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Abstract.

The distribution of the average noise power from a Poisson stream of vehicles is, properly normalised, shown to converge to a normal distribution although the corresponding stationary process is deterministic. The speed of convergence is estimated. Finally the asymptotic efficiency of a sampling procedure is discussed.

Keywords: central limit theorem, deterministic process, filtered Poisson Process, stationary process, traffic noise.

1. Introduction and Summary.

As a measure of traffic noise from highways one frequently uses, e.g. Kragh and Astrup (1973), the equivalent constant sound level defined as

$$L_{eq} = 10 \log_{10} \frac{1}{T} \int_0^T p^2(t)/p_0^2 \, dt,$$

where $p(t)/p_0$ denotes the ratio between the sound pressure measured and a reference pressure and $[0,T]$ is the time interval of observation. $10^{1/10\ L_{eq}}$ is the average power pr. surface unit in the period of observation.

In this connection one would like to give an estimate of the precision of such measurements. The variation due to the measuring instruments is negligible compared to the wild fluctuations in the actual noise level due to vehicles passing.

To describe the relevant uncertainty we therefore need a probabilistic model for the noise from a stream of vehicles on a highway. Such a model has recently been developed by Weiss (1970), Kurze (1971a), (1971b) and Marcus (1973), (1975). The assumptions of this model in a simple form are

i) vehicles travel independently of each other with a constant velocity v so that the arrivals of vehicles to the point of observation form a Poisson process with constant intensity μ (average number of vehicles pr. time unit);

ii) the relative power emitted by each vehicle is given by independent identically distributed random variables Z_i that are independent of the above Poisson process and have finite second order moments:

$$E\, Z_1^2 = P^2 < +\infty;$$

iii) if the observer is placed in the point (0,a) of the plane and the vehicles travel along the x-axis, then the impact on the observer due to a vehicle located in (x,0) and emitting a relative power Z is given as

$$\frac{Z}{a^2 + x^2} .$$

The assumption of constant velocity gives a one-to-one correspondance between time and position of a vehicle. Let X(t) denote the relative power observed at time t. On the basis of the assumptions one can now derive the following

a) X(t) has a distribution with characteristic function $\phi(s)$ given as

$$\log \phi(s) = 2\mu \int_0^\infty \left[\psi\left(\frac{s}{a^2 + v^2 t^2}\right) - 1\right] dt, \qquad (1)$$

where ψ is the characteristic function of Z, the relative power emitted by a single vehicle. This has been derived by Marcus (1975);

b) the process X(t) is strictly stationary with covariance function given as

$$R(\tau) = \text{Cov}\,(X(t), X(t+\tau)) = \frac{\pi\mu P^2}{2a^3 v} \frac{1}{(1 + \tau^2 (\frac{v}{2a})^2)}$$

and hence spectral density

$$f(\lambda) = \pi^2 P^2 \frac{\mu}{a^2 v^2} \exp(- \frac{2a}{v} |\lambda|). \qquad (2)$$

This result was given by Blumenfeld and Weiss (1975) for the case where Z has a degenerate distribution but the above generalisation is in this connection elementary.

Using the same argument as Marcus (1975) one can derive that the characteristic function γ_T of

$$Y_T = \frac{1}{2T} \int_{-T}^{T} X(t)\, dt$$

is given by

$$\log \gamma_T(s) = 2\mu \int_0^\infty \left[\psi\left(\frac{s}{2T} \int_{t-T}^{t+T} \frac{du}{a^2 + v^2 u^2}\right) - 1\right] dt. \qquad (3)$$

As demonstrated by Marcus (1975), (1) can be greatly simplified by an appropriate choice of ψ. This does not seem to be the case with (3).

Instead one could hope that the averaging procedure would make the distribution of Y_T approximately normal for large T. We have however that

$$\int_{-\infty}^{\infty} \frac{\log f(\lambda)}{1+\lambda^2} d\lambda = \int_{-\infty}^{\infty} \frac{c_1 - c_2 |\lambda|}{1+\lambda^2} d\lambda = -\infty$$

where f is the spectral density of the process. This implies that the process is <u>deterministic</u> (Rozanov (1967)). Thus none of the usual central limit theorems apply, cf. Ibragimov and Linnik (1971).

In the first place it seems rather surprising that a process with so much randomness can be deterministic, but the following heuristic argument for the result can be given: Suppose one has observed X(t) from $-\infty$ and up to, say, s. It is then possible from the signal to identify the position of all vehicles. Since the velocity is constant one can then predict the future position of all vehicles exactly and hence all future values of X(t).

Because of the relatively explicit expression for the characteristic function it is possible to prove asymptotic normality of Y_T directly and this is done in theorem 1.

Since one has the impression that much of the information obtained by continuous recording of the noise is redundant, it is convenient to consider a sampling procedure, i.e.

$$Y_N^* = \frac{1}{Nh} \sum_{j=0}^{N-1} \int_{jk}^{jk+h} X(t)dt, \tag{4}$$

where $k \geq h > 0$.

If h/k is small the computational work involved to get Y_N^* is considerably smaller than that to Y_{Nk}, whereas the precision of Y_N^* is not reduced that drastically.

Theorem 2 states that also Y_N^* has an asymptotic normal distribution and the efficiency problem above is discussed in the final section.

2. Limit Theorems.

Let F_T denote the distribution function of the normalised variable

$$U_T = \frac{Y_T - E\,Y_T}{V(Y_T)^{1/2}} = \frac{Y_T - m_T}{\sigma_T}.$$

Let Y_T have the characteristic function γ_T given by (3).

Let Φ denote the normal distribution function

$$\Phi(x) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du.$$

Then the following result holds:

<u>Theorem 1</u> <u>If</u> $E|Z|^3 < +\infty$ <u>then</u>

$$|F_T(x) - \Phi(x)| = O(T^{-\frac{1}{2}})$$

<u>uniformly in x for</u> $T \to \infty$.

<u>Proof:</u> The method of proof does not depend on the values of μ, a and v and for the sake of simplicity we shall therefore assume that $\mu = v = a = 1$.

The characteristic function for the normalised variable U_T is given by

$$\log G_T(s) = \log \gamma_T\left(\frac{s}{\sigma_T}\right) - i\,\frac{m_T\, s}{\sigma_T}.$$

If we introduce

$$A_T(t) = \frac{1}{2T\,\sigma_T} \int_{t-T}^{t+T} \frac{du}{1+u^2},$$

we get

$$\log G_T(s) = 2\int_0^{\infty} [E\,\exp\{i\,Z\,s\,A_T(t)\} - 1]\,dt - i\,\frac{m_T\, s}{\sigma_T}.$$

Taylor expansion yields

$$\log G_T(s) = -\frac{1}{2} s^2 - i \frac{s^3}{3} \int_0^\infty A_T(t)^3 \, E\, Z^3 \rho(Z)\, dt$$

$$= -\frac{1}{2} s^2 - i \frac{s^3}{3} R(s,t) \ ,$$

where

$$\rho(Z) = \sin(\theta_1 A_T(t)\, s\, Z) - i \cos(\theta_2 A_T(t)\, s\, Z)$$

for some $0 < \theta_1, \theta_2 < 1$, and

$$R(s,T) = \int_0^\infty A_T(t)^3 \, E\, Z^3 \rho(Z)\, dt.$$

$R(s,T)$ can be estimated by

$$|R(s,T)| \leqq \left|\int_0^{2T}\right| + \left|\int_{2T}^\infty\right| = |I_1| + |I_2|,$$

and

$$|I_1| \leq \sqrt{2}\, E|Z|^3 \int_0^{2T} A_T(t)^3 \, dt$$

$$= \frac{\sqrt{2}\, E|Z|^3}{8T^3 \sigma_T^3} \int_0^{2T} \left(\int_{t-T}^{t+T} \frac{1}{1+u^2} \, du \right)^3 dt$$

$$\leqq \frac{\sqrt{2}\, E|Z|^3}{8T^3 \sigma_T^3} \, 2T \, \pi^3 = c_1 \frac{1}{T^2 \sigma_T^3} \ .$$

It can be proved by elementary methods, but it also follows from theorem 18.3.1. p. 331 in Ibragimov and Linnik (1971) that

$$c_2 \, T^{-\frac{1}{2}} \leqq \sigma_T \leqq c_3 \, T^{-\frac{1}{2}} \ ,$$

which implies that

$$|I_1| = O(T^{-\frac{1}{2}}).$$

Further, since for t > 2T we have

$$\int_{t-T}^{t+T} \frac{du}{1+u^2} \leq 2T \frac{1}{1+(t-T)^2} \leq \frac{2T}{(t-T)^2} ,$$

we get

$$|I_2| \leq \sqrt{2}\, E|Z|^3 \int_{2T}^{\infty} \frac{dt}{\sigma_T^3 (t-T)^6} = \frac{\sqrt{2}\, E|Z|^3}{\sigma_T^3\, 5\, T^5} = O(T^{-7/2}).$$

From the smoothing lemma, Feller (1971) p. 538, we then get

$$|F_T(x) - \Phi(x)| \leq \frac{1}{\pi} \int_{-K}^{K} e^{-\frac{s^2}{2}} \left| \frac{1 - e^{i\frac{s^3}{3}R(s,T)}}{s} \right| ds + \frac{24}{\pi K\sqrt{2\pi}} .$$

Taylor expansion yields

$$\left| \frac{1 - e^{-i\frac{s^3}{3}R(s,T)}}{s} \right| \leq \frac{\sqrt{2}}{3} s^2 \, |R(s,T)|,$$

and thus

$$|F_T(x) - \Phi(x)| \leq \frac{\sqrt{2}}{3\pi} (K_1 T^{-1/2} + K_2 T^{-7/2}) \int_{-K}^{K} s^2 e^{-\frac{1}{2}s^2} ds + \frac{24}{\pi K\sqrt{2}}$$

$$\leq K_3 T^{-1/2} + K_4 T^{-7/2} + \frac{24}{\pi K\sqrt{2\pi}} .$$

Since K can be chosen arbitrarily and dependent on T, we have the desired result.□

The characteristic function γ_N^* for Y_N^* defined by (4) can easily be derived as

$$\log \gamma_N^*(s) = \mu \int_{-\infty}^{\infty} \left[\psi\left(\frac{s}{Nh} \sum_{j=0}^{N-1} \int_{t-jk-h}^{t-jk} \frac{du}{a^2 + v^2 u^2} \right) - 1 \right] dt.$$

Let now F_N^* denote the distribution function of the normalised variable U_N^*, where

$$U_N^* = \frac{Y_N^* - E\, Y_N^*}{V(Y_N^*)^{1/2}} = \frac{Y_N^* - m_N^*}{\sigma_N^*} .$$

We have the following

Theorem 2 If $E|Z|^3 < \infty$, then

$$|F_N^*(x) - \Phi(x)| = O(N^{-1/2})$$

uniformly in x for $N \to \infty$.

Proof: The proof is analogous to the preceding one. The crucial step is to get an estimate of the remainder term

$$R^*(s,N) = \int_{-\infty}^{\infty} A_N^*(t)^3 \; E\, Z^3 \rho^*(Z)\, dt \;,$$

where

$$\rho^*(Z) = \sin(\theta_1 A_N^*(t)\, s\, Z) - i \cos(\theta_2 A_N^*(t)\, s\, Z)$$

for some $0 < \theta_1, \theta_2 < 1$, and

$$A_N^*(t) = \frac{1}{\sigma_N^* hN} \sum_{j=0}^{N-1} \int_{t-jk-h}^{t-jk} \frac{du}{1+u^2} \;.$$

We get

$$|R^*(s,N)| \leq \left|\int_{-\infty}^{-2kN}\right| + \left|\int_{-2kN}^{2kN}\right| + \left|\int_{2kN}^{\infty}\right| = |I_1| + |I_2| + |I_3| \;,$$

and

$$|I_1| \leq \sqrt{2}\, E|Z|^3 \int_{-\infty}^{-2kN} |A_N^*(t)|^3\, dt \;.$$

Using that for $t < -2kN$ we have

$$\left|\int_{t-jk-h}^{t-jk} \frac{du}{1+u^2}\right| \leq \int_{t-h}^{t} \frac{du}{1+u^2} \leq \frac{h}{1+t^2} \leq \frac{h}{t^2} \;,$$

we get

$$|I_1| \leq \frac{\sqrt{2}\, E|Z|^3}{\sigma_N^{*3}} \int_{-\infty}^{-2kN} \frac{dt}{t^6} = \frac{\sqrt{2}}{160} \frac{E|Z|^3}{\sigma_N^{*3}\, k^5\, N^5} \;.$$

Again it can be proved directly but it follows from lemma 1 in the next section and theorem 18.2.1. on p. 322 in Ibragimov and Linnik (1971) that there exists constants c and C so that

$$c\, N^{-1/2} < \sigma_N^* < C\, N^{-1/2} .$$

Hence it follows that

$$|I_1| = O(N^{-7/2}) .$$

Further

$$|I_2| \leq \sqrt{2}\, E|Z|^3 \int_{-2kN}^{2kN} |A_N^*(t)|^3\, dt .$$

Using that

$$\left| \sum_{j=0}^{N-1} \int_{t-jk-h}^{t-jk} \frac{du}{1+u^2} \right| \leq \int_{-\infty}^{\infty} \frac{du}{1+u^2} = \pi ,$$

we get

$$|I_2| \leq \sqrt{2}\, \pi^3\, E|Z|^3 \int_{-2kN}^{2kN} \frac{dt}{h^3\, N^3\, \sigma_N^{*3}} = O(N^{-1/2}) .$$

By arguments completely analogous to those used for estimating I_1 we get

$$|I_3| = O(N^{-7/2}) ,$$

and therefore

$$|R^*(s,N)| = O(N^{-1/2}) .$$

The rest of the proof is a word by word repetition of the preceding proof.□

3. Asymptotic efficiency of sampling.

Since we have now proved that both Y_T and Y_N^* are asymptotically normally distributed it is reasonable to compare the asymptotic efficiency of the procedures by means of their second order properties.

It is clear that Y_T and Y_N^* have the same mean so we can concentrate on their asymptotic variances.

If we define for $j = 0,\pm1,\ldots$

$$X_j^* = \frac{1}{h} \int_{(j-1)k}^{(j-1)k+h} X(t)\, dt \qquad (5)$$

we have

$$Y_N^* = \frac{1}{N} \sum_{j=1}^{N} X_j^*$$

The process $(X_j^*,\ j = 0,\pm1,\ldots)$ is a strictly stationary process in discrete time and we shall need its spectral density:

<u>Lemma 1</u> <u>If</u> $X(t)$, $-\infty < t < \infty$ <u>has spectral density</u> f <u>given by</u>

$$f(\lambda) = A\, e^{-B|\lambda|}, \quad -\infty < \lambda < \infty$$

<u>then</u> X_j^*, $j = 0,\pm1,\ldots$ <u>defined by</u> (5) <u>has continuous spectral density</u> f^* <u>given by</u>

$$f^*(\lambda) = \frac{1}{k} \sum_{p=-\infty}^{\infty} g\left(h, \frac{\lambda + 2p\,\pi}{k}\right), \quad -\pi < \lambda < \pi$$

<u>where</u>

$$g(x,y) = A\, e^{-B|y|} \left|\frac{e^{ixy} - 1}{xy}\right|^2 .$$

<u>Proof:</u> We have

$$\mathrm{Cov}(X_1^*, X_{j+1}^*) = \frac{1}{h^2} \int_0^h \int_{jk}^{jk+h} \mathrm{Cov}(X(s),X(t))\, ds\, dt$$

$$= \frac{1}{h^2} \int_0^h \int_{jk}^{jk+h} \int_{-\infty}^{\infty} e^{-i(s-t)\lambda} f(\lambda)\, d\lambda\, ds\, dt$$

$$= \int_{-\infty}^{\infty} f(\lambda)\, e^{-ijk\lambda} \left|\frac{1}{h} \int_0^h e^{it\lambda}\, dt\right|^2 d\lambda$$

$$= \int_{-\infty}^{\infty} e^{-ijk\lambda} f(\lambda) |\phi(h\lambda)|^2\, d\lambda,$$

where ϕ is the characteristic function of the uniform distribution on the unit interval:

$$\phi(s) = \frac{e^{is} - 1}{i\,s} . \tag{6}$$

Define now

$$g(x,y) = f(y)\,|\phi(xy)|^2 , \tag{7}$$

substitute $u = k\lambda$ and we get

$$\mathrm{Cov}(X_1^*, X_{j+1}^*) = \frac{1}{k}\int_{-\infty}^{\infty} e^{-iju}\, g(h,\tfrac{u}{k})\, du$$

$$= \frac{1}{k}\sum_{p=-\infty}^{\infty}\int_{-\pi+2p\pi}^{\pi+2p\pi} e^{-iju}\, g(h,\tfrac{u}{k})\, du$$

$$= \int_{-\pi}^{\pi} e^{-iju}\,\frac{1}{k}\sum_{p=-\infty}^{\infty} g(h,\tfrac{u+2p\pi}{k})\, du .$$

Since we have

$$\left|g(h,\tfrac{u+2p\pi}{k})\right| \leqq f(\tfrac{u+2p\pi}{k}) = A\, e^{-B\left|\frac{u+2p\pi}{k}\right|} ,$$

the series under the integral is absolutely convergent with a continuous limit. This must then be f^* and the lemma follows by inserting (6) into (7).□

We now define the relative asymptotic efficiency of the sampling procedure as the limit of the ratio between the reciprocal variances of resp.

$$Y_N^* = \frac{1}{kN}\sum_{j=0}^{N-1}\int_{jk}^{jk+h} X(t)\, dt$$

and

$$Y_{Nk} = \frac{1}{kN}\sum_{j=0}^{N-1}\int_{jk}^{(j+1)k} X(t)\, dt$$

$$= \frac{1}{kN}\int_{0}^{kN} X(t)\, dt ,$$

i.e.

$$\text{eff}\ (h,k) = \lim_{N\to\infty} \frac{\sigma_{Nk}^2}{\sigma_N^{*2}} .$$

From theorems 18.2.1. and 18.3.1. on resp. p. 322 and 331 in Ibragimov and Linnik (1971) we have

$$\text{eff}\ (h,k) = \lim_{N\to\infty} \left(\frac{2\pi\ f(0)}{Nk} \cdot \frac{N}{2\pi\ f^*(0)} \right) = \frac{f(0)}{k\ f^*(0)} .$$

Inserting the expression for f* from lemma 1 we get

$$\text{eff}\ (h,k) = \left(1 + \frac{k^2}{2\pi^2\ h^2} \sum_{p=1}^{\infty} e^{-\frac{2a}{v}\frac{2p\pi}{k}} \left| \frac{e^{i\frac{2ph\pi}{k}} - 1}{p} \right|^2 \right)^{-1} \tag{8}$$

Formula (8) is suitable for direct numerical computation.

One should notice that the only traffic parameter entering into the expression for the sampling efficiency is 2a/v.

As an example we have tabulated the efficiency for a = 15 m and v = 60 km/h = 16 2/3 m/sec. (table 1).

Table 1

Efficiency of Interval Sampling

2a/v = 1.8 sec

break in seconds (k-h) \ Observation time in seconds (h)	1	5	10	15	20	30
10	0.494	0.580	0.676	0.738	0.780	0.833
30	0.189	0.265	0.357	0.430	0.487	0.574
40	0.144	0.207	0.287	0.354	0.409	0.495
50	0.116	0.170	0.240	0.300	0.352	0.434
60	0.097	0.144	0.206	0.261	0.308	0.387
90	0.065	0.099	0.145	0.187	0.225	0.291
120	0.049	0.075	0.112	0.146	0.177	0.233
180	0.033	0.051	0.076	0.101	0.124	0.167
240	0.025	0.038	0.058	0.077	0.096	0.130

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