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Preprint
March
1975

5

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STOCHASTIC STABLE POPULATION THEORY

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Preprint 1975 No. 5

INSTITUTE OF MATHEMATICAL STATISTICS

UNIVERSITY OF COPENHAGEN

March 1975

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ABSTRACT

This paper contains a systematic presentation of time-continuous stable population theory in modern probabilistic dress. The life-time births of an individual are represented by an inhomogeneous Poisson process stopped at death, and an aggregate of such processes on the individual level constitutes the population process. Forward and backward renewal relations are established for the first moments of the main functionals of the process and for their densities. Their asymptotic convergence to a stable form is studied, and the stable age distribution is given some attention. It is a distinguishing feature of the present paper that rigorous proofs are given for results usually set up by intuitive reasoning only.

CONTENTS

1. INTRODUCTION

1A. Background. Stable population theory in its classical deterministic formulation with continuous time and age parameters started with a paper by Sharpe and Lotka in 1911, and to this day it continues to form the backbone of a substantial chunk of population mathematics. (For recent reviews, see Keyfitz, 1968; Coale, 1972; Pollard, 1973.) The corresponding theory for the case of discrete time and age parameters got going much later (Bernadelli, 1941; Lewis, 1942; and particularly Leslie, 1945, 1948). It started out as a deterministic theory too, but the second half of the 1960- s saw the birth of a stochastic version of the time-discrete model (Pollard, 1966; Goodman, 1968). Reviews of the ensuing development have been given by Feichtinger (1971) and Pollard (1973).

No similar general reconciliation of deterministic and stochastic theory has been published so far for the time-continuous case, even though the results are latent in a fundamental paper which Kendall published in 1949 already. Crump and Mode (1968, 1969) and Jagers (1969, 1973, 1974) [see also Doney (1972)] have carried out the groundwork for a theory of generalized branching processes, however, and this tool turns out to be just what one needs to establish the basic results of a stochastic stable population theory. The present paper gives an account of such results.

In this presentation, we concentrate on the basic renewal equations of the theory. We establish forward equations, and we show how the classical Lotka integral equation is a backward relation which can be derived rigorously within the stochastic process context. The basic renewal theorem enables us to easily prove standard results concerning convergence of a population to the stable form. We also carry forward the work of Goodman (1967), Pollard (1969) and Keyfitz (1968, Section 8.2) by suggesting some strong analogies between the time-continuous and the time-discrete case.

The present paper clears up concepts and gives rigorous proofs of results which have been established previously, but by intuitive reasoning only. The full stochastic theory is much richer than this, and plans are in hand to present remaining parts in a companion paper. The discussion of "difficult" concepts such as age at childbearing and generation length will be postponed until then.

<u>1B. The flavour of classical stable population theory</u>. Lotka's fundamental integral equation has the following form:

$$B(t) = G^*(t) + \int_0^t B(t-x)p(x)m(x)dx. \tag{1.1}$$

Here, $B(t)$ is called the number of births at time t or the density of births at time t, $G^*(t)$ is "the number of births [at time t] to the initial population", $p(x)$ is the probability of surviving from birth to age x, and $m(x)dx$ is regarded as "the probability of a woman x years of age having a child in the interval x to $x + dx$". (See Preston, 1970, and Keyfitz, 1968, pp. 97-98.)

To give an impression of the flavour of the kind of reasoning we find within the classical, deterministic theory, we quote the argument which Keyfitz (1968, pp. 98-99) uses to establish (1.1):

"... the number of women of ages x to $x + dx$ at time t, born since time zero, will be the survivors of children born x years ago, $B(t-x)p(x)dx$, $x \leq t$. These women would have at time t a number of children equal to

$$B(t-x)p(x)m(x)dx$$

per year.

Integrating [this expression] through all x and adding the allowance $G^*(t)$ for births to those already alive at time zero gives the fundamental [integral] equation".

We regard this kind of reasoning as a commendable way of setting up relations between the various moments, distributions, and densities which appear in a theory of this sort.

Knowledge of the form of a relation makes it easier to find a proof for it. The pseudo-probabilistic argument above cannot be accepted as a rigorous proof, however. A proper proof of (1.1) and other similar relations will be given in this paper.

1C. The probabilistic approach. A population is an aggregate of individuals. In a probabilistic version of stable population theory, therefore, we start out by specifying a stochastic process on the individual level. This process should be regarded as a representation of the aspects which are taken into account of the lifetime reproductive behaviour of the individual. For this purpose, we use a straightforward Poisson process with time-dependent intensity $\{m(x): x \geq 0\}$, stopped at the death of the individual, in the basic model studied in this paper. The Poisson events correspond to births, and we get a birth-and-death model on the individual level. [Apart from what is contained in Kendall's 1949 paper, the seeds of these ideas can be found in papers by Joshi (1954) and Consaël and Lamens (1962). Hoem (1969) seems to be the first one to give this set-up any intensive attention.]

The random process on the population level consists of the aggregate of processes on the individual level. We take the individual processes to be stochastically independent. For an individual of age u at time 0, then, a Poisson process with intensity $\{m(x): x \geq u\}$ generates births until time $L - u$, when the parent individual dies. (The lifetime L is taken as independent of the Poisson process.) Each birth starts off a new and independent Poisson process, which generates new births until it is stopped at death, and so on. The study of this process constitutes our stochastic stable population theory, which covers all of the classical theory and goes far beyond it.

On the basis of this set-up, the present paper is organized as follows. The processes on the individual and the population level are presented in Sections 2 and 3, respectively. Forward and backward renewal relations are presented in

Section 4. In particular, Lotka's integral equation (1.1) is established in Theorem 4.4. Convergence to the stable form is studied in Section 5. In a final Section, we discuss the reproductive value and the stable age distribution, as well as their mathematical interpretation as eigenvectors for mean operators. This brings out a strong analogy with the discrete time theory.

In order to facilitate comparison with the classical theory, we have tried to stay as close as possible to the notation of Keyfitz (1968) and Coale (1972), even though we consider it unfortunate in some respects. We also use standard actuarial notation where possible unless the two sets of conventions conflict, in which case we stick to the demographic tradition.

1D. The possibility of extensions. The individual process of this paper provides a model for lifetime births. To get a similar model for liveborn offspring of either sex, say, one would specify an additional distribution for the number of such offspring in each birth and use a compound Poisson process (Feller, 1971) to represent the behaviour of the individual. Indeed, the specification of a more complex process on the individual level is the key to a generalized (single-sex) stable population theory which takes into account features beyond the mere births, such as birth order, marriage, residence, and so on. Compare Rogers 1966, 1974; Keyfitz, 1968, Chapter 14; Goodman, 1969; Feeney, 1970; Namboodiri, 1970; Le Bras, 1971; Keyfitz, 1973a;Feichtinger, 1974.

On the aggregate level, the tools would be an extension of the generalized branching processes and multivariate renewal theory. The former has not been worked out yet, but it should be possible to do so by known methods. For the latter, see Crump (1970).

The stochastic independence between individuals is a key assumption in this theory. Everything becomes much more difficult if the independence assumption is dropped. This is the reason why it is so hard to develop a satisfactory theory of genuinely two-sex populations. Although a number of authors have contributed to the subject, one still does not really seem to be anywhere as close to a general solution as in the case of a single sex. Recent work in stochastic processes primarily directed towards population genetics, such as that of Kesten (1970, 1971), may inspire further development in this part of demography, provided one can find mating rules which are realistic in human populations.

(For a review of the literature on two-sex demographic models up to 1971, see Keyfitz, 1973b. There are later contributions by McFarland, 1972; Parlett, 1972; Das Gupta, 1972; Bartlett, 1973; and Mode, 1972, 1974.)

2. THE PROCESS OF LIFETIME BIRTHS ON THE INDIVIDUAL LEVEL.

Let $\{K(t): t \geq 0\}$ be a Poisson process with a bounded and continuous intensity function $m(.)$. Let L be a random variable which is independent of $K(.)$ and which has the support $[0,w]$ (for $0 < w \leq \infty$) and the distribution function

$$q(x) = P\{L \leq x\} = 1 - \exp\{-\int_0^x \mu(s)ds\} \tag{2.1}$$

for $0 \leq x \leq w$, where $\mu(.)$ is non-negative and continuous on $[0,w]$. Since $q(w) = 1$, it follows that

$$\int_0^w \mu(s)ds = \infty.$$

Finally, let $x \wedge y = \min(x,y)$, $x \vee y = \max(x,y)$, $p(x) = 1 - q(x)$, ${}_tp_x = p(x+t)/p(x)$, ${}_tq_x = 1 - {}_tp_x$.

We then interpret $m(x)$ and $\mu(x)$ as the forces of fertility and mortality (or, equivalently, the birth and death intensities), respectively, of an individual of age x. We shall call $m(.)$ the <u>gross maternity function</u>, and $\phi(.)$, defined by

$$\phi(x) = m(x)p(x)$$

is known as the <u>net maternity function</u>. The <u>highest age possible</u> is w. We define

$$\Phi(x) = \int_0^x \phi(s)ds = \int_0^x m(s)p(s)ds$$

and

$$M(x) = \int_0^x m(s)ds.$$

The number of births by age x is $K(x \wedge L)$. We note that $M(x) = EK(x)$, and get

$$EK(x \wedge L) = EE\{K(x \wedge L)|L\} = EM(x \wedge L)$$

$$= \int_0^\infty \int_0^{x \wedge y} m(s)ds\, q(dy) = \int_0^x \int_s^\infty q(dy)m(s)ds$$

$$= \int_0^x [1-q(s)]m(s)ds,$$

so that

$$\Phi(x) = EK(x\wedge L). \tag{2.2}$$

Define

$$^{n}\Phi(x) = P\{K(x\wedge L) \geqq n\}. \tag{2.2}$$

Then, immediately,

$$\Phi(x) = \sum_{n=1}^{\infty} {}^{n}\Phi(x). \tag{2.4}$$

The quantity $\bar{R}_0 = \Phi(\infty) = EK(L)$ is called the <u>gross reproduction rate</u>, and, correspondingly, $R_0 = M(\infty) = EK(\infty)$ is called the <u>net reproduction rate</u>.

The following additional observations turn out to be useful. Assume that it is known that $L > u$, where $0 < u < w$, and concentrate on what happens after age u. Let E_u denote the expectation operator, conditional on the event $\{L > u\}$. The expected number of births between ages u and u + t, conditional on $\{L > u\}$, is

$$\Phi_u(t) = \int_0^t {}_sp_u \; m(u+s)ds, \tag{2.5}$$

as is seen by treating $E_u\{K[(u+t)\wedge L] - K(u)\}$ in the same manner as we proved (2.2). We define

$$\phi_u(t) = \frac{\partial}{\partial t}\Phi_u(t) = {}_tp_u \; m(u+t),$$

and see that $\phi_u(\circ)$ is continuous with support $[0,w-u]$. Notice that $\Phi(x) = \Phi_0(x)$ and $\phi(x) = \phi_0(x)$.

Let X(n) be the time of occurrence of the n-th event in the Poisson process K(.). Define

$$^{n}\Phi_u(t) = P\{u<X(n) \leqq (u+t) \wedge L|L > u\}.$$

Then

$$^{n}\Phi_u(t) = P\{K(u\wedge L) < n \leqq K[(u+t) \wedge L]|L > u\} =$$

$$=[^{n}\Phi(u+t) - {}^{n}\Phi(u)]/p(u). \tag{2.6}$$

The latter equality follows from the fact that for $t > 0$, $P\{(L \leqq u) \cap (K(u \wedge L) < n \leqq K[(u+t) \wedge L])\} = 0$. Summation in (2.6) finally gives

$$\Phi_u(t) = \sum_{n=1}^{\infty} {}^n\Phi_u(t), \tag{2.7}$$

of which (2.4) is a special case.

3. THE STOCHASTIC POPULATION PROCESS.

3A. The individuals. We now turn to the process on the population level. This is the age-dependent birth-and-death process introduced by Kendall (1949), and we shall restate its constructive definition in terms of family trees.

To begin with, we shall consider the situation where at time 0, the population consists of a single ancestor, whom we shall refer to as individual $\langle 0\rangle$. (Later on, we shall extend this to a population of ancestors.) Each birth gives rise to a single descendant, and for ease of exposition, we shall regard the original ancestor and all descendants as female. Let us refer to $\langle 0\rangle$'s first descendant after time 0 as individual $\langle 1\rangle$, let us call her second daughter $\langle 2\rangle$, and so on. In turn, the daughters of $\langle k\rangle$ are individuals $\langle k,1\rangle$, $\langle k,2\rangle$, and so on. The i_{n+1}-st daughter of individual $\langle i\rangle = \langle i_1,i_2,\ldots,i_n\rangle$ is called $\langle i_1,i_2,\ldots,i_{n+1}\rangle = \langle i,i_{n+1}\rangle$. Depending on its interpretation, the symbol i may mean a non-negative integer or a sequence of positive integers. We define $\langle 0,i\rangle$ and $\langle i,0\rangle$ as identical to $\langle i\rangle$.

Let $\underset{\sim}{N}$ be the set of positive integers and let $\underset{\sim}{J} = \{0\} \cup \underset{\sim}{N} \cup \underset{\sim}{N}^2 \cup \ldots$ be the set of possible individuals consisting of the original ancestor $\langle 0\rangle$ and all her descendants.

Finally, we let ${}^m\underset{\sim}{J} = \{m,i \mid i\in\underset{\sim}{J}\}$ denote the set of possible individuals in the subfamily generated by $\langle m\rangle$, for m = 1,2,... .

3B. Family histories. To each individual $\langle i\rangle$ there corresponds a life-length L_i and a reproduction measure $K_i(\cdot)$. The former is a positive random variable, and the latter is a nonhomogeneous Poisson process over $[0,\infty)$. Assume that the elements of $\underset{\sim}{J}$ are given in some fixed enumeration, e.g., $\langle 0\rangle$, $\langle 1\rangle$, $\langle 2\rangle$, $\langle 1,1\rangle$, $\langle 3\rangle$, $\langle 2,1\rangle$, $\langle 1,2\rangle$, $\langle 1,1,1\rangle$, ... A family history is then given by the sequence

$$\omega = \begin{pmatrix} L_0 & L_1 & L_2 & L_{1,1} & \cdots \\ K_0(\cdot) & K_1(\cdot) & K_2(\cdot) & K_{1,1}(\cdot) & \cdots \end{pmatrix}.$$

At time 0, the original ancestor will have some age u, which we shall regard as given. We assume that $q(u) < 1$ (compare (2.1)).

For any $i \in \underset{\sim}{J}$, we define $X_i(1)$, $X_i(2)$, ... by $X_i(k) = \inf\{t \geq 0 | K_i(t) \geq k\}$. Thus, $X_i(k)$ is defined as the age of $\langle i \rangle$ at the birth of her k-th daughter. We do not count any births to $\langle 0 \rangle$ before age u. What we register, therefore, is that $\langle 0 \rangle$ gives birth to $\langle K_0(0) + 1 \rangle$, $\langle K_0(0) + 2 \rangle$, ... at ages $X_0(K_0(0) + 1)$, $X_0(K_0(0) + 2)$, ..., until at time $L_0 - u$, $\langle 0 \rangle$ dies at age L_0. Similarly, individual $\langle K_0(0) + n \rangle$ gives birth to individuals $\langle K_0(0) + n, 1 \rangle$, $\langle K_0(0) + n, 2 \rangle$, ... at ages $X_{K_0(0)+n}(1)$, $X_{K_0(0)+n}(2)$, ..., until she dies at age $L_{K_0(0)+n}$ at time $X_0(K_0(0) + n) - u + L_{K_0(0)+n}$, and so on.

The generations $I_0(\omega)$, $I_1(\omega)$, ... are defined recursively as follows:

$$I_0(\omega) = \{\langle 0 \rangle\}, \quad I_1(\omega) = \{\langle i \rangle | u < X_0(i) \leq L_0\},$$

$$I_{k+1}(\omega) = \{\langle i,j \rangle | \langle i \rangle \in I_k(\omega),\ X_i(j) \leq L_i\}.$$

The family $I(\omega) = I_0(\omega) \cup I_1(\omega) \cup \ldots$ then consists of the original ancestor and all her descendants actually born after time 0.

Let $Z_i^a(t)$ be the indicator of the event that $\langle i \rangle \in \underset{\sim}{J}$ is born, alive, and of age not exceeding a at time t. If $\langle i \rangle = \langle i_1, \ldots, i_k \rangle$, then

$$S_i = X_0(i_1) + X_{i_1}(i_2) + \ldots + X_{i_1,i_2,\ldots,i_{k-1}}(i_k) - u$$

is the time at which $\langle i \rangle$ is born (if ever), and $Z_i^a(t) = 1$ if and only if

$$u < X_0(i_1) \leq L_0, X_{i_1}(i_2) \leq L_{i_1}, \ldots, X_{i_1,i_2,\ldots,i_{k-1}}(i_k)$$

$\leq L_{i_1, i_2, \ldots, i_k}$,

$t - a \leq S_i \leq t \leq S_i + L_i$.

3C. The stochastic process. Define $Z_i(t) = Z_i^{\infty}(t)$, and let

$$Z^a(t) = Z^a(t,\omega) = \sum_{i \in I(\omega)} Z_i^a(t).$$

Then $Z^a(t)$ denotes the number of individuals of age not exceeding a alive in the population at time t, and $\{Z^a(t) | 0 \leq a \leq \infty, 0 \leq t < \infty\}$ is defined as a stochastic process by the following assumptions.

Let Ω be the set of family histories ω as defined in Subsection 3B above. Assume that L_0, $K_0(\cdot)$, L_1, $K_1(\cdot)$ ... are independent and that their distributions are given as follows.

(a) The life-lengths L_1, L_2, ... are all distributed as L in Section 2. If $q(u) < 1$, then L_0 has the conditional distribution of L, given that $L > u$, i.e., $P\{L_0 \leq u + t\} = {}_tq_u$.

(b) The reproduction measures $K_1(\cdot)$, $K_2(\cdot)$, ... are all distributed as $K(\cdot)$ in Section 2. The reproduction measure $K_0(\cdot)$ of the original ancestor is distributed as $K(\cdot)$ restricted to $[u,\infty)$.

3D. Some functionals and their first moments. Let $Y_n(t)$ be the number of individuals born before time t in the n-th generation I_n. Utilizing the common convolution notation, we shall prove the following theorem.

Theorem 3.1. $E_uY_n(t) = \Phi_u * \Phi^{*(n-1)}(t)$ for $n \geq 1$.

Proof. The proof is by induction. We note that $E_uY_1(t) = \Phi_u(t)$ by (2.5). Assume that the formula in the theorem holds for $n \leq N$, and let ${}^mY_n(t)$ be the number of individuals in I_n who descend from <m> in I_1. Then by (2.6) and (2.7), and since <m> starts off on independent population with the same structure,

$$E_u Y_{N+1}(t) = \sum_{m=1}^{\infty} E_u {}^m Y_{N+1}(t)$$

$$= \sum_m \int_0^t E_u\{{}^m Y_{N+1}(t) \,|\, \langle m\rangle \text{ was born at time } s\} \, {}^m\Phi_u(ds)$$

$$= \sum_m \int_0^t E_0 Y_N(t-s) {}^m\Phi_u(ds) = \int_0^t E_0 Y_N(t-s)\Phi_u(ds).$$

The theorem then follows from the induction assumption.□

We let $Z^*(t)$ denote the number of individuals who have been members of the population during the period $[0,t]$, i.e., the number of births during $(0,t]$ plus 1 for the original ancestor. Similarly, we let $Z^\dagger(t)$ denote the number of deaths during the same period. The population size at time t is then

$$Z(t) = Z^\infty(t) = Z^*(t) - Z^\dagger(t). \tag{3.1}$$

The rigorous definition of $Z^*(t)$ and $Z^\dagger(t)$ is straightforward as is the proof of the fact that they are both nondecreasing.

The corresponding first moments are

$$N_u^a(t) = E_u Z^a(t), \; N_u^*(t) = E_u Z^*(t), \; N_u^\dagger(t) = E_u Z^\dagger(t),$$

and

$$N_u(t) = N_u^\infty(t) = E_u Z(t).$$

<u>Theorem 3.2</u>. $N_u^*(t) = 1 + \sum_{m=0}^{\infty} \Phi_u * \Phi^{*m}(t).$

<u>Proof</u>. Since $Z^*(t) = 1 + \sum_{m=1}^{\infty} Y_m(t)$, the theorem is a consequence of Theorem 3.1.□

<u>Corollary 1</u>. For $t \geq 0$, $u \geq 0$, and $0 \leq a \leq \infty$, we have

$$N_u^a(t) < \infty, \; N_u^*(t) < \infty, \; N_u^\dagger(t) < \infty,$$

as well as

$$Z^a(t) < \infty, \; Z^*(t) < \infty, \text{ and } Z^\dagger(t) < \infty$$

(a.s., conditional on $L_0 > u$).

Proof. If $\Phi(\infty) = 1$, then $N_0^*(t) < \infty$ is a standard result in renewal theory. A proof is found in Feller (1971, Chapter XI), and it is easily generalized to the case where $\Phi(\infty)$ is arbitrary.

By comparing our process with one started at time $-u$ with a single ancestor aged 0 where all births are counted, we see that $N_u^*(t) \leqq N_0^*(u+t)$. Furthermore, $N_u^a(t) \leqq N_u^*(t)$ and $N_u^\dagger(t) \leqq N_u^*(t)$, which proves the first statement of the corollary. The second statement follows from the finiteness of the expectations. □

Corollary 2. $N_u^*(\cdot)$ is continuously differentiable. If

$$n_u^*(t) = \frac{\partial}{\partial t} N_u^*(t),$$

we see that

$$n_u^*(t) = \sum_{m=0}^{\infty} \phi_u * \phi^{*m}(t)$$

for $0 < u < w$, $0 < t < \infty$.

Remark: In what follows, we shall derive a number of relations between the moment functions defined in this Subsection. For every relation we can prove for $N_u(t)$, there is an exactly corresponding relation for $N_u^*(t)$ which appears if we disregard the mortality of the descendants of the original ancestor. Once this observation has been made, separate proofs for $N_u^*(t)$ are superfluous if proofs are given for $N_u(t)$.

4. THE FORWARD AND BACKWARD RENEWAL EQUATIONS

4A. Forward renewal equations. Let us denote the indicator function of an event A by χ_A.

Theorem 4.1. For $0 \leqq a \leqq \infty$,

$$N_u^a(t) = \chi_{[0,a]}(u+t)\,{}_tp_u + \int_0^t {}_sp_u m(u+s) N_0^a(t-s)\,ds, \tag{4.1}$$

$$N_u^*(t) = 1 + \int_0^t {}_sp_u m(u+s) N_0^*(t-s)\,ds, \tag{4.2}$$

and

$$N_u^\dagger(t) = {}_tq_u + \int_0^t {}_sp_u m(u+s) N_0^\dagger(t-s)\,ds. \tag{4.3}$$

Remark 1. We shall prove the theorem in a minute, but before we do, let us note that these relations have straightforward deterministic interpretations. Let us take (4.1) as an example. It counts the total number of no-more-than-a-year-olds at time t as the expected survival proportion of the original ancestor, provided she has not become over a years old, plus the number of no-more-than-a-year-olds descending from the daughters born to the original ancestor in the period [0,t], including these daughters themselves unless they have reached an age over a at time t. Similarly for (4.2) and (4.3).

Remark 2. The formula in Theorem 3.2 is the standard solution of the renewal equation (4.2). We needed a direct proof of that theorem, however, in order to prove the finiteness of the process.

Proof of Theorem 4.1. It suffices to prove (4.1), for then (4.2) follows from the Remark at the end of Section 3 and (4.3) follows from (3.1).

To prove (4.1), then, first note that

$$Z^a(t) = Z_0^a(t) + \sum_{n=1}^{\infty} {}^nZ^a(t), \text{ where } {}^nZ^a(t) = \sum_{i \in \underset{\sim}{J}_n} Z_i^a(t).$$

(See the end of Subsection 3A for the definition of $\underset{\sim}{J}_n$.) The first right hand term in (4.1) follows from the fact that $E_u Z_0^a(t) = \chi_{[0,a]}(u+t)P\{L_0 \geqq u+t \mid L_0 > u\}$.

Furthermore, by (2.6),

$$E_u {}^n Z^a(t) = \int_u^{u+t} E_u\{{}^n Z^a(t) \mid X_0(n) = v,\ u<X(n)\leqq L_0\}{}^n\Phi_u(dv)$$

$$= \int_u^{u+t} N_0^a(t-v+u){}^n\Phi_u(dv),$$

so that by (2.7),

$$\sum_{n=1}^{\infty} E_u {}^n Z^a(t) = \int_u^{u+t} N_0^a(t-v+u)\Phi_u(dv).$$

From this and (2.5), (4.1) follows.□

<u>Corollary</u>. For each $a \in (0,\infty]$ and $t \in (0,\infty)$, $N_u^a(t)$ is continuous and bounded in u on $[0,w)$, except for a discontinuity of $-{}_t p_{a-t}$ at $a - t$ (if $a > t$).

<u>Proof</u>. The continuity results are immediate from (4.1). To show boundedness, we use (4.1) to get

$$N_u^a(t) \leqq 1 + \int_0^t \hat{m} N_0^a(t-s)ds < \infty$$

for all u, since $\hat{m} = \sup m(s) < \infty$ by assumption.□

<u>4B. Backward renewal equations</u>. At the end of Section 3, we defined a density of births, $n_u^*(t) = \frac{\partial}{\partial t} N_u^*(t)$. Another possibility for such a density is to define it as the density of individuals of age 0 at time t, i.e., as $n_u^0(t) = \frac{\partial}{\partial a} N_u^a(t)\big|_{a=0}$. We shall now prove that the latter density exists and that the two densities are equal.

The existence and continuity of the density

$$n_u^a(t) = \frac{\partial}{\partial a} N_u^a(t) \text{ for } 0 \leqq a < t$$

is proved and a formula for it is given in the following theorem. Note that (4.4) again has a straightforward deterministic interpretation.

Theorem 4.2.

$$N_u^a(t) = \chi_{[0,a]}(u+t)\,{}_tp_u + \int_0^{a\wedge t} n_u^*(t-v)p(v)dv \text{ for } 0 \leq a \leq \infty, \qquad (4.4)$$

and

$$n_u^a(t) = n_u^*(t-a)p(a) \text{ for } 0 \leq a < t. \qquad (4.5)$$

Proof. The first term in (4.4) is $E_u Z_0^a(t)$. Furthermore, if $\underset{\sim}{J}' = \underset{\sim}{J} \setminus \{0\}$,

$$\Delta = N_u^a(t) - E_u Z_0^a(t) = \sum_{i \in \underset{\sim}{J}'} E_u Z_i^a(t).$$

The individual term here equals

$$P\{\langle i\rangle \text{ is ever born, and } 0\vee(t-a) \leq S_i \leq t \leq S_i + L_i \mid L_0 > u\}$$

$$= E_u[\chi_{\{\langle i\rangle \text{ is ever born and } 0\vee(t-a) \leq S_i \leq t\}}$$

$$\cdot P\{t \leq S_i + L_i \mid \langle i\rangle \text{ is ever born, } L_0 > u,\ S_i\}]$$

$$= E_u[\chi_{\{\langle i\rangle \text{ is ever born, and } 0\vee(t-a) \leq S_i \leq t\}}\, p(t-S_i)]$$

$$= \int_{0\vee(t-a)}^{t} p(t-s)F_{i,u}(ds),$$

where $F_{i,u}(s) = P\{\langle i\rangle \text{ is ever born, and } S_i \leq s \mid L_0 > u\}$.

Since, evidently, $\sum_{i \in \underset{\sim}{J}'} F_{i,u}(s) = N_u^*(s) - 1$, we get

$$\Delta = \int_{0\vee(t-a)}^{t} p(t-s)N_u^*(ds),$$

from which (4.4) follows. Then (4.5) follows by differentiation, since $n_u^*(\cdot)$ is continuous by Corollary 2 of Theorem 3.2.□

Corollary 1. $n_u^0(t) = n_u^*(t)$.

Corollary 2. $N_u(t) = {}_tp_u + \int_0^t n_u^*(t-v)p(v)dv$.

Note that Corollary 2 implies the continuous differentiability of $N_u(\cdot)$.

We now turn to $N_u^\dagger(\cdot)$. It is continuously differentiable by the following Corollary. We let

$$n_u^\dagger(t) = \frac{\partial}{\partial t} N_u^\dagger(t).$$

<u>Corollary 3.</u> $N_u^\dagger(t) = {}_tq_u + \int_0^t n_u^*(t-v)q(v)dv.$

<u>Proof.</u> This follows since $N_u^\dagger(t) = N_u^*(t) - N_u(t)$. □

In addition to (4.2), we have the following relation for $N_u^*(t)$.

<u>Theorem 4.3.</u>

$$N_u^*(t) = 1 + \Phi_u(t) + \int_0^t [N_u^*(t-y) - 1]p(y)m(y)dy.$$

<u>Proof.</u> Insert the formula of Theorem 3.2 into this equation. □

This is essentially Lotka's integral equation for the present case. To see this, differentiate with respect to t and get

$$n_u^*(t) = {}_tp_u\, m(u+t) + \int_0^t n_u^*(t-y)p(y)m(y)dy,$$

which has the same type of interpretation as (1.1) has.

<u>4C. Lotka's integral equation.</u> So far, we have studied the case where there is only a single ancestor at time 0. Let us now extend this to the case where there is an arbitrary initial population at time 0, with ages distributed according to some integer valued function $\{Z^a(0): a \geq 0\}$, where $Z^a(0)$ denotes the number of persons of age at most a at time 0. We assume that $Z^\cdot(0)$ is the outcome of a point process which with probability 1 gives a finite initial population, i.e., $Z(0) < \infty$ a.s. We assume that the mean $N^a(0) = EZ^a(0)$ exists, is finite, and is absolutely continuous (as a function of a) for $0 \leq a \leq \infty$. The corresponding initial population density at age a is $n^a(0) = d\,N^a(0)/da$. We assume that $n^a(0) = 0$ for $a \geq w$, and define $N^a(t) = EZ^a(t)$, $N(t) = EZ(t) = N^\infty(t)$, and similarly for the other means. It is easily seen that they satisfy integral equations obtained from those of Theorem 4.1

by integration with respect to $N^{du}(0)$.

In everything which follows in this paper, the above assumptions will be taken to hold.

Theorem 4.4. Given the identification $B(t) = n^*(t)$, Lotka's integral equation holds:

$$B(t) = G^*(t) + \int_0^t B(t-x)p(x)m(x)dx,$$

with

$$G^*(t) = \int_0^\infty n^u(0)\,{}_tp_u m(u+t)du$$

Proof. Multiply the formula in Theorem 4.3 by $n^u(0)$ and integrate with respect to u, to get

$$N^*(t) = N(0) + \int_0^\infty \Phi_u(t)n^u(0)du + \int_0^t [N^*(t-y)-N(0)]p(y)m(y)dy.$$

Then differentiate this with respect to t.□

Remark. To see that $G^*(t)$ is always finite under the stated assumptions, note that, by assumption, ${}_tp_u m(u+t) \leq$ sup $m(s) < \infty$ and $\int n_0^u du = N(0) < \infty$.

Lotka's equation is a relation in terms of the density of births. A similar relation holds for the density of deaths, $D(t) = n^\dagger(t)$.

Theorem 4.5. Suppose that

$$G^\dagger(t) = \int_0^\infty n^u(0)\,{}_tp_u \mu(u+t)du < \infty.$$

Then

$$D(t) = G^\dagger(t) + \int_0^t B(t-x)p(x)\mu(x)dx.$$

Proof. Multiply the relation in Corollary 3 of Theorem 4.2 with $n^u(0)$ and integrate to get

$$N^\dagger(t) = \int_0^\infty n^u(0)\,{}_tq_u du + \int_0^t B(s)q(t-s)ds.$$

Then differentiate with respect to t and reorganize slightly.□

Remark. Since $\{p(t+u)\mu(t+u)\colon u \geq 0\}$ is the upper tail of the density of the life-time distribution, $G^{\dagger}(t) = 0$ for $t \geq w$ if $w < \infty$. For $t < w$, the condition that $G^{\dagger}(t) < \infty$ implies that $n^{u}(0)/p(u)$ should have a finite integral with respect to the tail $[t,\infty)$ of the life-time distribution, as it will if it is bounded, say.

4D. Crude rates. We now define the (deterministic) crude birth and death rates at time t as

$$b(t) = B(t)/N(t) \text{ and } d(t) = D(t)/N(t).$$

We shall prove a formula for each of these, and then need the following result.

Theorem 4.6. $N^{a}(t)$ is absolutely continuous in a, and its density $n^{a}(t)$ satisfies

$$n^{a}(t) = \begin{cases} B(t-a)p(a) & \text{for } 0 \leq a \leq t, \\ n^{a-t}(0)\,{}_{t}p_{a-t} & \text{for } a > t. \end{cases} \tag{4.6}$$

Proof. If $a > t$, $N_0^{a}(t-s) = N_0(t-s)$ for all $s \in [0,t]$. If we multiply (4.1) by $n^{u}(0)$ and integrate with respect to u, we get, therefore,

$$N(t) - N^{a}(t) = \int_{a-t}^{\infty} n^{u}(0)\,{}_{t}p_{u}\,du,$$

which proves the second line of (4.6). The first line follows from Theorem 4.2.□

Theorem 4.7.

$$B(t) = \int_0^{\infty} n^{a}(t)m(a)\,da \text{ and } D(t) = \int_0^{\infty} n^{a}(t)\mu(a)\,da.$$

Proof. By (4.6) and Theorem 4.4,

$$B(t) = \int_t^{\infty} n^{a-t}(0)\,{}_{t}p_{a-t}m(a)\,da + \int_0^{t} n^{a}(t)m(a)\,da,$$

which gives the formula for $B(t)$. The proof of that of $D(t)$ is similar. $\square$

Let us finally introduce the (<u>deterministic</u>) <u>rate of increase of the population at time t</u> as

$$r(t) = b(t) - d(t) = n(t)/N(t),$$

with $n(t) = \frac{d}{dt}N(t)$.

5. ASYMPTOTIC GROWTH

5A. Some basic concepts. We now come to a discussion of the behaviour as $t \to \infty$ of the means and their densities introduced above. A fundamental concept of this theory is the intrinsic growth rate, in that asymptotically, all means and densities grow (or decline) at a speed determined by this rate. It is defined in the following manner.

Let

$$I(\rho) = \int_0^\infty e^{-\rho x} p(x) m(x)\, dx.$$

As we shall see in a moment, the equation

$$I(\rho) = 1 \tag{5.1}$$

has at most a single real solution. If such a solution exists, we call it the intrinsic growth rate (or Malthusian parameter) of the process, and we denote it by r.

In discussing the existence of r, we start by noting that $I(0) = \Phi(\infty) = R_0$, the net reproduction rate. If $R_0 < \infty$, then $I(\rho) \to 0$ as $\rho \to \infty$. Thus, if $1 < R_0 < \infty$, the intrinsic growth rate will exist and be positive. Similarly, if $R_0 = 1$, then $r = 0$. Conversely, if r exists and is positive (zero), then $R_0 > 1$ $(R_0 = 1)$.

If $R_0 = \infty$ or $R_0 < 1$, then r may or may not exist, depending on the form of $\phi(\cdot) = p(\cdot)m(\cdot)$. If $R_0 < 1$ and r exists, then $r < 0$.

In all human and animal populations, $\phi(x) = 0$ for all sufficiently large x. In such a case, we shall say that ϕ eventually vanishes. If ϕ has this property, then $R_0 < \infty$, r exists, and we get

$$R_0 \gtreqless 1 \Leftrightarrow r \gtreqless 0. \tag{5.2}$$

Adopting some branching process terminology we shall call the process supercritical if $R_0 > 0$, critical if $R_0 = 1$, and

<u>subcritical</u> if $R_0 < 1$.

Now assume that r exists, and define

$$h(x) = e^{-rx}p(x), \quad H(x) = \int_0^x h(s)ds,$$

$$\psi(x) = e^{-rx}p(x)m(x), \quad \Psi(x) = \int_0^x \psi(s)ds.$$

Here $H(\infty)$ may be finite or infinite. If $H(\infty) < \infty$, let

$$c(x) = h(x)/H(\infty), \quad C(x) = \int_0^x c(s)ds = H(x)/H(\infty).$$

If it exists, $C(\cdot)$ is a probability distribution function and $c(\cdot)$ its density. The corresponding distribution is called <u>the stable age distribution</u>.

If p eventually vanishes, then $H(\infty) < \infty$. Similarly, if $r > 0$, $H(\infty) < \int_0^\infty e^{-rx}dx = 1/r < \infty$.

If $r = 0$ and $H(\infty) < \infty$, then $C(\cdot)$ is called the <u>stationary age distribution</u>. In this case,

$$H(\infty) = \int_0^\infty p(x)dx = \overset{\circ}{e}_0 = E(L)$$

is the expected lifetime of a new-born.

By the definition of r as the real valued solution of (5.1), ψ is a probability density, and the corresponding distribution is called the distribution of the <u>age at childbearing in the stable population.</u> The mean of this distribution,

$$A = \int_0^\infty xe^{-rx}p(x)m(x)dx,$$

is called <u>the mean age at childbearing in the stable population</u>. There exist cases where $A = \infty$, but A is finite if ϕ eventually vanishes.

Finally, let

$$V(x) = \int_0^\infty e^{-rt}{}_tp_x m(x+t)dt = e^{rx}\{1-\Psi(x)\}/p(x) \quad \text{for } 0 \leq x < w,$$

while $V(x) = 0$ for $x \geqq w$ (when $w < \infty$). We call $V(x)$ the <u>reproductive value</u> of an individual of age x. Note that $V(0) = 1$.

Let $\beta = \sup\{x: m(x) > 0\}$, so that β is the upper limit of the fertile age span. Then $V(x) = 0$ for $x \geqq \beta$. If $\beta < w$, as it is in human populations, $V(\cdot)$ is a continuous function which vanishes for $x \geqq \beta$. In particular, $V(\cdot)$ is bounded in this case.

In the general case, it may happen that $V(\cdot)$ is unbounded and that it is discontinuous at w, though examples of this are rather pathological. A simple sufficient condition for $V(\cdot)$ to be bounded when $r \geqq 0$ is that the expected remaining life-time

$$\overset{o}{e}_x = \int_0^\infty {}_tp_x \, dt$$

stays bounded.

The names of the concepts introduced in the present subsection are motivated by their interpretation in the theory of stable populations. We are not prepared to introduce the concept of a stable population yet, however, and shall postpone it to Subsection 6B. A complete discussion of a concept like the mean age at childbearing in the stable population requires theory beyond that of the present paper. Hoem (1971), Keiding (1973) and Jagers (1973,1974) have contributed to this discussion and we plan to include it in a companion paper.

We introduce these concepts here in spite of this, because we need some of their properties in the following Subsections. What we need, does not rest on a deeper understanding of their interpretation.

<u>5B. The renewal theorem</u>. The convergence theorems which we shall prove, are straightforward corollaries of a basic renewal theorem. For easy reference we shall state this theorem in the form which is most convenient here. We build on the formulation given by Feller (1971, Chapter XI). This formula-

tion uses the concept of direct Riemann integrability. A function f defined on $[0,\infty)$ is called directly Riemann integrable if the generalized Riemann sums formed from a partitioning of the whole half axis $[0,\infty)$ converge appropriately as the partitioning becomes finer. Feller (1966, 1971) gives several sufficient conditions for direct Riemann integrability. In demographic applications, this is no real problem since any Riemann integrable function on $[0,\infty)$ which eventually vanishes, is directly Riemann integrable.

Theorem 5.1. Let F be an absolutely continuous distribution function on $[0,\infty)$ with $F(0) = 0$. Let g and G be real functions vanishing on $(-\infty,0)$, and assume that g is directly Riemann integrable. Let G satisfy the renewal equation

$$G(x) = g(x) + \int_0^x G(x-y)F(dy). \tag{5.3}$$

Then

$$G(x) \to \int_0^\infty g(y)dy/\int_0^\infty yF(dy) \quad \text{as } x \to \infty, \tag{5.4}$$

the limit being interpreted as 0 if the denominator is ∞.

5C. Population growth. We shall call our process (as well as the population whose growth it represents) Malthusian if the intrinsic growth rate r exists, $A < \infty$, $H(\infty) < \infty$, and $V(\cdot)$ is bounded. Any population where $\beta < w < \infty$, as in any human population, will then be Malthusian.

Theorem 5.2. As $t \to \infty$ in a Malthusian process,

$$N_u^a(t)e^{-rt} \to C(a)V(u)H(\infty)/A \tag{5.5}$$

for $0 \leqq u < w$ and $0 \leqq a \leqq \infty$.

Furthermore,

$$N^a(t)e^{-rt} \to C(a)\overline{V}H(\infty)/A \tag{5.6}$$

for $0 \leqq a < \infty$. Finally,

$$N(t)e^{-rt} \to \overline{V}H(\infty)/A \tag{5.7}$$

if either (a) $r \geqq 0$, or (b) $w < \infty$, or (c) $n^u(0)$ eventually vanishes. In (5.6),

$$\overline{V} = \int_0^\infty V(u)n^u(0)du.$$

Proof. 1^0. Let

$$g(t) = \chi_{[0,a]}(t)p(t)e^{-rt} \text{ and } G(t) = N_0^a(t)e^{-rt}.$$

Then g will be directly Riemann integrable, as we see by repeatedly invoking some examples in Feller (1966, p. 349), as follows:

For all $a<\infty$, g is directly Riemann integrable since it vanishes on (a,∞). For $a = \infty$, $g = h$, which is directly Riemann integrable for $r \geqq 0$, since it is then decreasing and Riemann integrable (because $H(\infty) < \infty$). For $a = \infty$ and $r < 0$, we start by noting that $h(x) \to 0$ as $x \to \infty$ since $H(\infty) < \infty$. Thus, $h(\cdot)$ is bounded. It is also nonnegative and continuous. For $n \leqq x \leqq n+1$,

$$e^r h(n+1) \leqq h(x) \leqq e^{-r}h(n).$$

Since

$$H(\infty) = \int_0^\infty h(x)dx \geqq e^r \sum_{n=1}^\infty h(n),$$

the latter sum converges. Let $\mu_n = \max\{h(x): n \leqq x \leqq n+1\}$. Then $\Sigma\mu_n \leqq e^{-r}\Sigma h(n) < \infty$, so g is directly Riemann integrable.

2^0. We now prove (5.5) for $u = 0$. Rewrite (4.1) with $u = 0$ as

$$G(t) = g(t) + \int_0^t G(t-s)\Psi(ds).$$

Since $\psi(\infty) = 1$, we may use Theorem 5.1 to get $N_0^a(t)e^{-rt} \to H(a)/A$ as $t \to \infty$.

3^0. We then prove (5.5) for a general u. By (4.1),

$$N_u^a(t)e^{-rt} = \chi_{[0,a]}(u+t)\,{}_tp_u e^{-rt} + \int_0^t N_0^a(t-s)e^{-r(t-s)} \cdot {}_sp_u m(u+s)e^{-rs}ds. \tag{5.8}$$

Now let $t \to \infty$. Since $G(t)$ is bounded and converges, we get the limiting value of the integral to be $H(a)V(u)/A$ by dominated convergence. Call the first right hand side element of (5.8) $g(a,u,t)$. For $a < \infty$, $g(a,u,.)$ eventually vanishes. Finally, $g(a,u,t) \to 0$ as $t \to \infty$, because $H(\infty) < \infty$. Thus, (5.5) has been proved.

4^0. We now prove that for each $a \in [0,\infty]$, the integral in (5.8) is bounded by some constant independent of t and u. Since the integral is non-decreasing as $a \to \infty$, it suffices to take $a = \infty$. By (4.4), $N_0(x)e^{-rx}$ is continuous, and by 2^0 it converges to the finite constant $H(\infty)/A$ as $x \to \infty$. Thus it is bounded above by some constant K, which implies that the integral in (5.8) is bounded by $KV(u)$, which is bounded itself by assumption.

5^0. We note that

$$N^a(t)e^{-rt} = \int_0^\infty N_u^a(t)\ e^{-rt} n^u(0)du,$$

and intend to prove the rest of the theorem by dominated convergence. By 4^0, it remains to discuss the conditions under which $g(a,u,t)$ is bounded by some constant $K(a)$ independent of (u,t).

For $a < \infty$, $g(a,u,t) = 0$ at least for $t > a$, and we can take $K(a) = \max\{1,e^{-ra}\}$. This proves (5.6). Now let $a = \infty$.

If $r \geq 0$, then $g(\infty,u,t) \leq 1$, which proves (5.7) under condition (a) of the Theorem.

If $w < \infty$, then $g(\infty,u,t) = 0$ at least for $t \geq w$, so we can take $K(\infty) = \max\{1,e^{-rw}\}$, which proves (5.7) under condition (b).

If $r < 0$ and $w = \infty$ but $n^u(0) = 0$ for $u \geq w'$, we can take

$K(\infty) = \max\{1, e^{-rw'}\}$, as it suffices to bound $g(\infty,u,t)$ for $u \in [0,w']$. This proves (5.7) under (c).

The problem with the case where $r < 0$ and $w = \infty$ is that it may happen that

$$\lim_{u\to\infty} \sup_{t\to\infty} {}_t p_u e^{-rt} = \infty,$$

and condition (c) makes this unimportant.□

In a Malthusian process, therefore,

$$N(t) \sim e^{rt}\overline{V} \frac{\int_0^\infty e^{-rx} p(x)\,dx}{\int_0^\infty x e^{-rx} p(x) m(x)\,dx},$$

and

$$N^a(t)/N(t) \to C(a),$$

under either of assumptions (a) to (c) in Theorem 5.2. In this sense, the asymptotic age distribution is $C(\cdot)$. In $\overline{V}$, an initial individual of age u enters with weight $V(u)$. The latter quantity measures the number of children to be born to this member of the initial population, discounted at the rate of interest r (Fisher, 1958, p.27).

5D. The sequences of births and deaths. Starting with (1.1), classical deterministic stable population theorists argue that $B(t)$ will ultimately grow as e^{rt}. (Compare, e.g., Keyfitz, 1968, p.103, or Coale, 1972, pp. 64-65.) We shall prove this under the condition that $\{n^u(0): u \geq 0\}$ is directly Riemann integrable, as it will be at least whenever $w < \infty$, since $n^u(0) = 0$ for $u \geq w$.

Theorem 5.3. In a Malthusian process where $n^u(0)$ is directly Riemann integrable,

$$B(t)e^{-rt} \to \overline{V}/A \text{ as } t \to \infty.$$

Proof. Let $G^*(t)$ be defined as in Theorem 4.4, and let

$g(t) = e^{-rt}G^*(t)$, $G(t) = e^{-rt}B(t)$. By Theorem 4.4,

$$G(t) = g(t) + \int_0^t G(t-x)\Psi(dx).$$

We note that $\int_0^\infty g(t)dt = \int_0^\infty V(u)n^u(0)du$, so g is directly Riemann integrable. (Remember that $V(\cdot)$ is bounded in a Malthusian process.) Then invoke Theorem 5.1.□

Theorem 5.4. In a Malthusian process with finite maximal life-length w,

$$D(t)e^{-rt} \to \overline{V}\,\frac{\int_0^\infty e^{-rx}p(x)\mu(x)dx}{\int_0^\infty xe^{-rx}p(x)m(x)dx} = \overline{V}\,\frac{1-rH(\infty)}{A}.$$

Proof. 1^0. Use partial integration to prove that the integral in the numerator equals $1 - rH(\infty)$ and is therefore finite.

2^0. From Theorem 4.5, we get

$$D(t)e^{-rt} = \int_0^w \frac{n^u(0)e^{ru}}{p(u)}p(u+t)\mu(u+t)e^{-r(u+t)}du + \int_0^t B(t-x)e^{-r(t-x)}p(x)\mu(x)e^{-rx}dx$$

The first term here is 0 for $t > w$. By Corollary 2 to Theorem 3.2, $B(\cdot)$ is continuous. Since $n^u(0)$ vanishes for $u > w$, the assumption of Theorem 5.3 holds. Thus, $B(t)e^{-rt}$ is bounded by a constant. The present theorem then follows by 1^0 and dominated convergence.□

Corollary. In a Malthusian process with finite maximal life-length w,

$$n(t)e^{-rt} \to r\,\overline{V}\,H(\infty)/A$$

as $t \to \infty$. (Recall that $n(t) = dN(t)/dt$ by definition.)

Proof. $n(t) = B(t) - D(t)$.□

<u>Remark</u>. The convergence in Theorem 5.4 and its Corollary may be shown to hold for $w = \infty$ also, but only under various conditions upon which we shall not elaborate. Similarly in Theorem 5.5 below.

<u>5E. The crude rates</u>. We shall close this Section with a Theorem on the convergence of the crude rates introduced in Subsection 4D.

<u>Theorem 5.5</u>. In a Malthusian process where $w < \infty$, as $t \to \infty$,

$$b(t) \to 1/\int_0^\infty e^{-rx}p(x)dx = 1/H(\infty), \tag{5.9}$$

$$d(t) \to \int_0^\infty e^{-rx}p(x)\mu(x)dx/\int_0^\infty e^{-rx}p(x)dx = \frac{1}{H(\infty)} - r, \tag{5.10}$$

and

$$r(t) \to r. \tag{5.11}$$

<u>Proof</u>. These relations follow from Theorems 5.2, 5.3 and 5.4.□

6. STABLE POPULATIONS. THE OPERATOR SEMIGROUP.

6A. Time decomposition of $N_u^a(s+t)$. The present section is devoted to a study of some properties of stable populations. It turns out to be useful to decompose $N_u^a(s+t)$ into a contribution from the period $[0,s]$ and a subsequent contribution from the period $(s,s+t]$ before we embark on our account proper, so we shall first prove the following theorem, which holds for any population of the kind studied in this paper.

Theorem 6.1. $N_u^a(s+t) = \int_0^\infty N_u^{dx}(s)N_x^a(t)$.

Proof. 1^0. The proof goes as follows: We first prove that

$$E_u\{Z^a(t+s)\,|\,(Z^b(s):0\leq b\leq\infty)\} = \int_0^\infty Z^{dx}(s)N_x^a(t). \tag{6.1}$$

(See 2^0 for the definition of this integral.) This implies that

$$N_u^a(s+t) = E_u\int_0^\infty Z^{dx}(s)N_x^a(t).$$

We then note that $E_uZ^x(s) = N_u^x(s)$, so that once (6.1) is established, it remains to prove that

$$E_u\int_0^\infty Z^{dx}(s)N_x^a(t) = \int_0^\infty N_u^{dx}(s)N_x^a(t).$$

To show this, we prove that for any bounded function f,

$$E_u\int_0^\infty Z^{dx}(s)f(x) = \int_0^\infty N_u^{dx}(s)f(x). \tag{6.2}$$

Thus, once (6.1) and (6.2) have been established, our proof is complete.

2^0. To prove (6.1), let the ages of the almost surely finite number $Z(s)$ of individuals in the population at time s be $A_1,A_2,\ldots,A_{Z(s)}$. The right hand side in (6.1) equals

$$\sum_{i=1}^{Z(s)} N_{A_i}^a(t),$$

the sum being empty if $Z(s) = 0$. Since each individual alive and of age A at time s gives rise to an expected number of $N_A^a(t)$ live individuals of age at most a at time s+t, (6.1) follows from the fact that the conditioning process $\{Z^b(s): 0 \leq b \leq \infty\}$ determines the A_i completely.

3^0. To prove (6.2), check first that it holds (in that both sides equal $N_u^a(s)$) if $f(x) = \chi_{[0,a]}(x)$. The relation then holds for all step functions by linearity and for general bounded f by dominated convergence.□

6B. Stable populations. In Subsection 5C we proved that in a Malthusian process where $w < \infty$, say, $N^a(t)/N(t) \to C(a)$ as $t \to \infty$, and in Subsection 5A we called $C(\cdot)$ the stable age distribution. One would expect $N^a(t)/N(t)$ to be equal to $C(a)$ for all a and t if the initial age distribution were $C(\cdot)$, so that in this sense the age distribution would be stable. This turns out to be true, as will appear in a Corollary to Theorem 6.2 below. Let us agree, therefore, to call a Malthusian process stable if the initial population density is proportional to the density $c(\cdot)$, i.e., if

$$n^a(0) = kc(a) \text{ for all } a, \tag{6.3}$$

for some $k > 0$. Similarly, we shall call the population itself stable.

Theorem 6.2. In a Malthusian process, we have that

$$\int_0^\infty c(x)N_x^a(t)dx = e^{rt}C(a) \text{ for } 0 \leq a \leq \infty. \tag{6.4}$$

(Note that we do not assume (6.3) here.)

Proof. Use (5.5) to conclude that

$$N_0^a(s+t)/N_0(s) \to e^{rt}C(a) \text{ as } s \to \infty.$$

Let $u = 0$ in Theorem 6.1 and divide by $N_0(s)$ to get

$$N_0^a(s+t)/N_0(s) = \int_0^\infty N_x^a(t)N_0^{dx}(s)/N_0(s).$$

As $s \to \infty$, the distribution function $F_s(x) = N_0^x(s)/N_0(s)$ converges to the continuous distribution function $C(x)$ for all x. The theorem therefore follows from the continuity properties of $N^a_\cdot(t)$, proved in the Corollary of Theorem 4.1, and a standard result on weak convergence. (See, e.g., Billingsley, 1968, pp. 11-12.)□

Corollary. In a stable Malthusian process the following relations hold for all $t \geq 0$:

$$N^a(t) = e^{rt}N^a(0) \text{ and } N^a(t)/N(t) = C(a) \text{ for } 0 \leq a \leq \infty,$$

$$B(t) = e^{rt}B(0),\ D(t) = e^{rt}D(0),$$

$$b(t) = b(0),\ d(t) = d(0), \text{ and } r(t) = r.$$

Here,

$$B(0) = N(0)/H(\infty) = \overline{V}/A,$$

$$D(0) = N(0)\left(\frac{1}{H(\infty)} - r\right) = \overline{V}(1 - rH(\infty))/A,$$

$$b(0) = \frac{1}{H(\infty)}, \text{ and } d(0) = \frac{1}{H(\infty)} - r.$$

Proof. The two first relations follow immediately from (6.3) and (6.4). By differentiation with respect to a, we see that $n^a(t) = e^{rt}n^a(0)$. The rest of the Corollary is now easily obtained from Theorem 4.7.□

Remark. Notice that the average reproductive value per individual in a stable population is $\overline{V}/N(0) = A/H(\infty)$, and compare with the occurrence of these quantities in the theorems of Section 5.

6C. The reproductive value. We introduced the reproductive value $V(x)$ at the end of Subsection 5A, and it turned out to play a central role in the limit expressions in Theorems 5.2 to 5.4. We now prove a stability relation for it, similar to the one for $C(x)$ in (6.4).

<u>Theorem 6.3</u>. In a Malthusian process,

$$\int_0^\infty V(x)N_u^{dx}(t) = e^{rt}V(u).$$

<u>Proof</u>. Let $0 < a < \infty$ so that $C(a) > 0$. By Theorem 6.1,

$$\int_0^\infty e^{-rs}N_x^a(s)N_u^{dx}(t) = e^{rt}e^{-r(s+t)}N_u^a(s+t). \qquad (6.6)$$

By (5.5), the right hand side here converges to $e^{rt}V(u)\ C(a)H(\infty)/A$ as $s \to \infty$. The rest of the proof consists in showing that the left hand side of (6.6) converges to

$$\int_0^\infty V(x)N_u^{dx}(t)C(a)H(\infty)/A.$$

The latter integral exists and is finite because $V(\cdot)$ is bounded by the definition of a Malthusian process. By (5.5) again,

$$e^{-rs}N_x^a(s) \to V(x)C(a)H(\infty)/A.$$

Given the proof of Theorem 5.2, our present Theorem follows by dominated convergence. □

<u>6D. The operator semigroup</u>. We shall close this paper by pointing out the strong formal analogy between the theory for discrete time and age parameters on the one hand and the continuous time situation considered in the present paper on the other hand. We shall not go into the details of functional analysis, but one may obviously define an operator $\acute{N}_t$ on a suitable space of functions on $[0,\infty)$ by letting

$$\acute{N}_t f(x) = \int_0^\infty f(u)N_x^{du}(t).$$

The adjoint operator $\acute{N}_t^*$ is given by

$$\acute{N}_t^* F(a) = \int_0^\infty N_u^a(t)F(du)$$

for positive measures represented by the distribution function F.

Then $\{\acute{N}_t : 0 \leqq t < \infty\}$ satisfies the semigroup property $\acute{N}_{t+s} = \acute{N}_t \acute{N}_s$ by Theorem 6.1. Furthermore, in a Malthusian process with intrinsic growth rate r, e^{rt} is an eigenvalue of $\acute{N}_t$ and the reproductive value $V(\cdot)$ is the corresponding eigenvector by Theorem 6.3. The eigenvector for $\acute{N}_t^*$ corresponding to the eigenvalue e^{rt} is the stable age distribution $C(\cdot)$.

The convergence theorems may also be paraphrased in the language of the operator semigroup. Define the operator $V \otimes C$ and its adjoint $C \otimes V$ by letting

$$V \otimes Cf(x) = V(x)\int_0^\infty f(u)C(du)$$

and

$$C \otimes VF(a) = C(a)\int_0^\infty V(u)F(du).$$

Then (5.5) corresponds to the statements that, as $t \to \infty$,

$$e^{-rt}\acute{N}_t \to \frac{H(\infty)}{A} V \otimes C$$

and

$$e^{-rt}\acute{N}_t^* \to \frac{H(\infty)}{A} C \otimes V.$$

It seems an interesting task to make these concepts precise and to investigate the possibility of a direct operator-theoretical treatment of the asymptotic theory using results such as those by Karlin (1959).

ACKNOWLEDGEMENTS

We are grateful to Peter Jagers for making a pre-publication version of his forthcoming book (Jagers, 1973) available to us. Some discussions with Martin Jacobsen were helpful also.

A preliminary version of this paper was published as Technical Report No. 60, NSF Grant GP-30711X-2, at the Department of Statistics, Stanford University, USA.

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