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Limit Theorems for a Branching Process with Disasters

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LIMIT THEOREMS FOR A BRANCHING PROCESS WITH DISASTERS

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ABSTRACT

A Bellman-Harris process is considered where the population is subjected to disasters which occur at random times. Each particle alive at the time of the disaster, survives it with probability p. In the situation when explosion can occur, several limit theorems are proven. In particular, we prove that the age-distribution converges to the same stable distribution as the Bellman-Harris process and that the population size continues to be asymptotically exponential.

1. INTRODUCTION

In a recent paper [3], the following population model with disasters was formulated. Assume that a population grows according to a Bellman-Harris process. At random times (τ_i), disasters beset the population and each particle alive at the time of the disaster survives with probability p and the survival of any particle is assumed independent of the survival of any other particle. Let Z(t) denote the number of particles surviving at time t.

In [3] some basic facts about this process were established. In particular, necessary and sufficient conditions were proven for extinction. Our aims in this note are to prove some limit theorems when extinction does not occur. A typical result deals with the limiting behavior of the age-distribution of the Z process on the set of explosion. Define

$$Z(x,t) = \text{number of particles alive at time } t \text{ of age} \leq x.$$

and

$$A(x,t) = \frac{Z(x,t)}{Z(t)} \text{ providing } Z(t) > 0.$$

For the Bellman-Harris process without disasters, it has recently been proven [1] that a) A(x,t) converges to a deterministic function A(x) in probability on the set of explosion with assumptions slightly more than finiteness of the mean and b) the convergence holds w.p.1 if the usual 'j log j' condition holds. Our aim here is to prove analogs of (a) and (b) when disasters are present. As a consequence of this result, we are able to show under suitable conditions, that Z(t), properly normalized, converges a.e. on the set of explosion to a nondegenerate limit. This result was obtained in [3] for the special case when the (τ_i) formed a renewal sequence and the population grew as a Markov branching process.

2. NOTATION, RESULTS AND PROOFS.

We denote the growth process by $X(t)$, $t \geq 0$, which as already noted is assumed to be a Bellman-Harris process. Let G be its life time distribution and $f(s) = \sum_{j=0}^{\infty} p_j s^j$ its offspring p.g.f. We always assume G nonlattice, $G(0+) = 0$, $f'(1) = m > 1$ and finite. In this case we can define α to be the solution of the equation

$$m\int_0^\infty e^{-\alpha t}\, dG(t) = 1$$

Without loss of generality we assume that the X process has as its state space the collection of all family histories [see 2, Ch. VI].

Let $\acute{G} = \{g$: g is bounded, measurable and $g(x)(1 - G(x))e^{-\alpha x}$ is directly Riemann integrable$\}$

There is then no difficulty of defining for $0 \leq y < \infty$ and $g \in \acute{G}$,

$$X_y(g,t) = \sum_{i=1}^{X_y(t)} g(x_i)$$

where $x_1, x_2, \ldots, x_{X_y(t)}$ are the ages of the particles alive at time t given that we started with one particle at time zero of age y. One can imagine that to any particle of age a, we assign its 'g-value', $g(a)$. $X_y(g,t)$ is then the sum of the g-values of the particles alive at time t.

We put $\tilde{M}_y(g,t) = E(X_y(g,t))$, $0 \leq y < \infty$, $g \in \acute{G}$. We will always assume that $P(X(0) < \infty) = 1$, and write $M(t) = E(X(t))$.

The (τ_i) process is assumed independent of the X process and satisfies the assumptions:

i) $\tau_1 < \tau_2 < \ldots$ and $\lim_{i\to\infty} \tau_i = \infty$ w.p.1.

ii) there exists a constant λ_0 such that $\lim_{i\to\infty} \frac{\tau_i}{i} = \lambda_0 < \infty$ w.p.1.

Let $N(\cdot)$ be defined by $N(t) = k$ iff $\tau_k \leq t < \tau_{k+1}$. Then,

iii) For any $s > 0$, $P\{\overline{\lim_{t\to\infty}}[N(t+s) - N(t)] < \infty\} = 1$

It is not difficult to check that if the (τ_i) form a renewal sequence with interarrival distribution F satisfying $F(\varepsilon) = 0$ for some $\varepsilon > 0$, then i) ii) and iii) are satisfied. Condition iii) states that disasters cannot happen too frequently.

In this paper we assume that p is constant from disaster to disaster. This is only for convenience. There would be no difficulty in assuming that p is random, providing some kind of ergodicity is required; for example the $\{p_i\}$ could be i.i.d.

Unless otherwise stated all the assumptions made up to now, will always hold.

Let $Z(t)$ denote the number of particles surviving at time t. $Z(t)$ can be expressed as

$$Z(t) = \sum_{i=1}^{X(t)} \delta_i(t) \tag{1}$$

where

$$\delta_i(t) = \begin{cases} 1 & \text{if the ancestors of the } i^{th} \text{ particle (including itself) alive at time } t \text{ in the X process survived all the disasters that occurred up to time } t. \\ 0 & \text{otherwise} \end{cases}$$

Just as for the X process we write for any $g \in \bar{G}$,

$$Z(g,t) = \sum_{i=1}^{Z(t)} g(x_i)$$

where $x_1, x_2, \ldots, x_{Z(t)}$ are the ages of the particles surviving at time t. We also put

$$A(g,t) = \frac{Z(g,t)}{Z(t)} \text{ if } Z(t) \neq 0.$$

$$V(x) = \frac{me^{\alpha x}\int_x^\infty e^{-\alpha y}dG(y)}{1-G(x)}; \quad A(g) = \frac{\int_0^\infty g(y)e^{-\alpha y}(1-G(y))dy}{\int_0^\infty e^{-\alpha y}(1-G(y))dy}$$

$$n_1 = \frac{\int_0^\infty e^{-\alpha y}(1-G(y))dy}{m\int_0^\infty ye^{-\alpha y}dG(y)}$$

The key to the analysis of the Z process is to condition on the σ-field $\acute{F}$ generated by the (τ_i) process. By doing this we can assume that the times of disasters are deterministic and as a consequence distinct particles have independent lines of descent.

To denote this conditional measure we write $P(.|\acute{F})$. The relevant expectations then are those conditioned on $\acute{F}$. Thus we put

$$M(t) = E\{Z(t)|\acute{F}\}$$

$$M(g,t) = E(Z(g,t)|\acute{F}) \quad g \in \acute{G},\ t \geq 0.$$

We will add a subscript (y,s) to all the above random variables and their expectations to indicate when we start at time s with a particle of age y and ask what happens at some future time t+s.

It was observed in [3] that there exists a simple relation between $M(t)$ and $\tilde{M}(t)$. Indeed it is not difficult to show, using the independence assumptions of Section 1 that

$$M(t) = p^{N(t)}\tilde{M}(t) \tag{2}$$

Similarly,

$$M_{(y,s)}(g,t) = p^{N(t+s)-N(s)}\tilde{M}_y(g,t) \tag{3}$$

The following result was proven in [3].

<u>Theorem A.</u>

Let $\rho = \lambda_0 \log p + \alpha$. Then

$$\rho < 0 \Rightarrow P\{\lim_{t\to\infty} Z(t) = 0 | \acute{F}\} = 1 \qquad \text{w.p.1.}$$

$$\rho > 0 \Rightarrow P\{\lim_{t\to\infty} Z(t) = 0 | \acute{F}\} < 1 \qquad \text{w.p.1.}$$

For the remainder of this paper we assume $\rho > 0$.

We now state our main result:

Theorem 1. Assume $\rho > 0$ and $\inf_{y\in \mathrm{supp}G} V(y) > 0$ (supp G = support of G). Then for each $g \in \acute{G}$,

$$\lim_{t\to\infty} P(|A(g,t)-A(g)|>\varepsilon;\ \lim_{t\to\infty} Z(t) = \infty | \acute{F}) = 0 \text{ w.p.1.}$$

for every $\varepsilon > 0$.

It should be noted that the limit $A(g)$ is independent of p.

The assumption that $\inf_{y\in \mathrm{supp}\, G} V(y) > 0$ is a technical one and we conjecture that it is not needed. We note in passing that the condition is satisfied if G has bounded support or if G is negative exponential.

As an immediate corollary of Theorem 1, we have the convergence of the age distribution. Indeed let $A(x,t) = A(I_{[0,x]},t)$ and $A(x) = A(I_{[0,x]})$. We then have:

Corollary 1. Let the conditions of Theorem 1 hold. Then

$$\lim_{t\to\infty} P(|A(x,t)-A(x)|>\varepsilon;\ \lim_{t\to\infty} Z(t)=\infty|\acute{F}) = 0 \qquad \text{w.p.1.}$$

for every $\varepsilon > 0$.

The proof of Theorem 1 is very similar in structure to its counterpart for the Bellman-Harris process, which is given in detail in [1]. So rather than repeating all the details we will only sketch the main ideas. Since we are conditioning on $\acute{F}$, we can assume that disasters occur at the deterministic times τ_i, $i \geq 1$.

By the additive property of branching processes we can write

$$Z(g,t+s) = \sum_{i=1}^{Z(t)} Z_{(x_i,t)}(g,s) \tag{4}$$

where $\{x_i,\ i = 1,\ldots,Z(t)\}$ is the age chart of the Z process at time t and $Z_{(x_i,t)}(g,s)$ denotes the sum of the g-values of the particles surviving at time t+s in the line of descent initiated by a particle of age x_i at time t. Conditioned on the age chart at time t, the $\{Z_{(x_i,t)}(g,s)\}_{i=1}^{Z(t)}$ are independent random variables, and for each i, $Z_{(x_i,t)}(g,s)$ has the same distribution as was defined earlier.

We now rewrite (4) as

$$\frac{e^{-\alpha s}}{Z(t)} Z(g,t+s) =$$

$$\frac{1}{Z(t)} \sum_{i=1}^{Z(t)} [Z_{(x_i,t)}(g,s) - M_{(x_i,t)}(g,s)]e^{-\alpha s}$$

$$+ \frac{1}{Z(t)} \sum_{i=1}^{Z(t)} [M_{(x_i,t)}(g,s)e^{-\alpha s} - n_1 p^{N(t+s)-N(t)} V(x_i)A(g)]$$

$$+ p^{N(t+s)-N(t)} \frac{n_1 A(g)}{Z(t)} \sum_{i=1}^{Z(t)} V(x_i)$$

$$= a_t(g,s) + b_t(g,s) + c_t A(g)$$

Hence,

$$A(g,t+s) = \frac{a_t(g,s) + b_t(g,s) + c_t A(g)}{a_t(1,s)+b_t(1,s)+c_t}$$

The idea now is to show that $b_t(g,s)$ and $b_t(1,s)$ can be made small uniformly with respect to t by choosing s large. This causes no difficulty since by (3)

$$M_{(x_i,t)}(g,s) = p^{N(t+s)-N(t)} \tilde{M}_{x_i}(g,s)$$

Since $g \in \bar{G}$, $\lim_{s\to\infty} M_{x_i}(g,s)e^{-\alpha s} = n_1 V(x_i)A(g)$, and so we write

$$b_t(g,s) \le$$

$$p^{N(t+s)-N(t)}\{\frac{1}{Z(t)} \sum_{i=1}^{Z(t)} (\tilde{M}_{x_i}(g,s)e^{-\alpha s} - n_1 V(x_i)A(g))\}$$

Corollary 1 of [1] can now be applied. We then fix s and show using the law of large numbers that $a_t(g,s)$ and $a_t(1,s)$ go to zero in probability as $t \to \infty$, and that c_t is bounded below in probability. To show $a_t(g,s)$ behaves properly, the arguments of lemma 2 [1] can be used verbatium. To handle c_t, assumption (iii) on the disaster time process is needed, as well as the assumption that $\inf_{y \in \text{supp } G} V(y) > 0$. This is the only part of the proof where either of those assumptions are needed.

Remarks 1). For Theorem 1 to be of interest, we must know when $P\{\lim_{t\to\infty} Z(t) = \infty | \acute{F}\} > 0$ with positive probability. It is proven in [3] that for fairly general (τ_i), (for example if the (τ_i) are a renewal sequence)

$$P\{\lim_{t\to\infty} Z(t) = 0|\acute{F}\} + P\{\lim_{t\to\infty} Z(t) = \infty|\acute{F}\} = 1 \text{ w.p.1.}$$

2). In view of the proof of Theorem 1 it is not surprising that p does not appear in the limit. The only thing that is really relevant is the behavior of the mean functions. As already noted:

$$M(t) = p^{N(t)}\tilde{M}(t)$$

$$M(g,t) = p^{N(t)}\tilde{M}(g,t)$$

Hence

$$\lim_{t\to\infty} \frac{M(g,t)}{M(t)} = \lim_{t\to\infty} \frac{\tilde{M}(g,t)}{\tilde{M}(t)}$$

which is independent of p.

It turns out that Theorem 1 allows us to study the asymptotic behavior of Z(t). Towards this end we prove the next lemma.

Recall $V(x) = \frac{me^{\alpha x}}{1-G(x)} \int_x^\infty e^{-\alpha y} dG(y)$ and set

$$Y(t) = e^{-\beta(t)} Z(V,t)$$

where $\beta(t) = N(t) \log p + \alpha t$. Also let $\acute{F}_t$ denote the σ-algebra generated by $\acute{F}$ and the family history of the Z process up to time t.

<u>Lemma 1. Let $\rho > 0$. Then $\{Y(t), \acute{F}_t\}_{t \geq 0}$ is a positive martingale and hence converges w.p.1.</u> If $f''(1) < \infty$, the limit <u>random variable Y is nondegenerate. In fact,</u>

$$P(Y>0|\acute{F}) > 0 \quad \text{w.p.1.} \tag{5}$$

<u>If in addition the (τ_i) form a renewal sequence</u>, G has a density, and $\inf_{y \in \text{supp}G} V(y) > 0$,

$$P(Y=0|\acute{F}) = P(\lim_{t\to\infty} Z(t) = 0|\acute{F}) \quad \text{w.p.1.} \tag{6}$$

<u>Proof</u>. We first establish the martingale property. Fix $t,s>0$. It follows from (3) and (4) that

$$E(Z(V,t+s)|\acute{F}_t) = E\left(\sum_{i=1}^{Z(t)} Z_{(x_i,t)}(V,s)|\acute{F}_t\right)$$

$$= p^{N(t+s)-N(t)} \sum_{i=1}^{Z(t)} \tilde{M}_{x_i}(V,s) \tag{7}$$

It is well known [2, Ch. IV] that

$$\tilde{M}_x(V,s) = e^{\alpha s} V(x), \quad 0 \leq x < \infty, \; s > 0. \tag{8}$$

From (7) and (8) we conclude that

$$E(Y(t+s)|\acute{F}_t) = Y(t)$$

Thus $(Y(t),\acute{F}_t)_{t \geq 0}$ is a martingale, and since the Y(t) are positive, it follows from the martingale convergence theorem that $\lim_{t\to\infty} Y(t) = Y$ exists w.p.1.

To establish (5) it suffices (by Doob's Theorem) to prove that $\sup_t E(Y^2(t)|\acute{F}) < \infty$ a.e. or equivilently, since $\tau_i \to \infty$ a.e. and are measurable with respect to $\acute{F}$ and $E(Y^2(t)|\acute{F})$ is increasing in t, $\sup_i E(Y^2(\tau_i)|\acute{F}) < \infty$ a.e.. Using the martingale property, it is easy to check that

$$E(Y^2(\tau_{i+1})|\acute{F}) = E(Y^2(\tau_i)|\acute{F}) + E((Y(\tau_{i+1})-Y(\tau_i))^2|\acute{F})$$

But

$$E((Y(\tau_{i+1}) - Y(\tau_i))^2|\acute{F})$$

$$= e^{-2\beta(\tau_{i+1})} E(E((Z(V,\tau_{i+1})-pe^{\alpha(\tau_{i+1}-\tau_i)} Z(V,\tau_i))^2|\acute{F}_{\tau_i})|\acute{F})$$

$$= e^{-2\beta(\tau_{i+1})} E(\sum_{j=1}^{Z(\tau_i)} \mathrm{Var}(Z_{(x_j,\tau_i)}(V,\tau_{i+1}-\tau_i))|\acute{F})$$

The last equality follows since by conditioning on the age chart

$$Z(V,\tau_{i+1}) - pe^{\alpha(\tau_{i+1}-\tau_i)} Z(V,\tau_i)$$

can be considered as a sum of $Z(\tau_i)$ independent components each having mean zero. We also note that when $f''(1) < \infty$,

$$\mathrm{Var}(Z_{(x_j,\tau_i)}(V,\tau_{i+1}-\tau_i)) \le E(X^2_{x_j}(\tau_{i+1}-\tau_i))$$

$$\le Ke^{2\alpha(\tau_{i+1}-\tau_i)}$$

where K is some constant independent of i and j. There then is a constant K' such that

$$E(Y^2(\tau_{i+1})|\acute{F}) \le E(Y^2(\tau_i)|\acute{F}) + K'e^{-\beta(\tau_i)}[e^{-\beta(\tau_i)} E(Z(\tau_i)|\acute{F})]$$

and hence,

$$\overline{\lim_{i\to\infty}}\, E(Y^2(\tau_i)|\acute{F}) \le K'\overline{\lim_{i\to\infty}}\{\sum_{j=1}^{i} [e^{-\beta(\tau_j)} E(Z(\tau_j)|\acute{F})]e^{-\beta(\tau_j)}\}$$

$$< \infty \qquad \text{w.p.1.}$$

The last inequality follows since

$$\overline{\lim_{j\to\infty}}\, e^{-\beta(\tau_j)} E(Z(\tau_j)|\acute{F}) < \infty \qquad \text{w.p.1.}$$

and

$$\sum_{j=1}^{\infty} e^{-\beta(\tau_j)} < \infty \qquad \text{w.p.1.}$$

This completes the proof of (5).

It remains only to prove (6). Define the random variables,

$$\theta_y = P_{(y,0)}(I_{(Y=0)}|\bar{F}),\ 0 \leq y < \infty.$$

(i.e. we start at time zero with one particle of age y). It is convenient to write $\theta_y(\omega)$ for θ_y, thereby indicating the underlying sample point ω, which can be taken to be a realization of the (τ_i) process.

Let $\bar{F}(i)$ denote the σ-algebra generated by $\tau_1,\ldots,\tau_i$. A simple conditioning argument then shows that

$$P(Y=0|\bar{F}(i)) = P(Z(\tau_i)=0|\bar{F}(i))$$

$$+ E(I_{(Z(\tau_i)\neq 0)}\prod_{j=1}^{Z(\tau_i)} \theta_{y_j}(\omega_i)|\bar{F}(i))$$

where $y_1,\ldots,y_{Z(\tau_i)}$ are the ages of the particles alive at time τ_i, and ω_i is a typical sample path of the renewal process $(\tau_{j+i}-\tau_i)_{j\geq 1}$. Since the (τ_i) form a renewal sequence, $\bar{F}(i)$ is independent of the σ-field generated by $(\tau_{j+i}-\tau_i)_{j\geq 1}$.

Hence,

$$E(\prod_{j=1}^{Z(\tau_i)} \theta_{y_j}(\omega_i)|\bar{F}(i)) = \int \prod_{j=1}^{Z(\tau_i)} \theta_{y_j}(\omega)dP(\omega).$$

where P is the measure induced by the (τ_i), which is clearly the same as the measure induced by $(\tau_{j+i}-\tau_i)_{j\geq 1}$. Thus

$$P(Y=0) = P(Z(\tau_i)=0) + E(\int \prod_{j=1}^{Z(\tau_i)} \theta_{y_j}(\omega)dP(\omega); Z(\tau_i) \neq 0)$$

By Fubini's Theorem,

$$E(\int \prod_{j=1}^{Z(\tau_i)} \theta_{y_j}(\omega)dP(\omega);\ Z(\tau_i) \neq 0)$$

$$= \int E(\prod_{j=1}^{Z(\tau_i)} \theta_{y_j}(\omega);\ Z(\tau_i) \neq 0)dP(\omega).$$

Let f denote the density of G. It is not difficult to prove by looking at the time when the initial particle dies that,

$$\theta_y = 1 - {}_0\int^{\infty} p^{N(u)} (1 - A(u))\, dG_y(u)$$

$$= 1 - \frac{{}_0\int^{\infty} p^{N(u)} (1 - A(u)) f(u + y)\, du}{1 - G(y)}$$

where A is independent of y, and $0 \leq A(u) < 1$ for all u.

It follows from Theorem 13.24 [4] that ${}_0\int^{\infty} p^{N(u)} (1 - A(u)) f(u+y)\, du$ is continuous in y and so θ_y is continuous. With this knowledge it is a simple matter to verify that for a.e. ω, $\theta_{\cdot}(\omega) \in \acute{G}$.

It follows then from Theorem 1, and dominated convergence that for a.e. ω

$$\lim_{i\to\infty} E\left(\prod_{i=1}^{Z(\tau_i)} \theta_{y_i}(\omega);\ Z(\tau_i) \neq 0\right) = 0.$$

Thus

$$\lim_{i\to\infty} \int E\left(\prod_{i=1}^{Z(\tau_i)} \theta_{y_i}(\omega);\ Z(\tau_i) \neq 0\right) dP(\omega) = 0$$

and so

$$P(Y=0) = \lim_{i\to\infty} P(Z(\tau_i)=0)$$

$$= P(\lim_{t\to\infty} Z(t)=0)$$

This implies (6) since

$$P(Y=0|\acute{F}) \geq P(\lim_{t\to\infty} Z(t)=0|\acute{F}) \quad \text{w.p.1.} \qquad \text{Q.E.D.}$$

Since $e^{-\beta(t)} Z(t) = Y(t) A(V,t)$, we obtain the next result as a direct consequence of Theorem 1 and Lemma 1.

Theorem 2. Let the conditions of Theorem 1 hold. Then

$$\lim_{t\to\infty} e^{-\beta(t)} Z(t) = W \text{ exists in probability.}$$

If $f''(1) < \infty$, then $P(W>0|\acute{F}) > 0$ w.p.1. If in addition the (τ_i) form a renewal sequence and G has a density

$$P(W=0|\acute{F}) = P(\lim_{t\to\infty} Z(t)=0|\acute{F}) \text{ w.p.1.}$$

We now consider the problem of whether the convergence in Theorems 1 and 2 can be strengthened to hold w.p.1. Toward this goal we say that the stochastic process Z grows exponentially, if for a.e. sample path for which the process explodes there exists constants γ and $\delta > 0$, possibly depending on the sample path such that $Z(t) \geq \gamma e^{\delta t}$ for all t. Since $Y(t) < Z(t)e^{-\beta(t)}$, it follows from Lemma 1 and the properties of N(t) that if $f''(1) < \infty$ and the (τ_i) form a renewal sequence, then the Z process does grow exponentially. If the (τ_i) do not form a renewal sequence, then all we can conclude from Lemma 1 is that the set of sample paths for which $\{Z(t)\}$ does grow exponentially has positive probability.

We then have,

Theorem 3. Let $\rho > 0$, $\inf_{y\in \mathrm{supp}G} V(y) > 0$, $\sum_{j=2}^{\infty} j \log j \, p_j < \infty$, and assume that the Z process grows exponentially. Then for any bounded continuous function g,

$$P(\lim_{t\to\infty} |A(g,t)-A(g)|>\varepsilon;\ \lim_{t\to\infty} Z(t)=\infty|\acute{F}=0) \quad \text{w.p.1.}$$

for every $\varepsilon > 0$.

As an immediate consequence of Theorem 3 and Lemma 1 we have

Theorem 4. Let $\rho > 0$, $\inf_{y\in \mathrm{supp}G} V(y) > 0$, $\sum_{j=2}^{\infty} j \log j \, p_j < \infty$ and assume that the Z process grows exponentially. Then

$$\lim_{t\to\infty} e^{-\beta(t)} Z(t) = W \text{ exists w.p.1.}$$

<u>If $f''(1) < \infty, P(W>0|\acute{F}) > 0$.</u> w.p.1. <u>If in addition the</u> (τ_i) <u>form a renewal sequence and G has a density, then the Z process automatically grows exponentially and</u>

$$P(W=0|\acute{F}) = P(\lim_{t\to\infty} Z(t)=0|\acute{F}) \text{ w.p.1.}$$

The proof of Theorem 3 is similar to that of Theorem B in [1] and the reader is refered there for details.

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