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An Extension of a Result of Seneta and Heyde to p-Dimensional Galton-Watson Processes

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AN EXTENSION OF A RESULT OF SENETA AND HEYDE TO p-DIMENSIONAL GALTON-WATSON PROCESSES

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ABSTRACT

It is proven that a supercritical p-dimensional Galton-Watson process can always be normalized to obtain a.e. convergence to a nondegenerate random vector Wv where v is a deterministic vector and W is a scalar random variable with the property that it is a.e. positive on the set of explosion. Additional properties of W are also investigated.

KEY WORDS: p-dimensional Galton-Watson process, supercritical

AMS Classification 60J85.

1. INTRODUCTION

For the one-dimensional supercritical Galton-Watson process $\{Z_n\}$, Seneta and Heyde [5], [2] have proven that it is always possible to find a sequence of constants $\{c_n\}$ such that $c_n \to 0$ and $c_n Z_n$ converges almost surely to a random variable W which is positive on the set of explosion. The purpose of this paper is to prove the p-dimensional ($p \geq 2$) analog of this result. It has recently been brought to our attention that results analogous to ours were announced by Hoppe [7]. We were however, unable to obtain details of his work and so we felt it worth while to present our proofs.

Before stating our results, it is convenient to first give some notation. Let,

X = set of all p-tuples $i = (i_1, i_2, \ldots, i_p)$ whose elements are nonnegative integers

C = p-dimensional cube of points $s = (s_1, s_2, \ldots, s_p)$ such that $0 \leq s_i \leq 1$

$\underset{\sim}{0} = (0,0,\ldots,0)$, $\underset{\sim}{1} = (1,1,\ldots,1)$

$e^{(\alpha)} = (\delta_{1\alpha}, \delta_{2\alpha}, \ldots, \delta_{p\alpha})$, $1 \leq \alpha \leq p$ and $\delta_{\alpha\beta}$ is the usual Kronecker delta function.

$s^i = \prod_{j=1}^{p} s_j^{i_j}$ for $s \in C$ and $i \in X$

For any two elements s, t of either C or X we write

$$\langle s,t \rangle = \sum_{j=1}^{p} s_j t_j$$

$$s \underset{(\geq)}{\overset{(<)}{\leq}} t \quad \text{iff} \quad s_j \underset{(\geq)}{\overset{(<)}{\leq}} t_j \quad 1 \leq j \leq p$$

$$||s|| = \max_{1 \leq j \leq p} |s_j|$$

If A is any $p \times p$ matrix, then sAt is the obvious bilinear form.

Let $\{Z_n = (Z_{n1}, Z_{n2}, \ldots, Z_{np})\}_{n \geq 0}$ be a p-type

Galton-Watson process where a particle of type α, $1 \leq \alpha \leq p$, produces offspring according to $\{p^{(\alpha)}(i)\}_{i\in X}$. If $Z_0 = \underset{\sim}{e}^{(\alpha)}$ w.p.1, we write $\{Z_n^{(\alpha)}\}$, $1 \leq \alpha \leq p$.

As usual, it is more convenient to deal with probability generating functions and so we write

$$f(s) = (f^{(1)}(s), f^{(2)}(s), \ldots, f^{(p)}(s))$$

where

$$f^{(\alpha)}(s) = \sum_{i\in X} s^i p^{(\alpha)}(i) \qquad 1 \leq \alpha \leq p,\ s\in C$$

We denote the n^{th} composition of f with itself by f_n i.e. $f_n(s) = f_{n-1}(f(s))$, $n \geq 1$ ($f_0(s) \equiv s$, $f_1(s) \equiv f(s)$).

Let $m_{\alpha\beta} = \frac{\partial f^{(\alpha)}}{\partial s_\beta}(\underset{\sim}{1})$, $1 \leq \alpha,\beta \leq p$ and denote the matrix $(m_{\alpha\beta})_{1\leq\alpha,\beta\leq p}$ by M. As is customary we will always assume that M satisfies the following condition.

<u>Condition A.</u> The elements of M are all finite and M is nonsingular and positively regular.

(The reader can consult [1, pg. 184] for the appropriate definitions).

When Condition A holds, it is well known [1,pg.185] that M has a maximum eigenvalue ρ which is positive, simple and has associated positive right and left eigenvectors u and v which are normalized so that $\langle u,\underset{\sim}{1}\rangle = 1$ and $\langle u,v\rangle = 1$.

In this paper we assume that $1 < \rho < \infty$. This is what is meant by the process being supercritical. It is well known [1,pg.186] that in this situation

$$P(\lim_{n\to\infty} Z_n = \underset{\sim}{0} | Z_0 = e^{(\alpha)}) + P(\lim_{n\to\infty} \|Z_n\| = \infty | Z_0 = e^{(\alpha)}) = 1$$

$$q^{(\alpha)} = P(\lim_{n\to\infty} Z_n = \underset{\sim}{0} | Z_0 = e^{(\alpha)}) < 1 \qquad 1 \leq \alpha \leq p$$

and

$$f(q) = q \text{ where } q = (q^{(1)}, q^{(2)}, \ldots, q^{(p)}).$$

We now state our results.

Theorem 1. *Let $\{Z_n\}_{n\geq 0}$ be a p-dimensional Galton-Watson process satisfying Condition A and $1 < \rho < \infty$. Then there always exists a sequence of constants $\{c_n\}$ with $c_n \to 0$ and $c_n/c_{n+1} \to \rho$ as $n \to \infty$, such that the sequence of random variables $\{W_n = c_n\langle u,Z_n\rangle\}_{n\geq 0}$ converges a.s. to a finite random variable W having the following properties:*

i) $P(W=0|Z_0=e^{(\alpha)}) = q^{(\alpha)}$ $\qquad 1 \leq \alpha \leq p$

ii) Let $\theta^{(\alpha)}(z) = E(e^{-izW}|Z_0 = e^{(\alpha)})$ $\qquad 1 \leq \alpha \leq p$, z real.

Then

$$\theta(z) = f(\theta(\tfrac{z}{\rho})).$$

iii) *There exist nonnegative measurable functions $w^{(\alpha)}(x)$ such that for $0<a<b<\infty$*

$$P(a<W<b|Z_0=e^{(\alpha)}) = \int_a^b w^{(\alpha)}(x)dx \qquad 1 \leq \alpha \leq p$$

iv) $E(W|Z_0=e^{(\alpha)}) < \infty$ for some $1 \leq \alpha \leq p$ iff

$\sum_{i\in X} i_\beta \log i_\beta \, p^{(\alpha)}(i) < \infty$, $1 \leq \alpha,\beta \leq p$ iff

$c_n \sim \frac{1}{\rho^n}$.

Theorem 2. *Let the assumptions of Theorem 1 hold. Then,*

$$\lim_{n\to\infty} \left|\left|\frac{Z_n}{\langle u,Z_n\rangle} - v\right|\right| = 0 \qquad \text{a.e. on the set of explosion}$$

where v is the left eigenvector of M corresponding to ρ.

Combining Theorems 1 and 2 we obtain:

Theorem 3. *Let the assumptions of Theorem 1 hold. Then there exists a sequence of constants $\{c_n\}$ with $c_n \to 0$ and $c_n/c_{n+1} \to \rho$ such that*

$$\lim_{n\to\infty} c_n Z_n = Wv \qquad \text{w.p.1.}$$

where v is the left eigenvector of M corresponding to ρ *and W is the random variable given in Theorem 1.*

2. PROOFS.

For ease of exposition, the proof of Theorem 1 will be carried out in a series of lemmas.

Lemma 1. *There exists a sequence of vectors* $\{x_n\}_{n\geq 0}$ *such that* $q < x_0 < \underset{\sim}{1}$, *and* $x_n = f(x_{n+1})$, $n \geq 0$.

Proof. Let R_n be the range of f_n, $n \geq 1$, and set $\hat{R} = \bigcap_n R_n$. Since q and $\underset{\sim}{1}$ are fixed points of f, they necessarily belong to $\hat{R}$. Also, since $q < \underset{\sim}{1}$ and each R_n is arcwise connected, there exists an $x_0 \in R$ such that $q < x_0 < \underset{\sim}{1}$, and by our choice of x_0, there exists a sequence of vectors $\{y_n\}$ such that $y_n \in R_{n-1}$ and $x_0 = f(y_n)$. By the Bolzano-Wierstrauss Theorem there exists some point $x_1 \in C, x_1 \neq q$, $x_1 \neq \underset{\sim}{1}$, and a subsequence of the $\{y_n\}$ say $\{y_{n'}\}$ such that $\lim_{n\to\infty} y_{n'} = x_1$. By continuity, $x_0 = f(x_1)$. If we can show that $x_1 \in \hat{R}$, then the proof of the lemma will follow by simple induction. Thus, we need to show $x_1 \in R_k$ for all k. Pick a $k \geq 1$. Since $y_{n'} \in R_k$ for $n' > k$, we can find vectors $w_{n'}$ such that $f_k(w_n') = y_{n'}$, $n' > k$. Since the $\{w_{n'}\}$ has some limit point $w \in C$, we conclude by continuity that $f_k(w) = x_1$. This implies $x_1 \in R_k$ and completes the proof. Q.E.D.

Remark. In the 1-dimensional case, it is not difficult to see that f has an inverse f^{-1} and as a consequence, $\hat{R} = [q,1]$ and $x_n = f_n^{-1}(x_0)$. Naturally in the higher dimensional case, it is not as easy to describe $\hat{R}$, nor to make the evaluation of the $\{x_n\}$.

For $n \geq 0$ define

$$z_n = (z_{n1}, z_{n2}, \ldots, z_{np}) \text{ where } z_{n\alpha} = -\ln x_{n\alpha}, 1 \leq \alpha \leq p$$

$$c_n = \langle v, \underset{\sim}{1} - x_n \rangle$$

$$W_n = \langle z_n, Z_n \rangle, \quad Y_n = e^{-W_n}$$

F_n = σ-algebra generated by $Z_0, Z_1, \ldots, Z_n$

Our next lemma proves some properties of the $\{z_n\}$ and the $\{c_n\}$.

<u>Lemma 2. The following properties are true.</u>

i) $\lim_{n\to\infty} z_n = \underset{\sim}{0}$

ii) $\lim_{n\to\infty} \frac{z_{n\alpha}}{z_{n+1\alpha}} = \rho \qquad 1 \le \alpha \le p.$

iii) $\lim_{n\to\infty} \frac{z_n}{c_n} = u$, where u is the right eigenvector of M corresponding to ρ.

<u>Proof</u>. To prove (i) it is sufficient to show that $\lim_{n\to\infty} x_n = \underset{\sim}{1}$. It follows from the construction of the $\{x_n\}$ that $x_0 = f_n(x_n)$. Suppose now, that say $x_{n1} \not\to 1$. There then exists a subsequence say $\{x_{n'1}\}$ such that $\sup_{n'} x_{n'1} = \delta < 1$. Thus,

$$q^{(1)} < x_{01} = f_{n'}^{(1)}(x_{n'}) \le f_{n'}^{(1)}(\delta, 1, \ldots, 1).$$

But it is well known [1,pg.186] that $f_{n'}^{(1)}(\delta,1,\ldots,1) \to q^{(1)}$ and so $q^{(1)} = x_{01}$ which is a contradiction. To prove (ii) and (iii) we use arguments similar to those in [3]. It is proven in [3] that there exists a family of matrices $\{E(s)\}_{s\in C}$ such that

$$0 \le E(s) \le M, \; t \le s \Rightarrow E(t) \ge E(s)$$

$E(t) \to 0$ componentwise as $t \to \underset{\sim}{1}$ in C, and

$$\underset{\sim}{1} - f(s) = (M - E(s))(\underset{\sim}{1} - s). \; s \in C \tag{1}$$

Consider now $\{z_{n1} = -\log x_{n1}\}$. The arguments for the other $\{z_{ni}\}$ are the same. Since $x_{n1} \to 1$, we can find a sequence $\varepsilon_n \to 0$ such that

$$(1-\varepsilon_n)(1-x_{n1}) \le z_{n1} \le (1+\varepsilon_n)(1-x_{n1}) \tag{2}$$

Using Lemma 1 and (1) we can write $1 - x_{n1}$ as

$$1 - x_{n1} = e^{(1)}\left(\prod_{k=1}^{m} M - E(x_{n+k})\right)\left(\underset{\sim}{1} - x_{n+m}\right) \qquad (3)$$

for any $m \geq 1$.

Let $R = (u_\alpha v_\beta)_{1\leq\alpha,\beta\leq p}$. It follows from the Frobenious Theorem[1,pg.185] that $\lim_{n\to\infty} \rho^{-n}M^n = R$. Thus we can find a sequence $\delta_n \to 0$ such that

$$R(1-\delta_n) \leq \rho^{-n}M^n \leq R(1+\delta_n) \qquad (4)$$

Since $E(t) \to 0$ as $t \to \underset{\sim}{1}$, we can find a sequence $\eta_n \to 0$ such that

$$E(x_n) \leq \eta_n R \qquad (5)$$

Note also that since $\rho^{-1}M\,R = R\,\rho^{-1}M = R$, we have as in [3] for any real arbitrary numbers $\gamma_1,\ldots,\gamma_n$,

$$\prod_{k=1}^{n} (\rho^{-1}M-\gamma_k R) = \rho^{-n}M^n - \{1- \prod_{k=1}^{n} (1-\gamma_k)\}R$$

$$\geq \rho^{-n}M^n - \sum_{k=1}^{n} \gamma_k R \qquad (6)$$

Combining (2) - (6) we obtain

$$(1-\delta_m - \sum_{k=1}^{m} \eta_{n+k})e^{(1)}R(\underset{\sim}{1}-x_{n+m}) \leq \rho^{-m}(1-x_{n1}) \leq$$

$$\leq (1+\delta_m)e^{(1)}R(\underset{\sim}{1}-x_{n+m}) \qquad (7)$$

and so for $m \geq 1$,

$$\frac{(1-\varepsilon_n)(1-\delta_m - \sum_{k=1}^{m} \eta_{n+k})\rho}{(1+\varepsilon_{n+1})(1+\delta_{m-1})} \leq \frac{z_{n1}}{z_{n+11}} \leq \frac{(1+\varepsilon_n)(1+\delta_m)\rho}{(1-\varepsilon_{n+1})\,(1-\delta_{m-1} - \sum_{k=1}^{m-1} \eta_{n+k})} \qquad (8)$$

By letting first n tend to ∞ and then m, we see from (8) that (ii) is true.

To prove (iii) we use (7) to write

$$\frac{(1-\varepsilon_n)(1-\delta_m-\sum_{k=1}^{m}\eta_{n+k})e^{(1)}R(\underset{\sim}{1}-x_{n+m})}{(1+\varepsilon_n)(1+\delta_m)\langle v,R(\underset{\sim}{1}-x_{n+m})\rangle} \leq \frac{z_{n1}}{c_n} \tag{9}$$

$$\leq \frac{(1+\varepsilon_n)(1+\delta_m)e^{(1)}R(\underset{\sim}{1}-x_{n+m})}{(1-\varepsilon_n)(1-\delta_m-\sum_{k=1}^{m}\eta_{n+k})\langle v,R(\underset{\sim}{1}-x_{n+m})\rangle}$$

But because of the normalization $(u,v) = 1$, $\frac{Rw}{\langle v,Rw\rangle} = u$ for any choice of $w \in C$, Thus (iii) follows from (9) by letting first n tend to ∞ and then m. Q.E.D.

As an immediate consequence of Lemma 2 we have

Corollary 1. $\lim_{n\to\infty} c_n = 0$ and $\lim_{n\to\infty} c_n c_{n+1}^{-1} = \rho$

Our next result proves that the $\{Y_n\}$ converge w.p.1.

Lemma 3. *The family* $\{Y_n, \acute{F}_n\}_{n\geq 1}$ *is a positive martingale.* *Hence,*

$$\lim_{n\to\infty} Y_n = Y$$

exists w.p.1, and

$$\lim_{n\to\infty} E(Y_n^r|Z_0=e^{(\alpha)}) = E(Y^r|Z_0=e^{(\alpha)}) \quad \text{for all } r>0$$
$$1 \leq \alpha \leq p.$$

Proof. Since the $\{Y_n\}$ are positive, all we need do is verify the martingale property. Thus for $n \geq 1$,

$$\begin{aligned} E(Y_{n+1}|\acute{F}_n) &= E(e^{-\langle z_{n+1},Z_{n+1}\rangle}|\acute{F}_n) \\ &= \prod_{i=1}^{p} f^{(i)}(e^{-z_{n+11}},\ldots,e^{-z_{n+1p}})^{Z_{ni}} \\ &= \prod_{i=1}^{p} e^{-z_{ni}Z_{ni}} \\ &= Y_n \end{aligned}$$

since $f^{(i)}(e^{-z_{n+11}},\ldots,e^{-z_{n+1p}}) = f^{(i)}(x_{n+1}) = x_{ni} = e^{-z_{ni}}$
$1 \leq i \leq p.$
The convergence of the moments follows by bounded convergence.

Q.E.D.

The next lemma allows us to conclude that $-\log Y$ is finite and not degenerate at 0.

Lemma 4. Let Y be as in Lemma 3. Then

$$P(Y=1|Z_0=e^{(\alpha)}) = q^{(\alpha)}$$

$$P(Y>0|Z_0=e^{(\alpha)}) = 1 \qquad 1 \leq \alpha \leq p.$$

Proof. Essentially the same as Lemma 3. Sec. 10 Ch. 1 of [1] and so omitted.

It follows from Lemma 4 that $\lim_{n\to\infty} W_n = W$ w.p.1. where $P(W<\infty|Z_0=e^{(\alpha)}) = 1$ and $P(W=0|Z_0=e^{(\alpha)}) = q^{(\alpha)}$, $1 \leq \alpha \leq p$. Furthermore in view of (iii) of Lemma 2 we have that $\lim_{n\to\infty} c_n\langle Z_n,u\rangle = W$ w.p.1.

To complete the proof of Theorem 1 it remains only to examine the random variable W. (i) has already been proven. To prove (ii) let $\theta_n^{(\alpha)}(z) = E(e^{-iz\langle Z_n,u\rangle}|Z_0=e^{(\alpha)})$ and $\theta_n(z) = (\theta_n^{(1)}(z),\ldots,\theta_n^{(p)}(z))$.

It is not hard to check that

$$\theta_n(z) = f(\theta_{n-1}(\frac{c_n}{c_{n-1}}z))$$

and so by Corollary 1, we have

$$\theta(z) = f(\theta(\frac{z}{\rho}))$$

This proves (ii). The proof of the absolute continuity of W follows much along the same lines as the one-dimensional case and the reader is referred to Lemmas 7-9 Sec. 10, Ch. 1 of [1] for details. Finally we note that the proof of (iv) follows directly from Khintchin's Theorem and Theorems 1 - 5 in Sec. 6, Ch. 5 of [1]. We leave the reader to check the details. This completes the proof of Theorem 1.

Remark. We note in passing that the second part of (iv) can be proven directly without recourse to the results in [1].

We now turn to the proof of Theorem 2. Our proof is similar in spirit to that in [6]. For definiteness we assum $Z_0 = e^{(1)}$ w.p.1. Following [1,pg.195], we have the following representation for $X_{n+m} = Z_{n+m} / \langle u, Z_{n+m}\rangle$

$$X_{n+m} = \frac{\sum_{j=1}^{p} Z_{nj} e^{(j)} M^m + \sum_{j=1}^{p} \sum_{l=1}^{Z_{nj}} \left(Z_m^{(j)l}(n) - e^{(j)} M^m \right)}{\rho^m \langle u, Z_n\rangle + \sum_{j=1}^{p} \sum_{l=1}^{Z_{nj}} \left(\langle u, Z_m^{(j)l}(n)\rangle - \rho^m u_j \right)} \tag{10}$$

where $Z_{mi}^{(j)l}(n)$ denotes the number of particles of type i in the $(m+n)^{th}$ generation descending from the l^{th} particle of type j in the n^{th} generation. It follows from the definition of the process that the collection of random vectors $\{Z_m^{(j)l}(n);\ l=1,\ldots,Z_n^{(j)},\ j=1,\ldots,p\}$ are conditionally independent when given F_n. Dividing through in (10) by $\rho^m \langle u, Z_n\rangle$ and subtracting v, we obtain

$$X_{n+m} - v = \frac{(X_n M^m \rho^{-m} - v) - r_{n,m} v + \alpha_{n,m}}{1 + r_{n,m}}$$

where

$$r_{n,m} = \frac{\langle \underset{\sim}{1}, Z_n\rangle}{\langle u, Z_n\rangle} \frac{1}{\langle \underset{\sim}{1}, Z_n\rangle} \sum_{j=1}^{p} \sum_{l=1}^{Z_{nj}} \left(\langle u, Z_m^{(j)l}(n)\rangle - \rho^m u_j \right) \rho^{-m}$$

and

$$\alpha_{n,m} = \frac{\langle \underset{\sim}{1}, Z_n\rangle}{\langle u, Z_n\rangle} \frac{1}{\langle \underset{\sim}{1}, Z_n\rangle} \sum_{j=1}^{p} \sum_{l=1}^{Z_{nj}} \left(Z_m^{(j)l}(n) - e^{(j)} M^m \right) \rho^{-m}$$

Let $\varepsilon > 0$. It follows from Lemma 1 [1,pg.194] that there exists an m_0 such that for $m > m_0$

$$\sup_n \| X_{n+m} M^m \rho^{-m} - v \| < \varepsilon \tag{11}$$

Fix $m > m_0$. To prove that $|r_{n,m}|$ and $\|\alpha_{n,m}\|$ converge to zero almost surely on the set of explosion we need the next two lemmas.

Lemma 5. Let $X_1, X_2, \ldots, X_n$ be independent random variables with mean zero such that

$$P(|X_i|>t) \leq \int_t^\infty dQ(x) \text{ for all } t > 0 \text{ and } i=1,\ldots,n$$

where Q is a distribution on $[0,\infty)$ with finite mean. Then for $\delta > 0$.

$$P\left(\left|\frac{1}{n}\sum_{i=1}^n X_i\right|>\delta\right) \leq C\{n\int_n^\infty dQ(x)+\frac{1}{n}\int_0^n x^2 dQ(x)\}$$

where C is a constant that does not depend on n.

For a proof of Lemma 5, the reader is refered to Lemma 1 of [4].

Lemma 6. Let Q be a probability measure on $(0,\infty)$ with finite mean and let $\{m_k\}$ be a sequence of numbers that ultimately increase monotonically to ∞, such that $\lim_{k\to\infty} \frac{m_k}{m_{k-1}} > 1$.

Then

$$\sum_{k=1}^\infty [m_k \int_{m_k}^\infty dQ(x) + \frac{1}{m_k} \int_0^{m_k} x^2 dQ(x)] < \infty$$

Proof. Since all terms are positive, it suffices by Fubini's Theorem to show

$$\sum_{k=1}^\infty m_k I_{[m_k,\infty)}(x) + \sum_{k=1}^\infty \frac{x^2}{m_k} I_{[0,m_k)}(x) = O(x).$$

(For any set A, I_A is the indicator function of the set A).

We will only prove that the first sum is $O(x)$ since the proof for the second is similar. By omitting a finite number of terms we can assume W.L.O.G. that the $m_k \uparrow$. Let k_0 be such that $m_{k_0} \leq x < m_{k_0+1}$.

Then $$\sum_{k=1}^{\infty} m_k I_{(m_k,\infty)}(x) = \sum_{k=1}^{k_0} m_k$$

$$\leq \frac{x}{m_{k_0}} \sum_{k=1}^{k_0} m_k$$

But in view of our assumptions on the $\{m_k\}$ it is not difficult to show that

$$\overline{\lim_{k\to\infty}}\, m_k^{-1} \sum_{j=1}^{k} m_j < \infty. \qquad \underline{\text{Q.E.D.}}$$

We now prove that $\lim_{n\to\infty} |r_{n,m}| = 0$ w.p.1. The proof for $||\alpha_{n,m}||$ is similar and omitted. Let $\{V^{(j)}\}_{j=1,\ldots,p}$ be positive independent random vectors each distributed as $|<u,Z_m^{(j)}>-u_j\rho^m|$, $1 \leq j \leq p$, and define

$$Q(t) = P(\sum_{j=1}^{p} V_1^{(j)} \leq t).$$

It then follows that

$$P(|<u,Z_m^{(j)1}(n)> - \rho^m u_j|>t) \leq \int_t^\infty dQ(x) \tag{12}$$

for $1 = 1,\ldots,Z_n^{(j)}$, $j = 1,\ldots,p$ and $t > 0$.

Furthermore, from Theorem 1 and Lemma 2 we have for every $k \geq 1$.

$$\lim_{n\to\infty} \frac{<\underset{\sim}{1},Z_{nk}>}{<\underset{\sim}{1},Z_{(n-1)k}>} \geq \rho^k \frac{\min u_i}{\max u_i} \tag{13}$$

almost surely on the set of explosion. Choose k so that $\rho^k \frac{\min u_i}{\max u_i} > 1$. It then follows from (12) and (13) that the conditions of Lemmas 5 and 6 are satisfied and so we have for any $\varepsilon > 0$

$$\sum_{n=1}^{\infty} P(|r_{nk,m}|>\varepsilon|\acute{F}_{nk}) < \infty$$

almost surely on the set of explosion. Applying the extended Borel-Cantelli lemma we conclude that

$$\lim_{n\to\infty} |r_{nk,m}| = 0$$

almost surely on the set of explosion.

In a totally analogous way we have for $1 \leq j \leq k$

$$\lim_{n\to\infty} |r_{nk+j,m}| = 0 \tag{14}$$

almost surely on the set of explosion.

The convergence of $|r_{nm}|$ to zero follows directly from (14). This completes the proof of Theorem 2.

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