{-\alpha t} \; Z_t[\eta] \leqq \overline{\lim_{n\to\infty}} \; e^{-\alpha n\delta} \sum_{i=1}^{|Z_{n\delta}|} \overline{Y}^{n\delta, i} =$$

$$\overline{\lim_{t\to\infty}} \; e^{-\alpha n\delta} \; Z_{n\delta}[\overline{\xi}_\delta] \leqq \overline{\lim_{n\to\infty}} \; e^{-\alpha n\delta} \; Z_{n\delta}[\overline{\xi}^*_\delta] = WA[\overline{\xi}^*_\delta] .$$

If we take the paths right-continuous, then $P^x(\lambda \leqq \delta) \to 0$, $\delta \downarrow 0$, so that $\overline{\xi}^*_\delta \to \eta$. This proves $\overline{\lim} \leqq$ in (5.4) and $\underline{\lim} \geqq$ is similar. □

It remains to prove that $W = 0$ if $(x \log x)$ fails. A short and self-contained proof of this fact is given in [5]. We present here a different proof along the lines of § 2 for the following reasons. First, the basic step, Lemma 5.11 below, seems to us to be a major step in extending the convergence rate results of § 3 to the Bellman-Harris process, though we have not worked out the details. Second our proof uses not specific properties of the process quite as heavily as [5] and might thus be somewhat better suited for generalizations.

5.11 LEMMA <u>There exists a set</u> $B \subseteq [0,\infty[$ <u>of positive</u> A-<u>measure and</u> $c_1 > 0$, $c_2 < \infty$ <u>such that for all</u> $x \in B$, $u > 0$

$$E^x W_1 I(W_1 > u) \geqq c_1 \int_{c_2 u}^{\infty} x\, dF(x) .$$

PROOF Let $\gamma_1, \gamma_2, \ldots$ denote constants with $0 < \gamma_i < \infty$. We choose B such that

$$\inf_{x\in B} P^x(\lambda \leqq 1) = \inf_{x\in B} \frac{G(x+1) - G(x)}{1 - G(x)} = \gamma_1 > 0 \quad . \tag{5.12}$$

For example, choose first $y > 0$ in the support of G and next $z > 0$ such that $y-1 < z < y$ and that both z and z+1 are continuity points of G. Then for x in a suitable open neighbourhood B of z (which has positive A-measure)

$$\frac{G(x+1) - G(x)}{1 - G(x)} \geqq \frac{1}{2} \frac{G(z+1) - G(z)}{1 - G(z)} > 0 \quad .$$

We next remark, that if $S_N = U_1 + \ldots + U_N$ with the U_i i.i.d., $U_i \geqq 0$, $EU_i > 0$, then for some γ_2, γ_3 and all N,u

$$E\, S_N I(S_N > u) \geqq \gamma_2 N I(N > \gamma_3 u) . \tag{5.13}$$

To see this, choose γ_4, γ_5 such that $P(S_{N-1}/N > \gamma_4) \geqq \gamma_5$ for all $N \geqq 2$ and note that a lower bound for the left-hand side of (5.13) is

$$N\, EU_1 P(U_2 + \ldots + U_N > u) \geqq N\, EU_1 I(N > u/\gamma_4)\, \gamma_5 .$$

Now let $N = |Z_\lambda|$, $U_i = \inf_{0\leqq t\leqq 1} W_t^{\lambda,i}$. Then $W_1 \geqq \gamma_6 S_N$ on $\{\lambda \leqq 1\}$ and thus for $x \in B$

$$E^x W_1 I(W_1 > u) \geqq E^x W_1 I(W_1 > u) I(\lambda \leqq 1) \geqq$$

$$\gamma_6 E^x[I(\lambda \leqq 1)\, E(S_N\, I(\gamma_6 S_N > u) \mid N,\lambda)] \geqq$$

$$\gamma_7 E^x[I(\lambda \leqq 1) N I(N > \gamma_8 u)] \geqq \gamma_7 \gamma_1 \int_{\gamma_8 u}^{\infty} x\, dF(x) . \ \square$$

PROOF OF THE NECESSITY OF $(x\log x)$. Choosing $\tilde{W}_{n+1}$, R_n as for the sufficiency, $\Sigma R_n < \infty$ follows from (5.10) exactly as in § 2 . Writing $Z_n = \langle x_1 \ldots x_n \rangle$, we have

$$R_n = e^{-\alpha n} \sum_{i=1}^{|Z_n|} E^{x_i} W_1 I(W_1 > e^{\alpha n}) \geqq e^{-\alpha n} Z_n[I_B]\, c_1 \int_{c_2 e^{\alpha n}}^{\infty} x\, dF(x) \quad .$$

If $(x\log x)$ fails then

$$\sum_{n=0}^{\infty} \int_{c_2 e^{\alpha n}}^{\infty} x\, dF(x) = \infty$$

so that on $\{W > 0\}$ it follows from $e^{-\alpha n} Z_n[I_B] \to W A[I_B] > 0$ that $\Sigma R_n = \infty$ a.s. Thus $P(W > 0) = 0$.□

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